Igor Ciganović

Parabolic induction from two segments, linked under contragredient, with a one half cuspidal reducibility, a special case

Manuscript accepted
September 5, 2023.
PARABOLIC INDUCTION FROM TWO SEGMENTS,
LINKED UNDER CONTRAGREDIENT, WITH A ONE HALF
CUSPIDAL REDUCIBILITY, A SPECIAL CASE

Igor Ciganović
University of Zagreb, Croatia

Abstract. In this paper, we determine the composition series of the
induced representation \( \delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^b\rho,\nu^c\rho]) \rtimes \sigma \) where \( a, b, c \in \mathbb{Z}^+ \)
such that \( \frac{1}{2} \leq a < b < c \), \( \rho \) is an irreducible cuspidal unitary
representation of a general linear group and \( \sigma \) is an irreducible cuspidal
representation of a classical group such that \( \nu^{\frac{1}{2}}\rho \times \sigma \) reduces.

1. Introduction

The problem of determining the composition series of induced representa-
tions is important for the representation theory. Namely, classes of representa-
tions of certain interest, like irreducible, unitary, tempered representations or
discrete series, are often placed inside parabolically induced representations,
raising a question of their position. Despite the interest, complete description
of the composition factors of induced representations is known only in some
special cases, such as [2], [7], [9], [14] and [24], and for some low-rank groups.

This paper is a continuation of the effort from [3] and [4] to study classes of
parabolically induced representations similar to ones appearing in the Möglin-
Tadić classification of discrete series ([11], [13]). To explain this we introduce
some notation. Fix a local non-archimedean field \( F \) of characteristic different
than two. Let \( \rho \) be an irreducible cuspidal unitary representation of some
\( GL(m, F) \), and \( x, y \in \mathbb{R} \), such that \( y - x + 1 \in \mathbb{Z}_{\geq 0} \). By Zelevinsky classification,
the set \( \Delta = [\nu^x\rho,\nu^y\rho] = \{\nu^x\rho, \ldots, \nu^y\rho\} \) is called a segment. We have a

2020 Mathematics Subject Classification. Primary 22D30, Secondary 22E50, 22D12,
11F85.

Key words and phrases. Classical group, composition series, induced representations,
p-adic field, Jacquet module.
unique irreducible subrepresentation

$$\delta(\Delta) = \delta([\nu^x \rho, \nu^y \rho]) \hookrightarrow \nu^y \rho \times \cdots \times \nu^x \rho,$$

of the parabolically induced representation. If $$\Delta \subseteq \Delta',$$ then $$\delta(\Delta) \times \delta(\Delta') \cong \delta(\Delta') \times \delta(\Delta)$$ is irreducible. Set $$e(\Delta) = (x + y)/2.$$ Given a sequence of segments $$\Delta_1, \ldots, \Delta_k,$$ such that $$e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$$ and an irreducible tempered representation $$\tau,$$ of a symplectic or (full) orthogonal group, we have a unique irreducible quotient, called the Langlands quotient,

$$\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \times \tau \rightarrow L(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \times \tau),$$

of the parabolically induced representation. Let $$\rho$$ be as above, and $$\sigma$$ an irreducible cuspidal representation of a symplectic or (full) orthogonal group such that $$\nu^{\pm} \rho \times \sigma$$ reduces. Let $$a, b, c \in \mathbb{Z} + \frac{1}{2}$$ such that $$\frac{1}{2} \leq a < b < c.$$ In [3], we determined composition series of induced representation

$$\delta([\nu^b \rho, \nu^c \rho]) \times \delta([\nu^a \rho, \nu^b \rho]) \times \sigma.$$

The search for composition factors relied on decomposing kernels of intertwining operators using results of [14]. This approach held in [4], where we considered an arbitrary number of segments $$\Delta_i.$$ There, one of conditions was on segments involved: induced representations $$\delta(\Delta_i) \times \delta(\Delta_j)$$ and $$\delta(\Delta_i) \times \delta(\Delta_j), i \neq j,$$ are to be irreducible, where $$\tilde{\cdot}$$ stands for the contragredient. As a result, all discrete series there appeared as subrepresentations. Neither this property nor the condition holds in the present paper, where we consider composition series of induced representation

$$\delta([\nu^{-b} \rho, \nu^c \rho]) \times \delta([\nu^{\pm} \rho, \nu^b \rho]) \times \sigma,$$

even though all reducible subquotients, including discrete series, are preserved. Moreover, decomposing kernels of intertwining operators requires more basic tools developed in [14], complicating our search.

