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Janko Marovt and Dijana Mosić

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ON 2MP-, MP2-, AND CMP2-INVERSES IN $*$-RINGS

Janko Marovt and Dijana Mosić

Abstract. The notions of a 2MP-inverse, a MP2-inverse, and a C2MP-inverse are extended from the set of all $m \times n$ complex matrices to the set $\mathcal{R}^\dagger$ of all Moore-Penrose invertible elements in a unital $*$-ring $\mathcal{R}$. We study properties of these hybrid generalized inverses and thus generalize some known results. We apply the (b, c)-inverse of $a \in \mathcal{R}^\dagger$ to determine a special case of a 2MP- or MP2-inverse of a and then use these inverses to solve certain equations which lead to least-squares solutions and the normal equation.

1. Introduction

Let $\mathcal{R}$ be a $*$-ring, i.e., a ring equipped with an involution $*$. There are many generalized inverses that may be defined on $\mathcal{R}$ and two of the best known are the Moore-Penrose inverse and an inner generalized inverse. We call an element $a \in \mathcal{R}$ *Moore-Penrose invertible* or $*$-*regular* with respect to $*$ if there exists $x \in \mathcal{R}$ that satisfies the following four equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa. \tag{1.1}$$

If such x exists, we write $x = a^\dagger$ and call it the *Moore-Penrose inverse* of a. It is known that $a^\dagger$ is unique if it exists. The set of all $*$-regular elements in $\mathcal{R}$ is denoted by $\mathcal{R}^\dagger$. We say that $a \in \mathcal{R}$ is *regular* if there exists $x \in \mathcal{R}$ that satisfies the first equation in (1.1). Such x, if it exists, is called an *inner generalized inverse* or $\{1\}$-*inverse* of a, and we write $x = a^-$, i.e., $aa^-a = a$. The set of all $\{1\}$-inverses of a is denoted by $a\{1\}$ and we denote the set of all regular elements in $\mathcal{R}$ by $\mathcal{R}^{(1)}$. If there exists $x \in \mathcal{R}$ that satisfies the second equation in (1.1), then such x is called an *outer generalized inverse* or $\{2\}$-*inverse* of a, and we write $x = a^{2-}$, i.e., $a^{2-}aa^{2-} = a^{2-}$. The set of all $\{2\}$-inverses of a is denoted by $a\{2\}$, and we denote the set of all elements in $\mathcal{R}$ that have an outer inverse by $\mathcal{R}^{(2)}$.

A ring $\mathcal{R}$ where every element is $*$-regular is called a $*$-*regular ring*. An example of a $*$-regular ring is the set $M_n(\mathbb{C})$ of all complex $n \times n$ matrices where A^* denotes the conjugate transpose of

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$A \in M_n(\mathbb{C})$. The above generalized inverses are defined in the same way on the set $M_{m,n}(\mathbb{C})$ of all $m \times n$ complex matrices, and it is known that every matrix $A \in M_{m,n}(\mathbb{C})$ has an inner generalized inverse $A^- \in M_{n,m}(\mathbb{C})$, an outer generalized inverse $A^{2-} \in M_{n,m}(\mathbb{C})$, and the unique Moore-Penrose inverse $A^\dagger \in M_{n,m}(\mathbb{C})$. Two new types of hybrid generalized inverses were introduced and studied in [3] on $M_{m,n}(\mathbb{C})$ (see also [9]). Let $A \in M_{m,n}(\mathbb{C})$. For each outer generalized inverse A^{2-} of A, the matrices

$$A^{2MP} = A^{2-}AA^\dagger \quad \text{and} \quad A^{MP2} = A^\dagger AA^{2-}$$

are called a *2MP-inverse* and a *MP2-inverse* of A, respectively. Observe that $A^{2MP}A = A^{2-}A$ and $AA^{MP2} = AA^{2-}$ and thus

$$AA^{2MP}A = AA^{2-}A = AA^{MP2}A. \tag{1.2}$$

Since there may be many outer generalized inverses A^{2-} of A, A^{2MP} and A^{MP2} are (in general) not unique. In the case when the range and the null space of A^{2-} are fixed, the 2MP-inverse and the MP2-inverse of A reduce to the unique OMP inverse and MPO inverse, respectively, proposed in [7, 8] as follows. Let $A^{2-}_{T,S}$ denote the (unique) outer generalized inverse of $A \in M_{m,n}(\mathbb{C})$ with the range T and the null-space S. Then

$$A^{(2),\dagger} = A^{2-}_{T,S}AA^\dagger \quad \text{and} \quad A^{\dagger,(2)} = A^\dagger AA^{2-}_{T,S}$$

are called *the outer Moore-Penrose* (or OMP) *inverse* and *the Moore-Penrose outer* (or MPO) *inverse* of A, respectively.

Recall that the *Drazin inverse* of $A \in M_n(\mathbb{C})$ is the unique matrix $X \in M_n(\mathbb{C})$ that satisfies

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k$$

for some nonnegative integer k. The Drazin inverse, which exists for every $A \in M_n(\mathbb{C})$, is denoted by A^D. Note that A^D is an outer generalized inverse of A. In [6], a *CMP-inverse* of a matrix $A \in M_n(\mathbb{C})$ was introduced as

$$A^{c\dagger} = A^\dagger A_1 A^\dagger$$

where A_1 is the core part in the core-nilpotent decomposition of A, i.e., $A_1 = AA^DA$. As a generalization of CMP-inverses from square to rectangular matrices, another generalized inverse was introduced and studied in [3]. For $A \in M_{m,n}(\mathbb{C})$ and for each outer generalized inverse A^{2-} of A, we call the matrix

$$C_2^A = AA^{2-}A$$

a *2MP core-part* of A (see (1.2)). For each outer generalized inverse A^{2-} of $A \in M_{m,n}(\mathbb{C})$, the matrix

$$A^{C2MP} = A^\dagger C_2^A A^\dagger$$

is called a *C2MP-inverse* of A. Note that for $A \in M_{m,n}(\mathbb{C})$, C_2^A and A^{C2MP} are not (in general) unique. Also, for $A \in M_n(\mathbb{C})$, take $A^{2-} = A^D$, and observe that then $A^{C2MP} = A^{c\dagger}$.

The aim of this paper is to extend the concepts of 2MP-, MP2-, and C2MP-inverses to the set $\mathcal{R}^\dagger$ of all $*$-regular elements in a $*$-ring $\mathcal{R}$, and present some characterizations and properties of these hybrid generalized inverses.

2. Preliminaries

In this section, let $\mathcal{R}$ be a $*$-ring with the (multiplicative) identity 1. If for $p \in \mathcal{R}$, $p^2 = p$, then p is said to be an *idempotent*. A *projection* $p \in \mathcal{R}$ is a self-adjoint idempotent, i.e., $p = p^2 = p^*$. The equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n$ are idempotents in $\mathcal{R}$ and $e_i e_j = 0$ for $i \neq j$, is called a *decomposition of the identity of* $\mathcal{R}$. Let $1 = e_1 + e_2 + \cdots + e_n$ and $1 = f_1 + f_2 + \cdots + f_n$ be two decompositions of the identity of $\mathcal{R}$. We have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j.$$

Then any $x \in \mathcal{R}$ can be uniquely represented in the following matrix form:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f} \tag{2.3}$$

where $x_{ij} = e_i x f_j \in e_i \mathcal{R} f_j$. With $e \times f$ we emphasize the use of the decompositions of the identity $1 = e_1 + e_2 + \cdots + e_n$ on the left side and $1 = f_1 + f_2 + \cdots + f_n$ on the right side of $x = 1 \cdot x \cdot 1$. If $x = (x_{ij})_{e \times f}$ and $y = (y_{ij})_{e \times f}$, then $x + y = (x_{ij} + y_{ij})_{e \times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is another decomposition of the identity of $\mathcal{R}$ and $z = (z_{ij})_{f \times g}$, then, by the orthogonality of the idempotents involved, $xz = \left(\sum_{k=1}^{n} x_{ik} z_{kj}\right)_{e \times g}$. Thus, if we have decompositions of the identity of $\mathcal{R}$, then the usual algebraic operations in $\mathcal{R}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{R}$. When $n = 2$ and $p, q \in \mathcal{R}$ are idempotents, we may write

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p \times q}.$$

Here $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$, $x_{22} = (1-p)x(1-q)$.

By (2.3) we may write

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{n1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{nn}^* \end{bmatrix}_{f^* \times e^*},$$

where this matrix representation of x^* is given relative to the decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_n^*$.

Let $a \in \mathcal{R}$ and let a° denote the right annihilator of a, i.e., the set $a^\circ = \{x \in \mathcal{R} : ax = 0\}$. Similarly we denote the left annihilator ${}^\circ a$ of a, i.e., the set ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$. Suppose that $p, q \in \mathcal{R}$ are such idempotents that ${}^\circ a = {}^\circ p$ and $a^\circ = q^\circ$. Observe (or see [1, Lemma 2.2]) that ${}^\circ p = \mathcal{R}(1-p)$ and $q^\circ = (1-q)\mathcal{R}$. It follows that then $a = paq$, i.e.,

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q}. \tag{2.4}$$

Let $a \in \mathcal{R}^{(2)}$, $x \in a\{2\}$, and let us represent x with

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{q \times p}.$$

Then

$$xax = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{q\times p} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{q\times p} = \begin{bmatrix} x_{11}ax_{11} & x_{11}ax_{12} \\ x_{21}ax_{11} & x_{21}ax_{12} \end{bmatrix}_{q\times p}.$$

Since $xax = x$, it follows that $x_{11} = x_{11}ax_{11}$, $x_{12} = x_{11}ax_{12}$, $x_{21} = x_{21}ax_{11}$, and $x_{22} = x_{21}ax_{12}$. Let $t = ax_{12}$ and $u = x_{21}a$. Then $x_{22} = (x_{21}a)x_{12} = ux_{12} = ux_{11}(ax_{12}) = ux_{11}t$ and thus

$$x = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p}.$$

Conversely, let

$$x = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p}$$

with $x_{11} = x_{11}ax_{11}$, $t \in p\mathcal{R}(1-p)$, and $u \in (1-q)\mathcal{R}q$. Then

$$\begin{aligned} xax &= \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} \\ &= \begin{bmatrix} x_{11}ax_{11} & x_{11}ax_{11}t \\ ux_{11}ax_{11} & ux_{11}ax_{11}t \end{bmatrix}_{q\times p} = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} = x. \end{aligned}$$

Let $a \in \mathcal{R}^{(2)}$ and suppose that there exist idempotents $p, q \in \mathcal{R}$ such that a has the matrix form (2.4). We showed that then $x \in a\{2\}$ if and only if

$$x = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p}$$

where $t \in p\mathcal{R}(1-p)$, and $u \in (1-q)\mathcal{R}q$ are arbitrary (but fixed) elements and $x_{11} = x_{11}ax_{11}$.