To describe our results, we introduce some discrete series, appearing as only irreducible subrepresentations in the following induced representations (for more details see Section 3):

$$\sigma_a \hookrightarrow \delta([\nu^{\pm} \rho, \nu^a \rho]) \times \sigma,$$

and similarly for $$\sigma_b$$ and $$\sigma_c.$$ Further

$$\sigma_{b,c}^+ \hookrightarrow \delta([\nu^{\pm} \rho, \nu^b \rho]) \times \sigma_c, \quad \sigma_{b,c}^- \hookrightarrow \delta([\nu^{-b} \rho, \nu^c \rho]) \times \sigma,$$

and similarly for $$\sigma_{a,c}^\pm.$$ Finally $$\sigma_{b,c,a}^\pm \hookrightarrow \delta([\nu^{\pm} \rho, \nu^a \rho]) \times \sigma_{b,c}^\pm.$$ Now we have
Theorem 1.1. Let $\psi = \delta([\nu^{-a}\rho, \nu^{c}\rho]) \times \delta([\nu^{b}\rho, \nu^{-b}\rho]) \rtimes \sigma$ and define representations

$$W_1 = \sigma_{b,c,a}^{+} + L(\delta([\nu^{b}\rho, \nu^{-b}\rho]) \rtimes \sigma_{b,c,a}^{+}) + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_{b,c,a}^{-}) + L(\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{b,c,a}) + \sigma_{b,c,a},$$

$$W_2 = L(\delta([\nu^{b}\rho, \nu^{a}\rho]) \rtimes \sigma_{b,c,a}^{+}) + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_{b,c,a}^{-}) + L(\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{b,c,a}) + \sigma_{b,c,a},$$

$$W_3 = L(\delta([\nu^{b}\rho, \nu^{a}\rho]) \rtimes \sigma_{b,c,a}^{+}) + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_{b,c,a}^{-}) + L(\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{b,c,a}) + \sigma_{b,c,a},$$

$$W_4 = L(\psi).$$

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \psi$, such that

$$V_i/V_{i-1} \cong W_i, \quad i = 1, \ldots, 4.$$  

Further, $W_1$ is chosen to be the largest possible, then $W_2$, and so on.

Now we describe the content of the paper. After Preliminaries, we fix the notation in Section 3 and collect some reducibility results. Intertwining operators and an approach to decompose the induced representation are considered in Section 4. In Section 5, we determine the occurring discrete series. The remaining non-tempered candidates, not provided in Section 4, are listed in Section 6. Their occurrence is confirmed in Sections 7-9. Composition factors are described in Section 10. To determine composition series, we decompose kernels of intertwining operators in Sections 11-13, and provide the main result in Section 14.

2. Preliminaries

Let $F$ be a local non-archimedean field of characteristic different than two. As in [13], fix a tower of symplectic or orthogonal non-degenerate $F$ vector spaces $V_n$, $n \geq 0$ where $n$ is the Witt index. We denote by $G_n$ the group of isometries of $V_n$. It has split rank $n$. Also, we fix the set of standard parabolic subgroups in the usual way. Standard parabolic proper subgroups of $G_n$ are in bijection with the set of ordered partitions of positive integers $m \leq n$:

$$\{ s = (n_1, \ldots, n_k) \mid n_1 + \cdots + n_k = m, k > 0 \} \leftrightarrow P_s,$$