3. 2MP-inverses in rings

Let $\mathcal{R}$ be a ring with identity 1 and let $a \in \mathcal{R}^{(2)}$. We next define a binary relation $\sim_l$ on the set $a\{2\}$ as follows. For $a^{2-}, a^{2=} \in a\{2\}$ we write

$$a^{2-} \sim_l a^{2=} \quad \text{if} \quad a^{2-}a = a^{2=}a.$$

Clearly, $\sim_l$ is an equivalence relation and for a given $a^{2-} \in a\{2\}$ its equivalence class is the set

$$\left[a^{2-}\right]_{\sim_l} = \left\{a^{2=} \in a\{2\} \colon a^{2=}a = a^{2-}a\right\}.$$

Suppose there exist idempotents $p, q \in \mathcal{R}$ such that ${}^\circ a = {}^\circ p$ and $a^\circ = q^\circ$. Let a have the matrix form (2.4) and let $a^{2-}, a^{2=} \in a\{2\}$ with

$$a^{2-} = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} \quad \text{and} \quad a^{2=} = \begin{bmatrix} x'_{11} & x'_{11}t' \\ u'x'_{11} & u'x'_{11}t' \end{bmatrix}_{q\times p}$$

where $t, t' \in p\mathcal{R}(1-p)$, $u, u' \in (1-q)\mathcal{R}q$, $x_{11} = x_{11}ax_{11}$, and $x'_{11} = x'_{11}ax'_{11}$. Suppose $a^{2=} \in \left[a^{2-}\right]_l$. Since then $a^{2=}a = a^{2-}a$, we obtain

$$\begin{bmatrix} x_{11}a & 0 \\ ux_{11}a & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} x'_{11}a & 0 \\ u'x'_{11}a & 0 \end{bmatrix}_{q\times q}$$

and hence $x_{11}a = x'_{11}a$ and $ux_{11}a = u'x'_{11}a$. It follows that $x_{11} - x'_{11} \in {}^\circ a$ and $ux_{11} - u'x'_{11} \in {}^\circ a$. From ${}^\circ a = {}^\circ p$ we get $x_{11}p = x'_{11}p$ and $ux_{11}p = u'x'_{11}p$, but since $x_{11}, x'_{11} \in q\mathcal{R}p$ we obtain that $x_{11} = x'_{11}$ and $ux_{11} = u'x'_{11}$. Conversely, if $x_{11} = x'_{11}$ and $ux_{11} = u'x'_{11}$, then $x_{11}a = x'_{11}a$ and

$ux_{11}a = u'x'_{11}a$, and thus $a^{2=}a = a^{2-}a$. We proved that $a^{2=} \in \left[a^{2-}\right]_{\sim_l}$ if and only if $x_{11} = x'_{11}$ and $ux_{11} = u'x'_{11}$. So, for $a^{2-} \in a\{2\}$ with

$$a^{2-} = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p}, \tag{3.5}$$

where $t \in p\mathcal{R}(1-p)$, $u \in (1-q)\mathcal{R}q$, and $x_{11} = x_{11}ax_{11}$, it follows that

$$\left[a^{2-}\right]_{\sim_l} = \left\{ \begin{bmatrix} x_{11} & x_{11}t' \\ ux_{11} & ux_{11}t' \end{bmatrix}_{q\times p} : t' \in p\mathcal{R}(1-p) \text{ is arbitrary} \right\}.$$

If we pick $t' = 0$, we get a representative

$$\begin{bmatrix} x_{11} & 0 \\ ux_{11} & 0 \end{bmatrix}_{q\times p}$$

of this equivalence class and hence a complete set of representatives of the partition of $a\{2\}$ induced by $\sim_l$ is given by

$$\text{Rep}_{\sim_l} = \left\{ \begin{bmatrix} x_{11} & 0 \\ ux_{11} & 0 \end{bmatrix}_{q\times p} : u \in (1-q)\mathcal{R}q \text{ is arbitrary and } x_{11}ax_{11} = x_{11} \right\}.$$

From now until the end of Section 3, let $\mathcal{R}$ be a ∗-ring with identity 1.

Remark 3.1. Suppose $a \in \mathcal{R}^\dagger$ and let $p = aa^\dagger$ and $q = a^\dagger a$. Then p and q are projections. Moreover, $pa = a$ and $aq = a$, and so ${}^\circ a = {}^\circ p$ and $a^\circ = q^\circ$. We may thus write a in the matrix form (2.4). Let $a^{2-} \in a\{2\}$ be represented with the matrix form (3.5). It follows that

$$a^{2-}aa^\dagger = a^{2-}p = \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} x_{11}p & 0 \\ ux_{11}p & 0 \end{bmatrix}_{q\times p} = \begin{bmatrix} x_{11} & 0 \\ ux_{11} & 0 \end{bmatrix}_{q\times p}.$$

Here $x_{11}ax_{11} = x_{11}$. Thus, every element of $\text{Rep}_{\sim_l}$ can be factorized as $a^{2-}aa^\dagger$ for $a \in \mathcal{R}^\dagger$ and for some $a^{2-} \in a\{2\}$.

We now extend the notion of a 2MP-inverse to the set of all ∗-regular elements in a ∗-ring.

Definition 3.2. *Let $a \in \mathcal{R}^\dagger$. For each $a^{2-} \in a\{2\}$ we call the element*

$$a^{2MP} = a^{2-}aa^\dagger$$

a 2MP-inverse of a. We denote

$$a\{2MP\} = \left\{a^{2-}aa^\dagger : a^{2-} \in a\{2\}\right\}.$$

Remark 3.3. Observe that a^{2MP} is the most simple representative of the equivalence class $\left[a^{2-}\right]_{\sim_l}$. Since $a^\dagger \in a\{2MP\}$, it follows that $a\{2MP\}$ is nonempty for every $a \in \mathcal{R}^\dagger$. Also, clearly, $0 \in a\{2MP\}$. Suppose $a \in \mathcal{R}^\dagger$ is written as (2.4) where $p = aa^\dagger$ and $q = a^\dagger a$. Then

$$a\{2MP\} = \left\{ \begin{bmatrix} x_{11} & 0 \\ ux_{11} & 0 \end{bmatrix}_{q\times p} : u \in (1-q)\mathcal{R}p \text{ is arbitrary and } x_{11}ax_{11} = x_{11} \right\}.$$

We prove the following result in the same way as [3, Proposition 2.3].