$$P_s = M_sN_s, \quad \text{Levi factorization with } M_s \text{ Levi factor},$$

$$M_s \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n-m}.$$  

By $\text{Alg } G_n$ we denote smooth representations of $G_n$, $\text{Irr } G_n$ irreducible representations, and subscript $f.l.$ means finite length, $u$ unitary, and $cusp$ cuspidal. Also denote $\text{Alg } G = \cup_{n \geq 0} \text{Alg } G_n$, and so on. We use a similar notation.
for $GL(n, F)$. For $\delta_i \in \text{Alg } GL(n_i, F), i = 1, \ldots, k$ and $\tau \in \text{Alg } G_{n-m}$, let
\[
\pi = \delta_1 \otimes \cdots \otimes \delta_k \otimes \tau \in \text{Alg } M_s
\]
and
\[
\delta_1 \times \cdots \times \delta_k \times \tau = \text{Ind}_{M_s}^{G_n}(\pi)
\]
be the representation induced from $\pi$ using normalized parabolic induction. If $\sigma \in \text{Alg } G_n$ we denote by $r_\sigma(\sigma) = r_{M_s}(\sigma) = \text{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of $\sigma$. We have the Frobenius reciprocity
\[
\text{Hom}_{G_n}(\sigma, \text{Ind}_{M_s}^{G_n}(\pi)) \cong \text{Hom}_{M_s}(\text{Jacq}_{M_s}^{G_n}(\sigma), \pi).
\]
Let $\rho, \nu \in \text{Irr}_{u, \text{cusp}} GL$ and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{\geq 0}$. The set
\[
\Delta = [\nu^x \rho, \nu^y \rho] = \{\nu^x \rho, \ldots, \nu^y \rho\}
\]
is called a segment. We have a unique irreducible subrepresentation
\[
\delta([\nu^x \rho, \nu^y \rho]) \hookrightarrow \nu^y \rho \times \cdots \times \nu^x \rho,
\]
of the induced representation, and it is essentially square integrable. We also denote $e([\nu^x \rho, \nu^y \rho]) = e(\delta([\nu^x \rho, \nu^y \rho])) = \frac{x + y}{2}$. For $y - x + 1 \in \mathbb{Z}_{<0}$ define $[\nu^x \rho, \nu^y \rho] = \emptyset$ and $\delta(\emptyset)$ is the irreducible representation of the trivial group. Let $\tilde{\Delta} = [\nu^{-y} \rho, \nu^{-x} \rho]$ where $\tilde{\rho}$ denotes the contragredient of $\rho$. We have $\delta(\Delta)^\gamma = \delta(\tilde{\Delta})$. By [24] if $\delta \in \text{Irr } GL$ is essentially square integrable, there exists a segment $\Delta$ such that $\delta = \delta(\Delta)$. Let $\delta_i = \delta(\Delta_i), e_i = e(\delta_i), i = 1, 2$. We have
\[
\delta_1 \times \delta_2 \text{ reduces } \Leftrightarrow \Delta_1 \cup \Delta_2 \text{ is a segment and } \Delta_1 \not\subseteq \Delta_2, \Delta_2 \not\subseteq \Delta_1.
\]
In that case, if $e_1 \geq e_2$, the induced representation has a unique irreducible quotient, called Langlands quotient, and a unique irreducible subrepresentation. They swap positions in $\delta_2 \times \delta_1$ and make composition factors. We have an exact sequence.
\[
\delta(\Delta_1 \cup \Delta_2) \times \delta(\Delta_1 \cap \Delta_2) \hookrightarrow \delta_1 \times \delta_2 \rightarrow L(\delta_1 \times \delta_2) = L(\delta_1, \delta_2).
\]
Given a sequence $\delta_i = \delta(\Delta_i), i = 1, \ldots, k$ such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ and $\tau \in \text{Irr } G$, tempered, the Langlands quotient is a unique irreducible quotient:
\[
\delta_1 \times \cdots \times \delta_k \times \tau \rightarrow L(\delta_1 \times \cdots \times \delta_k \times \tau),
\]
and it appears with multiplicity one in the induced representation. It is also a unique irreducible subrepresentation of
\[
\tilde{\delta}_1 \times \cdots \times \tilde{\delta}_k \times \tau \cong (\delta_1 \times \cdots \times \delta_k \times \tau)^\gamma.
\]
Permuting $\delta_i$-s and possibly taking contragredients does not change composition factors. Every irreducible representation of $G_n$ can be written as a Langlands quotient.
If \( \sigma \) is a discrete series representation of \( G_n \) then by the Mœglin-Tadić, now unconditional, classification ([11],[13]), it is described by an admissible triple

\[
(\text{Jord}, \sigma_{\text{cusp}}, \epsilon).
\]

Here Jord is a set of pairs \((a, \rho)\) where \( \rho \cong \rho \in \text{Irr}_{u, \text{cusp}} GL \) and \( a \in \mathbb{Z}_{> 0} \), of parity depending on \( \rho \), such that \( \delta([\nu^{-(a-1)/2} \rho, \nu^{(a-1)/2} \rho]) \times \sigma \) is irreducible, but there exists an integer \( a' > a \), of the parity same as \( a \) such that the induced representation reduces when we replace \( a \) with \( a' \). We write

\[
\text{Jord}_a = \{ a : (a, \rho) \in \text{Jord} \}
\]

and for \( a \in \text{Jord}_a \) let \( a_+ \) be the largest element of \( \text{Jord}_a \) strictly less than \( a \), if such exists. Next, there exists a unique, up to an isomorphism, \( \sigma_{\text{cusp}} \in \text{Irr}_{\text{cusp}} G \), such that there exists \( \pi \in \text{Irr} GL \) and \( \sigma \leftrightarrow \pi \rtimes \sigma_{\text{cusp}} \). It is called the partial cuspidal support of \( \sigma \). Finally, \( \epsilon \) is a function from a subset of \( \text{Jord} \cup (\text{Jord} \times \text{Jord}) \) into \( \{ \pm 1 \} \). Assume \((a, \rho) \in \text{Jord} \) and \( a \) is even. Then \( \epsilon(a, \rho) \) is defined, and if \( a = \min(\text{Jord}_a) \)

\[
\epsilon(a, \rho) \epsilon(a_-, \rho)^{-1} = 1 \iff \exists \pi'' \in \text{Irr} G, \quad \sigma \leftrightarrow \delta([\nu^{(a-1)/2} \rho, \nu^{(a-1)/2} \rho]) \times \pi''.
\]

Now we recall the Tadić formula for computing Jacquet modules of induced representations. Let \( R(G_n) \) be the Grothendieck group of the category of smooth representations of \( G_n \) of finite length. It is the free Abelian group generated by classes of irreducible representations of \( G_n \). If \( \sigma \) is a smooth finite length representation of \( G_n \) denote by \( \text{s.s.}(\sigma) \) the semisimplification of \( \sigma \), that is the sum of classes of composition series of \( \sigma \). Put \( R(G) = \oplus_{n \geq 0} R(G_n) \). Let \( R^+_n(G) \) be a \( \mathbb{Z}_{\geq 0} \) subspan of classes of irreducible representations. For \( \pi_1, \pi_2 \in R(G) \) we define \( \pi_1 \leq \pi_2 \) if \( \pi_2 - \pi_1 \in R^+_n(G) \). Similarly define \( R(GL(n, F)) = \oplus_{n \geq 0} R(GL(n, F)) \). We have the map \( \mu^* : R(G) \to R(GL) \otimes R(G) \) defined by

\[
\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^n \text{s.s.}(r_k(\sigma)), \quad \sigma \in R(G_n).
\]

The following result derives from Theorems 5.4 and 6.5 of [20], see also Section 1. in [13]. They are based on Geometrical Lemma (2.11 of [1]).

**Theorem 2.1.** Let \( \sigma \in \text{Alg}_{f,1} G \), and \( [\nu^xp, \nu^y\rho] \neq \emptyset \) a segment. Then

\[
\mu^*(\delta([\nu^xp, \nu^y\rho]) \times \sigma) = \sum_{\delta' \otimes \sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{y-x+1} \sum_{j=0}^{x-1} \delta([\nu^{y-j} \rho, \nu^{-x} \rho]) \times \delta([\nu^{y+1-j} \rho, \nu^y \rho]) \times \delta' \otimes \delta([\nu^{x+1-i} \rho, \nu^{y-j} \rho]) \times \sigma'.
\]
where \( \delta' \otimes \sigma' \) denotes an irreducible subquotient in the appropriate Jacquet module.

Now we provide results of [10], about Jacquet modules of some irreducible representations. Consider induced representation

\[
\pi = \delta([\nu^{-a}\rho, \nu^c\rho]) \times \sigma
\]

where \( \sigma \in \text{Irr}_{\text{cusp}} G, \ \tilde{\rho} \cong \rho \in \text{Irr}_{\text{cusp}} GL \), such that \( \nu^2 \rho \times \sigma \) reduces, and \( a, c \in \mathbb{Z} + \frac{1}{2} \), such that \( c - a \geq 0 \). We define terms \( \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) \), \( \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma) \), and \( L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma) \). Each of them is either an irreducible representation or zero.