Proposition 3.4. *Let $a \in \mathcal{R}^\dagger$. Then there exists a bijective map between the quotient set $a\{2\}/\sim_l$ of $a\{2\}$ by $\sim_l$ and the set $a\{2MP\}$.*

For $a \in \mathcal{R}^\dagger$ and for each $a^{2-} \in a\{2\}$ we define

$$c_2^a = aa^{2MP}a \tag{3.6}$$

and call this element the *2MP core-part of* a. Since $a^{2MP} \in \left[a^{2-}\right]_{\sim_l}$, we have $a^{2MP}a = a^{2-}a$ and thus

$$c_2^a = aa^{2-}a. \tag{3.7}$$

REMARK 3.5. Note that if $a \in \mathcal{R}^\dagger$ and $a^{2-} \in a\{2\}$ are represented with the matrix forms (2.4) and (3.5), respectively, then

$$c_2^a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \begin{bmatrix} x_{11} & x_{11}t \\ ux_{11} & ux_{11}t \end{bmatrix}_{q\times p} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} ax_{11}a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = ax_{11}a.$$

Many new generalized inverses have been introduced recently as solutions of certain systems of equations (see, e.g., [5, 9]). With the next result we characterize 2MP-inverses in terms of solutions of systems of equations. Observe that

$$c_2^a a^{2MP} = a\left(a^{2-}aa^{2-}\right)aa^\dagger = aa^{2-}aa^\dagger = c_2^a a^\dagger.$$

Since also $a^{2MP} \in a\{2\}$ and $a^{2MP}a = a^{2-}a$, the element a^{2MP} is a solution of the following system of equations: $xax = x$, $xa = a^{2-}a$, $c_2^a x = c_2^a a^\dagger$. It is easy to prove (or see [3, proof of Theorem 2.5]) that it is the unique solution of the system.

PROPOSITION 3.6. *Let $a \in \mathcal{R}^\dagger$. For each $a^{2-} \in a\{2\}$, the element a^{2MP} is the unique solution to the following system of equations*

$$\text{(i) } xax = x, \quad \text{(ii) } xa = a^{2-}a, \quad \text{(iii) } c_2^a x = c_2^a a^\dagger. \tag{3.8}$$

Let ImA and Ker$A$ denote the image (i.e., the column space) and the kernel (i.e., the null space) of $A \in M_n(\mathbb{C})$. Note that for $A, B \in M_n(\mathbb{C})$ we have (see [4, proof of Lemma 2.1])

$$\mathrm{Im}A \subseteq \mathrm{Im}B \quad \text{if and only if} \quad {}^\circ B \subseteq {}^\circ A$$

and

$$\mathrm{Ker}A \subseteq \mathrm{Ker}B \quad \text{if and only if} \quad A^\circ \subseteq B^\circ.$$

Let $A \in M_n(\mathbb{C})$ and let A^{2-} be an outer generalized inverse of A. By [3, Theorem 2.6], AA^{2MP} is an idempotent matrix with $\mathrm{Im}\left(AA^{2MP}\right) = \mathrm{Im}\left(AA^{2MP}A\right)$ and $\mathrm{Ker}\left(AA^{2MP}\right) = \mathrm{Ker}A^{2MP}$, and $A^{2MP}A$ is an idempotent matrix such that $\mathrm{Im}\left(A^{2MP}A\right) = \mathrm{Im}A^{2MP}$ and $\mathrm{Ker}\left(A^{2MP}A\right) = \mathrm{Ker}\left(AA^{2MP}A\right)$. We now extend this result to the $*$-ring setting.

THEOREM 3.7. *Let $a \in \mathcal{R}^\dagger$. For a given $a^{2-} \in a\{2\}$, the element a^{2MP} satisfies the following properties:*

(a) ${}^\circ a^{2-} = {}^\circ a^{2MP}$.

(b) *aa^{2MP} is the idempotent with ${}^\circ\left(aa^{2MP}\right) = {}^\circ c_2^a$ and $\left(aa^{2MP}\right)^\circ = \left(a^{2MP}\right)^\circ$.*

(c) *$a^{2MP}a$ is the idempotent with ${}^\circ\left(a^{2MP}a\right) = {}^\circ a^{2MP}$ and $\left(a^{2MP}a\right)^\circ = \left(c_2^a\right)^\circ$.*

PROOF. (a) Clearly, by Definition 3.2, ${}^\circ a^{2-} \subseteq {}^\circ a^{2MP}$. Let now $z \in {}^\circ a^{2MP}$ for some $z \in \mathcal{R}$. Then $0 = za^{2-}aa^\dagger$. Multiplying this equation from the right consequently first by a and then by a^{2-}, we get $0 = za^{2-}$. So, ${}^\circ a^{2-} = {}^\circ a^{2MP}$.

(b) Since $aa^{2MP}a = aa^{2-}a$ and $a^{2MP} = a^{2-}aa^\dagger$, we have

$$\left(aa^{2MP}\right)^2 = \left(aa^{2MP}a\right)a^{2MP} = a\left(a^{2-}aa^{2-}\right)aa^\dagger = aa^{2-}aa^\dagger = aa^{2MP}$$

and so aa^{2MP} is an idempotent. Let us now prove that ${}^{\circ}\left(aa^{2MP}\right) = {}^{\circ}c_2^a$. Let $z \in {}^{\circ}\left(aa^{2MP}\right)$ for some $z \in \mathcal{R}$. Then $0 = zaa^{2MP} = zaa^{2-}aa^{\dagger} = zc_2^a a^{\dagger}$ and thus $0 = zc_2^a a^{\dagger} a$. But since $c_2^a = aa^{2MP}a$, it follows that $c_2^a a^{\dagger} a = c_2^a$, and therefore, $z \in {}^{\circ}c_2^a$. Let now $z \in {}^{\circ}c_2^a$ for some $z \in \mathcal{R}$. Then $0 = zaa^{2-}a$ and thus $0 = zaa^{2-}aa^{\dagger} = zaa^{2MP}$. So, ${}^{\circ}\left(aa^{2MP}\right) = {}^{\circ}c_2^a$.

Let us now show that $\left(aa^{2MP}\right)^{\circ} = \left(a^{2MP}\right)^{\circ}$. Clearly, $\left(a^{2MP}\right)^{\circ} \subseteq \left(aa^{2MP}\right)^{\circ}$. If $aa^{2MP}z = 0$ for some $z \in \mathcal{R}$, then $0 = aa^{2-}aa^{\dagger}z$ and hence $0 = a^{2-}aa^{2-}aa^{\dagger}z = a^{2-}aa^{\dagger}z = a^{2MP}z$. So, $\left(aa^{2MP}\right)^{\circ} = \left(a^{2MP}\right)^{\circ}$.

We similarly prove (c). □

As a corollary to Theorem 3.7, we give another characterization of a 2MP-inverse. First, let us prove an auxiliary result.

Lemma 3.8. *Let $p_1, p_2 \in \mathcal{R}$ be two idempotent elements. If ${}^{\circ}p_1 = {}^{\circ}p_2$ and $p_1^{\circ} = p_2^{\circ}$, then $p_1 = p_2$.*

Proof. From ${}^{\circ}p_1 = {}^{\circ}p_2$ we have $(1 - p_1)p_2 = 0$ and so $p_2 = p_1 p_2$. By $p_1^{\circ} = p_2^{\circ}$ we obtain $p_1(1 - p_2) = 0$, i.e., $p_1 = p_1 p_2$. So, $p_1 = p_1 p_2 = p_2$. □

Corollary 3.9. *Let $a \in \mathcal{R}^{\dagger}$. For each $a^{2-} \in a\{2\}$, the 2MP-inverse a^{2MP} of a is the unique element x that satisfies the following conditions:*

(i) *ax is an idempotent with*

$$^{\circ}(ax) = {}^{\circ}c_2^a \quad \text{and} \quad (ax)^{\circ} = \left(c_2^a a^{\dagger}\right)^{\circ},$$

(ii) ${}^{\circ}a^{2-} \subseteq {}^{\circ}x$.

Proof. Condition (ii) is satisfied by Theorem 3.7. Also, aa^{2MP} is an idempotent with ${}^{\circ}\left(aa^{2MP}\right) = {}^{\circ}c_2^a$, and since $c_2^a a^{\dagger} = aa^{2-}aa^{\dagger} = aa^{2MP}$, the element $x = a^{2MP}$ satisfies also conditions in (i).

Let us prove the uniqueness. Suppose that $x_1, x_2 \in \mathcal{R}$ satisfy both (i) and (ii). Then ax_1 and ax_2 are idempotents with ${}^{\circ}(ax_1) = {}^{\circ}c_2^a = {}^{\circ}(ax_2)$ and $(ax_1)^{\circ} = \left(c_2^a a^{\dagger}\right)^{\circ} = (ax_2)^{\circ}$, and therefore by Lemma 3.8, $ax_1 = ax_2$. From (ii), ${}^{\circ}a^{2-} \subseteq {}^{\circ}x_1 \cap {}^{\circ}x_2$. So,

$$\left(1 - a^{2-}a\right) x_1 = 0 = \left(1 - a^{2-}a\right) x_2$$

and thus $x_1 = a^{2-}ax_1 = a^{2-}ax_2 = x_2$. □

The following characterizations of a 2MP-inverse can also be verified.

Theorem 3.10. *Let $a \in \mathcal{R}^{\dagger}$ and $x \in \mathcal{R}$. For a given $a^{2-} \in a\{2\}$, the following statements are equivalent:*

(i) $x = a^{2MP}$.
(ii) $x\mathcal{R} = a^{2-}\mathcal{R}$ *and* $ax = aa^{2-}aa^{\dagger}$.
(iii) $x\mathcal{R} \subseteq a^{2-}\mathcal{R}$ *and* $ax = aa^{2-}aa^{\dagger}$.
(iv) $x^*\mathcal{R} = aa^{\dagger}(a^{2-})^*\mathcal{R}$ *and* $xa = a^{2-}a$.
(v) $x^*\mathcal{R} \subseteq a\mathcal{R}$ *and* $xa = a^{2-}a$.

Proof. (i) ⇒ (ii): Since $x = a^{2MP} = a^{2-}aa^{\dagger}$, it follows that $ax = aa^{2-}aa^{\dagger}$ and

$$x\mathcal{R} = a^{2-}aa^{\dagger}\mathcal{R} = a^{2-}a\mathcal{R} = a^{2-}\mathcal{R}.$$

(ii) ⇒ (iii): This implication is clear.

(iii) $\Rightarrow$ (i): The hypothesis $x\mathcal{R} \subseteq a^{2-}\mathcal{R}$ implies $x = a^{2-}u$, for some $u \in \mathcal{R}$. Hence, by $ax = aa^{2-}aa^{\dagger}$,

$$x = a^{2-}a(a^{2-}u) = a^{2-}(ax) = (a^{2-}aa^{2-})aa^{\dagger} = a^{2MP}.$$

In a similar manner, we check the rest. □

Let $a, b \in \mathcal{R}$. If $a\mathcal{R} \subseteq b\mathcal{R}$, then $a = bu$ for some $u \in \mathcal{R}$, and thus ${}^{\circ}b \subseteq {}^{\circ}a$. Suppose now there exists $v \in \mathcal{R}$ such that $bvb = b$, i.e., $b \in v\{2\}$. If ${}^{\circ}b \subseteq {}^{\circ}a$, then $(1-bv)a = 0$ which implies $a\mathcal{R} \subseteq b\mathcal{R}$. Consequently, we obtain by Theorem 3.10 more characterizations of a 2MP-inverse.