- If \( \frac{1}{2} < -a \), then \( \pi \) is irreducible and \( L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma) = \pi \), while other terms are zero.
- If \( -a \leq \frac{1}{2} \) then \( \pi \) reduces. We denote by \( \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) \) a unique irreducible subquotient that has in its minimal standard Jacquet module at least one irreducible subquotient whose all exponents are non-negative.
- If \( -a = \frac{1}{2} \), then \( \pi \) is of length two, \( \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) \) is a discrete series subrepresentation and \( L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma) \) is the Langlands quotient of \( \pi \). The remaining term is zero.
- If \( -a \leq -\frac{1}{2} \), and \( a = c \), then \( \pi \) is a direct sum of two tempered representations, \( \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) \) and \( \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma) \). The remaining term is zero.
- If \( -a \leq -\frac{1}{2} \), and \( a \neq c \), then \( \pi \) is of length three. It has two discrete series representations, \( \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) \) and \( \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma) \), and \( L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma) \) is the Langlands quotient.

We have in \( R(G) \)

\[
\delta([\nu^{-a}\rho, \nu^c\rho]) \times \sigma = \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) + \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma) + L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma).
\]

where the right hand side equals to

\[
\begin{align*}
\frac{1}{2} < -a : & \quad L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma), \\
\frac{1}{2} = -a : & \quad \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) + L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma), \\
c = a, -a \leq -\frac{1}{2} : & \quad \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) + \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma), \\
c \neq a, -a \leq -\frac{1}{2} : & \quad \delta([\nu^{-a}\rho, \nu^c\rho]_+; \sigma) + \delta([\nu^{-a}\rho, \nu^c\rho]_-; \sigma) + L(\delta([\nu^{-a}\rho, \nu^c\rho]); \sigma).
\end{align*}
\]
Now we have

\[ \mu^* (\delta([\nu^{-a} \rho, \nu^c \rho]; \sigma)) = \pm \frac{1}{2} - \sum_{i=-a-1}^{a} \delta([\nu^{-i} \rho, \nu^a \rho]) \times \delta([\nu^{i+1} \rho, \nu^c \rho]) \otimes \sigma + \]

\[ \sum_{i=-a-1}^{a} \sum_{j=i+1}^{c} \delta([\nu^{-i} \rho, \nu^a \rho]) \times \delta([\nu^{i+1} \rho, \nu^c \rho]) \otimes \delta([\nu^{j+1} \rho, \nu^c \rho]; \sigma) \]

\[ \sum_{-a-1 \leq i \leq a} \sum_{i+1 \leq j \leq a} \delta([\nu^{-i} \rho, \nu^a \rho]) \times \delta([\nu^{i+1} \rho, \nu^c \rho]) \otimes L(\delta([\nu^{j+1} \rho, \nu^c \rho]; \sigma)). \]

If we write \( \delta([\nu^{\frac{1}{2}} \rho, \nu^{-\frac{1}{2}} \rho]; \sigma) \) for \( \sigma \), we have

\[ \mu^* (\delta([\nu^{\frac{1}{2}} \rho, \nu^c \rho]; \sigma)) = \sum_{j=-\frac{1}{2}}^{c} \delta([\nu^{j+1} \rho, \nu^c \rho]) \otimes \delta([\nu^{\frac{1}{2}} \rho, \nu^j \rho]; \sigma). \]

And for \( a < \frac{1}{2} \) or \( \frac{1}{2} \leq a < c \) we have

\[ \mu^* (L(\delta([\nu^{-a} \rho, \nu^c \rho]; \sigma))) = \sum_{i=\frac{1}{2}}^{c} L(\delta([\nu^{-i} \rho, \nu^a \rho]), \delta([\nu^{i+1} \rho, \nu^c \rho])) \otimes \sigma + \]

\[ \sum_{-a-1 \leq i \leq a} \sum_{i+1 \leq j \leq c} L(\delta([\nu^{-i} \rho, \nu^a \rho]), \delta([\nu^{i+1} \rho, \nu^c \rho])) \otimes L(\delta([\nu^{j+1} \rho, \nu^c \rho]; \sigma)). \]