Corollary 3.11. *Let $a \in \mathcal{R}^{\dagger}$ and $x \in v\{2\}$ for some $v \in \mathcal{R}$. For a given $a^{2-} \in a\{2\}$, the following statements are equivalent:*

(i) $x = a^{2MP}$.
(ii) ${}^{\circ}x = {}^{\circ}(a^{2-})$ *and* $ax = aa^{2-}aa^{\dagger}$.
(iii) ${}^{\circ}x \supseteq {}^{\circ}(a^{2-})$ *and* $ax = aa^{2-}aa^{\dagger}$.
(iv) ${}^{\circ}(x^*) = {}^{\circ}[aa^{\dagger}(a^{2-})^*]$ *and* $xa = a^{2-}a$.
(v) ${}^{\circ}(x^*) \supseteq {}^{\circ}a$ *and* $xa = a^{2-}a$.

Let us recall the definition of the (b, c)-inverse which is a special kind of the outer generalized inverse. For $a, b, c \in \mathcal{R}$, an element $x \in \mathcal{R}$ is a (b, c)-inverse of a if $xax = x$, $x\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}c$. The (b, c)-inverse of a is unique, if it exists, and denoted by $a^{||(b,c)}$ [2]. Applying a 2MP-inverse determined by the (b, c)-inverse $a^{||(b,c)}$ in place of an outer generalized inverse a^{2-}, we prove solvability of the next equation.

Theorem 3.12. *Let $a \in \mathcal{R}^{\dagger}$ and $b, c, d \in \mathcal{R}$. If $a^{||(b,c)}$ exists, the general solution to the equation*

$$cax = caa^{\dagger}d \tag{3.9}$$

is expressed as

$$x = a^{||(b,c)}aa^{\dagger}d + (1 - a^{||(b,c)}a)z, \tag{3.10}$$

for an arbitrary $z \in \mathcal{R}$.

Proof. By $\mathcal{R}a^{||(b,c)} = \mathcal{R}c$, notice that $c = ua^{||(b,c)}$ and $a^{||(b,c)} = vc$, for some $u, v \in \mathcal{R}$. Thus,

$$caa^{||(b,c)} = ua^{||(b,c)}aa^{||(b,c)} = ua^{||(b,c)} = c.$$

For x expressed by (3.10), we therefore get

$$cax = caa^{||(b,c)}aa^{\dagger}d + ca(1 - a^{||(b,c)}a)z = caa^{\dagger}d,$$

i.e., x is a solution to (3.9).

If equation (3.9) has a solution x, then, by $a^{||(b,c)} = vc$,

$$a^{||(b,c)}ax = v(cax) = (vc)aa^{\dagger}d = a^{||(b,c)}aa^{\dagger}d.$$

Thus, x has the form (3.10):

$$x = a^{||(b,c)}aa^{\dagger}d + x - a^{||(b,c)}ax = a^{||(b,c)}aa^{\dagger}d + (1 - a^{||(b,c)}a)x.$$

□

Since $c = ua^{||(b,c)}$ and $a^{||(b,c)} = vc$, for some $u, v \in \mathcal{R}$, note that equation (3.9) is satisfied if and only if

$$a^{||(b,c)}ax = a^{||(b,c)}aa^{\dagger}d.$$

Hence, any solution to (3.9) is a solution to $a^{||(b,c)}ax = a^{||(b,c)}aa^{\dagger}d$ and vice versa.

As a consequence of Theorem 3.12, we obtain the solvability of equation (3.10) with the constrain $d \in a\mathcal{R}$.

COROLLARY 3.13. *Let $a \in \mathcal{R}^\dagger$ and $b, c, d \in \mathcal{R}$. If $a^{||(b,c)}$ exists, the general solution to the equation*

$$cax = cd, \quad d \in a\mathcal{R}$$

is expressed as

$$x = a^{||(b,c)}d + (1 - a^{||(b,c)}a)z,$$

for an arbitrary $z \in \mathcal{R}$.

PROOF. The assumption $d \in a\mathcal{R}$ gives $d = aa^\dagger d$. The rest is clear by Theorem 3.12. □

We now study when equation (3.9) has the unique solution.

THEOREM 3.14. *Let $a \in \mathcal{R}^\dagger$ and $b, c, d \in \mathcal{R}$ such that $a^{||(b,c)}$ exists. Then $a^{||(b,c)}aa^\dagger d$ is the unique solution in $b\mathcal{R}$ to (3.9).*

PROOF. We firstly observe that $a^{||(b,c)}aa^\dagger d \in a^{||(b,c)}\mathcal{R} = b\mathcal{R}$. Theorem 3.12 implies that $a^{||(b,c)}aa^\dagger d$ is a solution to (3.9).

For two solutions $y \in b\mathcal{R}$ and $x = a^{||(b,c)}aa^\dagger d$ to equation (3.9), we get

$$cax = caa^\dagger d = cay$$

and thus

$$y - x \in (ca)^\circ \cap b\mathcal{R}.$$

Note that $\mathcal{R}a^{||(b,c)} = \mathcal{R}c$ implies $(ca)^\circ = (a^{||(b,c)}a)^\circ$, and that $a^{||(b,c)}aa^{||(b,c)} = a^{||(b,c)}$ yields $b\mathcal{R} = a^{||(b,c)}\mathcal{R} = a^{||(b,c)}a\mathcal{R}$. Thus,

$$y - x \in (a^{||(b,c)}a)^\circ \cap a^{||(b,c)}a\mathcal{R} = \{0\}.$$

Thus, $y = x = a^{||(b,c)}aa^\dagger d$ represents the unique solution in $b\mathcal{R}$ to (3.9). □