### 3. Notation and Basic Reducibilities

In this section, we fix the notation and prepare some reducibility results. Let \( \rho \) be an irreducible unitary cuspidal representation of \( GL(m_\rho, F) \) and \( \sigma \) an irreducible cuspidal representation of \( G_n \) such that \( \nu^{\frac{1}{2}} \rho \rtimes \sigma \) reduces. By Proposition 2.4 of [17], \( \rho \) is self-dual. We consider

\[ \frac{1}{2} \leq a, b, c \in \mathbb{Z} + \frac{1}{2}, \]

that need not be fixed, but when appearing together in a formula, we have, depending on which appears, \( a < b < c \). We denote the representation we want to decompose

\[ \psi = \delta([\nu^{-a} \rho, \nu^c \rho]) \times \delta([\nu^{\frac{1}{2}} \rho, \nu^b \rho]) \rtimes \sigma. \]

Further, we shorten some notations from (2.2):

\[ \sigma_a = \delta([\nu^{\frac{1}{2}} \rho, \nu^a \rho]; \sigma), \quad \sigma_{b,c} = \delta([\nu^{\frac{1}{2}} \rho, \nu^b \rho]; \sigma), \quad \sigma_{b,c}^+ = \delta([\nu^{-b} \rho, \nu^c \rho]; \sigma). \]

The following result is Theorem 2.3 from [14].
Theorem 3.1. With discrete series being subrepresentations, we have in $R(G)$
\[
\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma = \sigma_a + L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma),
\]
\[
\delta([\nu^b \rho, \nu^c \rho]) \times \sigma = \sigma_{b,c}^+ + \sigma_{b,c}^- + L(\delta([\nu^b \rho, \nu^c \rho]) \times \sigma).
\]
Here
\[
\text{Jord}(\sigma_a) = \{(2a + 1, \rho)\} \cup \text{Jord}(\sigma).
\]
\[
\text{Jord}(\sigma_{b,c}^+) = \text{Jord}(\sigma_{b,c}^-) = \{(2b + 1, \rho), (2c + 1, \rho)\} \cup \text{Jord}(\sigma).
\]
Further, $\epsilon_{\sigma_a}, \epsilon_{\sigma_{b,c}^+},$ and $\epsilon_{\sigma_{b,c}^-}$ extend $\epsilon_\sigma$ such that $\epsilon_{\sigma_a}(2a + 1, \rho) = 1,$ and $\epsilon_{\sigma_{b,c}^+}(2b + 1, \rho) = 1, \epsilon_{\sigma_{b,c}^-}(2b + 1, \rho) = \epsilon_{\sigma_{b,c}^-}(2c + 1, \rho) = -1.$

The next proposition follows from Theorem 2.1 of [14].

Proposition 3.2. We use $\sigma_{b,c,a}^+ = \sigma_{a,b,c}^+$ and $\sigma_{b,c,a}^-$ to denote non-isomorphic discrete series, such that in $R(G)$ we have
\[
\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_a = \sigma_{b,c,a}^+ + \sigma_{b,c,a}^- + L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_a) \text{ and}
\]
\[
\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_c = \sigma_{b,c,a}^+ + \sigma_{b,c,a}^- + L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_c).
\]
These discrete series appear as subrepresentations in induced representations. Also
\[
\text{Jord}(\sigma_{b,c,a}^+) = \text{Jord}(\sigma_{b,c,a}^-) = \text{Jord}(\sigma_{a,b,c}) = \\
\{ (2a + 1, \rho), (2b + 1, \rho), (2c + 1, \rho) \} \cup \text{Jord}(\sigma)
\]
and $\epsilon_{\sigma_{b,c,a}^+}, \epsilon_{\sigma_{b,c,a}^-}$ extend $\epsilon_\sigma$ such that
\[
\epsilon_{\sigma_{b,c,a}^+}(2a + 1, \rho) = 1, \quad \epsilon_{\sigma_{b,c,a}^+}(2b + 1, \rho) = 1, \quad \epsilon_{\sigma_{b,c,a}^-}(2c + 1, \rho) = 1,
\]
\[
\epsilon_{\sigma_{b,c,a}^-}(2a + 1, \rho) = 1, \quad \epsilon_{\sigma_{b,c,a}^-}(2b + 1, \rho) = -1, \quad \epsilon_{\sigma_{b,c,a}^-}(2c + 1, \rho) = -1,
\]
\[
\epsilon_{\sigma_{a,b,c}}(2a + 1, \rho) = -1, \quad \epsilon_{\sigma_{a,b,c}}(2b + 1, \rho) = -1, \quad \epsilon_{\sigma_{a,b,c}}(2c + 1, \rho) = 1.
\]