4. MP2-inverses in rings

Let $\mathcal{R}$ be a ring and $a \in \mathcal{R}$. Similarly to Section 3, we define an equivalence relation $\sim_r$ on the set $a\{2\}$ as follows. For $a^{2-}, a^{2=} \in a\{2\}$, we write

$$a^{2-} \sim_r a^{2=} \quad \text{if} \quad aa^{2-} = aa^{2=}.$$

Consider now a new ring $\mathcal{Q} = (\mathcal{R}, \circ)$ where

$$a \circ b := ba \tag{4.11}$$

for $a, b \in \mathcal{R}$. It is then easy to see that $b \in a\{2\}$ in the ring $\mathcal{R}$ if and only if $b \in a\{2\}$ in the ring $\mathcal{Q}$, and that $b \sim_r c$ in the ring $\mathcal{R}$ if and only if $b \sim_l c$ in the ring $\mathcal{Q}$. Also, if $\mathcal{R}$ is a $*$-ring, then $*$ is also an involution in $\mathcal{Q}$ which yields that $a \in \mathcal{R}^\dagger$ if and only if $a \in \mathcal{Q}^\dagger$.

From now on, let $\mathcal{R}$ be a $*$-ring with identity.

DEFINITION 4.1. *Let $a \in \mathcal{R}^\dagger$. For each $a^{2-} \in a\{2\}$ we call the element*

$$a^{MP2} = a^\dagger aa^{2-}$$

the MP2-inverse of a. We denote

$$a\{MP2\} = \left\{a^\dagger aa^{2-} : a^{2-} \in a\{2\}\right\}.$$

Note that $b \in a\{MP2\}$ in the ring $\mathcal{R}$ if and only if $b \in a\{2MP\}$ in the ring $\mathcal{Q}$. For $a \in \mathcal{R}$ observe that $z \in {}^{\circ}a$ in the ring $\mathcal{R}$ if and only if $z \in a^{\circ}$ in the ring $\mathcal{Q}$, and $z \in {}^{\circ}a$ in the ring $\mathcal{Q}$ if and only if $z \in a^{\circ}$ in the ring $\mathcal{R}$.

The next two results thus follow immediately if we apply (4.11) to Proposition 3.6 and Theorem 3.7, respectively.

Proposition 4.2. *Let $a \in \mathcal{R}^{\dagger}$. For each $a^{2-} \in a\{2\}$, the element a^{MP2} is the unique solution to the following system of equations:*

$$(i)\ xax = x, \quad (ii)\ xa = aa^{2-}, \quad (iii)\ xc_2^a = a^{\dagger}c_2^a.$$

Theorem 4.3. *Let $a \in \mathcal{R}^{\dagger}$. For a given $a^{2-} \in a\{2\}$, the element a^{MP2} satisfies the following properties:*

(a) $\left(a^{2-}\right)^{\circ} = \left(a^{MP2}\right)^{\circ}$.
(b) $a^{MP2}a$ *is the idempotent with* ${}^{\circ}\left(a^{MP2}a\right) = {}^{\circ}a^{MP2}$ *and* $\left(a^{MP2}a\right)^{\circ} = (c_2^a)^{\circ}$.
(c) aa^{MP2} *is the idempotent with* ${}^{\circ}\left(aa^{MP2}\right) = {}^{\circ}c_2^a$ *and* $\left(aa^{MP2}\right)^{\circ} = \left(a^{MP2}\right)^{\circ}$.

We may similarly obtain other results and observations, analogous to the ones from Section 3.

5. C2MP-inverses in rings

In this section, we extend the concept of C2MP-inverses to the set of all $*$-regular elements in a $*$-ring. Recall that for $a \in \mathcal{R}^{\dagger}$ and for each $a^{2-} \in a\{2\}$, c_2^a is defined with (3.6) (see also (3.7)).

Definition 5.1. *Let $a \in \mathcal{R}^{\dagger}$. For each outer generalized inverse a^{2-} of a, the element*

$$a^{C2MP} = a^{\dagger}c_2^a a^{\dagger}$$

is called a C2MP-inverse of a. We denote

$$a\{C2MP\} = \left\{a^{\dagger}\left(aa^{2MP}a\right)a^{\dagger} : a^{2MP} \in a\{2MP\}\right\}.$$

Since $a\{2MP\}$ is nonempty, it follows that $a\{C2MP\}$ is also nonempty for every $a \in \mathcal{R}^{\dagger}$. Also, since 2MP-inverses are not unique, the same holds also for C2MP-inverses.

Remark 5.2. Suppose that $a \in \mathcal{R}^{\dagger}$ and let $p = aa^{\dagger}$ and $q = a^{\dagger}a$. Let $a^{2-} \in a\{2\}$ be represented with the matrix form (3.5) with respect to projections p and q. By Remark 3.5 we have

$$a^{C2MP} = a^{\dagger}ax_{11}aa^{\dagger} = qx_{11}p$$

and since $x_{11} \in q\mathcal{R}p$, we may conclude that

$$a^{C2MP} = x_{11}.$$

Recall here that $x_{11}ax_{11} = x_{11}$.

We now list four propositions that may be proved in a similar way as corresponding results for matrices in $M_{m,n}(\mathbb{C})$. The proof of the first proposition is the same as the proof of [3, Proposition 4.3].

Proposition 5.3. *Let $a \in \mathcal{R}^{\dagger}$. For a given $a^{2MP} \in a\{2MP\}$, the element a^{C2MP} satisfies the following properties:*

(a) $a^{C2MP} = a^{MP2}aa^{2MP}$.
(b) $a^{C2MP} = a^{\dagger}aa^{2MP}aa^{\dagger}$.
(c) $a^{C2MP} \in a\{2\}$.

(d) $aa^{C2MP}a = c_2^a$.
(e) $aa^{C2MP} = c_2^a a^\dagger = aa^{2MP}$.
(f) $a^{C2MP}a = a^\dagger c_2^a = a^{MP2}a$.

By properties (c), (e), and (f) of Proposition 5.3, a^{C2MP} is a solution of the following system of equations: $xax = x$, $ax = c_2^a a^\dagger$, $xa = a^\dagger c_2^a$. It is easy to check (see [3, proof of Theorem 4.4]) that a^{C2MP} is the unique solution of this system.

PROPOSITION 5.4. *Let $a \in \mathcal{R}^\dagger$. For each $a^{2MP} \in a\{2MP\}$, the element a^{C2MP} is the unique solution to the following system of equations:*

$$\text{(i) } xax = x, \quad \text{(ii) } ax = c_2^a a^\dagger, \quad \text{(iii) } xa = a^\dagger c_2^a.$$

By using Proposition 5.3, we may prove the next proposition in the same way as [3, Proposition 4.6].

PROPOSITION 5.5. *Let $a \in \mathcal{R}^\dagger$ and let $p = aa^\dagger$ and $q = a^\dagger a$. For each $a^{2-} \in a\{2\}$, the element a^{C2MP} satisfies the following properties:*

(a) $a^{C2MP} \in a\{1\}$ *if and only if* $a^{2-} \in a\{1\}$.
(b) $a^{C2MP} = qa^{2-}p = qa^{2MP}p$.
(c) $a^{C2MP} \in c_2^a\{1\} \cap c_2^a\{2\}$.
(d) $c_2^a a^{C2MP} = aa^{C2MP}$.
(e) $a^{C2MP}c_2^a = a^{C2MP}a$.
(f) $pc_2^a q = c_2^a$.

PROPOSITION 5.6. *Let $a \in \mathcal{R}^\dagger$. For each $a^{2-} \in a\{2\}$, the following statements are equivalent:*

(i) $a^{C2MP} = a^\dagger$.
(ii) $c_2^a = a$.
(iii) $a^{2-} \in a\{1\}$.
(iv) $a^{2-} \in a\{1\} \cap a\{2\}$.
(v) $a^\dagger = x_{11}$ *where a and a^{2-} are represented with (2.4) and (3.5), respectively.*
(vi) $a^{C2MP} \in a\{1\}$.

PROOF. We may prove the equivalence of statements (i), (ii), (iii), (iv), and (vi) by Proposition 5.5 and the arguments from the proof of [3, Theorem 4.8]. Equivalence of statements (v) and (i) is a direct corollary of Remark 5.2. □

We end the paper with a result that extends [3, Theorem 4.10].

THEOREM 5.7. *Let $a \in \mathcal{R}^\dagger$ and let $p = aa^\dagger$ and $q = a^\dagger a$. For each $a^{2-} \in a\{2\}$ written as in (3.5), the following statements are satisfied:*

(a) $\left(a^{C2MP}\right)^\dagger = x_{11}^\dagger$.
(b) $\left(a^\dagger\right)^{C2MP} = z$ *where $z \in p\mathcal{R}q$ with $za^\dagger z = z$.*
(c) $\left(a^{C2MP}\right)^\dagger = \left(a^\dagger\right)^{C2MP}$ *if and only if $x_{11}^\dagger = z$ where $z \in p\mathcal{R}q$ with $za^\dagger z = z$.*
(d) $a^{C2MP} = a^*$ *if and only if* $a = x_{11}^*$.
(e) $a^{C2MP} = 0$ *if and only if* $a^{2-} = 0$ *if and only if* $c_2^a = 0$.

PROOF. Statement (a) follows directly by Remark 5.2.

(b) Note that

$$a^\dagger = \begin{bmatrix} a^\dagger & 0 \\ 0 & 0 \end{bmatrix}_{q \times p}.$$

Also, $\left(a^{\dagger}\right)^{\dagger} = a$ and so there exists $\left(a^{\dagger}\right)^{2-} \in \left(a^{\dagger}\right)\{2\}$. In accordance with (3.5), we write

$$\left(a^{\dagger}\right)^{2-} = \begin{bmatrix} z & zn \\ mz & mzn \end{bmatrix}_{p\times q},$$

where $m \in (1-p)\mathcal{R}p$, $n \in q\mathcal{R}(1-q)$, and $z = za^{\dagger}z$. Then

$$\left(a^{\dagger}\right)^{C2MP} = \left(a^{\dagger}\right)^{\dagger} c_2^{a^{\dagger}} \left(a^{\dagger}\right)^{\dagger} = aa^{\dagger}\left(a^{\dagger}\right)^{2-} a^{\dagger}a = p\left(a^{\dagger}\right)^{2-} q = z.$$

Statement (c) follows directly by statements (a) and (b).

(d) By Remark 5.2, $a^{C2MP} = x_{11}$, and thus $a^{C2MP} = a^*$ if and only if $a^* = x_{11}$ which is equivalent to $a = x_{11}^*$.

(e) Since $a^{C2MP} = x_{11}$, we have that $a^{C2MP} = 0$ if and only if $a^{2-} = 0$. Since $c_2^a = aa^{2-}a$, $a^{2-} = 0$ implies $c_2^a = 0$. If $c_2^a = 0$, then $aa^{2-}a = 0$ and thus

$$x_{11} = qa^{2-}p = a^{\dagger}\left(aa^{2-}a\right)a^{\dagger} = 0.$$

So, $a^{2-} = 0$. □

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J. Marovt
University of Maribor
Faculty of Economics and Business
Razlagova 14, SI-2000 Maribor, Slovenia,
and IMFM
Jadranska ulica 19, SI-1000 Ljubljana, Slovenia
E-mail: `janko.marovt@um.si`

D. Mosić
Faculty of Sciences and Mathematics
University of Niš
P.O. Box 224, 18000 Niš, Serbia
E-mail: `dijana@pmf.ni.ac.rs`