Observe that
\[
\mu^*(\sigma_{b,c,a}^+) \geq \delta([\nu^{-a} \rho, \nu^b \rho]) \otimes \sigma_c + \delta([\nu^{-b} \rho, \nu^c \rho]) \otimes \sigma_a,
\]
\[
\mu^*(\sigma_{b,c,a}^-) \geq \delta([\nu^{-b} \rho, \nu^c \rho]) \otimes \sigma_a,
\]
\[
\mu^*(\sigma_{a,b,c}) \geq \delta([\nu^{-a} \rho, \nu^b \rho]) \otimes \sigma_c.
\]
We finished introducing notation and state some more reducibility results.

Here is a consequence of Theorem 6.3 of [19], see also section 3 there.

Proposition 3.3. We have in $R(G),$ with multiplicity one:
\[
\nu^a \rho \times \cdots \times \nu^\frac{1}{2} \times \sigma \geq \sigma_a.
\]
The next lemma follows from Theorem 5.1 of [14], ii).

**Lemma 3.4.** We have in $R^G$,

\[
\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_b = \sigma_{a,b}^+ + L(\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_b), \quad \text{and}
\]

\[
\delta([\nu^2 p, \nu^b \rho]) \otimes \sigma_a = \sigma_{a,b}^+ + L(\delta([\nu^{-a} p, \nu^b \rho]) \otimes \sigma_a) +
\]

\[L(\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_b) + L(\delta([\nu^{-b} p, \nu^c \rho]) \otimes \sigma_a).
\]

By Proposition 2.4 of [3] we have

**Lemma 3.5.** We have in $R^G$

\[
\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_{b,c}^\pm = \sigma_{b,c,a}^\pm + L(\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_{b,c}^\pm), \quad \text{so}
\]

\[
(3.2) \quad \mu^*(\sigma_{b,c,a}^\pm) \geq \delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_{b,c}^\pm.
\]

The next is Proposition 3.2 of [8].

**Theorem 3.6.** With discrete series being a subrepresentation, we have in $R^G$

\[
\delta([\nu^{-a} p, \nu^c \rho]) \otimes \sigma_b = \sigma_{b,c,a}^\pm + L(\delta([\nu^{-a} p, \nu^c \rho]) \otimes \sigma_b) +
\]

\[L(\delta([\nu^{-b} p, \nu^c \rho]) \otimes \sigma_a) + L(\delta([\nu^{-b} p, \nu^c \rho]) \otimes \sigma_a).
\]

Finally we have the main result of [3].

**Theorem 3.7.** With discrete series being a subrepresentation, we have in $R^G$

\[
\delta([\nu^{-b} p, \nu^c \rho]) \otimes \delta([\nu^2 p, \nu^a \rho]) \otimes \sigma = L(\delta([\nu^2 p, \nu^a \rho]) \otimes \sigma_{b,c}^+) + L(\delta([\nu^2 p, \nu^r \rho]) \otimes \sigma_{b,c}^-) +
\]

\[+ \sigma_{b,c,a}^+ + \sigma_{b,c,a}^- + L(\delta([\nu^{-b} p, \nu^c \rho]) \otimes \sigma_a) + L(\delta([\nu^{-b} p, \nu^c \rho]) \otimes \delta([\nu^2 p, \nu^a \rho]) \otimes \sigma).
\]

4. **Decomposing mixed case**

As we are interested in the composition series of induced representations, we shall need a result that follows from proofs of Theorems 2-1 and 2-6 from [5].

**Theorem 4.1.** There exists a contravariant exact functor:

\[
\text{Alg } G_n \xrightarrow{\Delta} \text{Alg } G_n,
\]

such that

\[
\Delta \pi \cong \pi, \quad \pi \in \text{Irr } G_n,
\]

and if $\delta_i \in \text{Irr } GL(n_i, F)$, $i = 1, \ldots, k$, $m = n_1 + \cdots + n_k$ and $\tau \in \text{Irr } G_{n-m}$ we have

\[
(\delta_1 \times \cdots \times \delta_k \rtimes \tau)^\wedge \cong \tilde{\delta}_1 \times \cdots \times \tilde{\delta}_k \rtimes \tau.
\]
Proof. We follow the same lines as in proofs of Theorems 2-1 and 2-6 from [5]. If $G_n$ is orthogonal, then $\wedge$ is just contragredient. If $G_n$ is symplectic, by a result of Waldspurger ([12] Chapter 4. II.1), for any element $\eta \in GSp(2n)$ of similitude $-1$, and $\pi \in IrrG_n$ we have $\tilde{\pi} \cong \pi^\sigma$, where $\pi^\sigma(g) = \pi(\eta g \eta^{-1})$.

Now choose an element of the form $\eta = (id, \eta') \in GL(n, F) \times GSp(0, F) = GL(n, F) \times F^\times$, identified with the Levi subgroup of the appropriate maximal parabolic subgroup of $GSp(2n, F)$, where $\eta'$ is an element with similitude equal to $-1$, as $\eta$ is. For $\wedge = \eta^\circ$, we have

\[
(\delta_1 \times \cdots \times \delta_k \times \tau)^\wedge \cong (\tilde{\delta_1} \times \cdots \times \tilde{\delta_k} \times \tilde{\tau})^\eta \cong \tilde{\delta_1} \times \cdots \times \tilde{\delta_k} \times \tau.
\]

Consider some standard intertwining operators

\[
\delta([v^{-a} \rho, \nu^\circ \rho]) \times \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \sigma \xrightarrow{\cong} \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \delta([v^{-a} \rho, \nu^\circ \rho]) \times \sigma \\
\cong f_0 \cong g_0 \\
\delta([v^{1/2} \rho, \nu^\circ \rho]) \times \delta([v^{-a} \rho, \nu^\circ \rho]) \times \sigma \xrightarrow{\cong} \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \delta([v^{-a} \rho, \nu^\circ \rho]) \times \sigma \\
\downarrow f_1 \downarrow g_1 \\
\delta([v^{1/2} \rho, \nu^\circ \rho]) \times \delta([v^{-a} \rho, \nu^\circ \rho]) \times \sigma \xrightarrow{\cong} \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \delta([v^{-a} \rho, \nu^\circ \rho]) \times \sigma \\
\downarrow f_2 \downarrow g_2 \\
\delta([v^{-c} \rho, \upsilon^\circ \rho]) \times \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \sigma \xrightarrow{\cong} \delta([v^{-c} \rho, \upsilon^\circ \rho]) \times \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \sigma \\
\downarrow f_3 \downarrow g_3
\]

We denoted $\psi = \delta([v^{-a} \rho, \nu^\circ \rho]) \times \delta([v^{1/2} \rho, \nu^\circ \rho]) \times \sigma$. Also for all $i \geq 1$ denote $K_i = \text{Ker} f_i$, and $H_i = \text{Ker} g_i$. By Theorems 3.1 and 4.1, we have

\[
K_1 = \text{Ker} f_0 = \text{Ker} g_0 \\
H_1 = \text{Ker} f_1 = \text{Ker} g_1 \\
H_2 = \text{Ker} f_2 = \text{Ker} g_2
\]

and no kernel contains $L(\psi)$. Thus we have:

\[
\text{Im}(f_0 \circ \cdots \circ f_0) = L(\psi) = \text{Im}(g_0 \circ \cdots \circ g_0),
\]

and the diagram is commutative up to a constant. We have in $R(G)$:

\[
\forall i \quad K_i \leq \psi \leq K_1 + K_2 + K_3 + L(\psi).
\]

The composition factors of $K_2$ and $K_3$ are determined by Theorems 3.6 and 3.7. So in search of remaining subquotients, we need to decompose $K_1$. After
that, we determine the multiplicities of all subquotients of ψ. Note that by Lemma 8.1 of [13], all tempered subquotients of ψ are discrete series.

5. Discrete series subquotients

Here we determine discrete series subquotients in three induced representations

$$\delta([\nu^{-a} \rho, \nu^{c} \rho]) \times \delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \sigma \geq \delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \sigma_{a,c}^{+} + \delta([\nu^{-a} \rho, \nu^{b} \rho]) \times \sigma_{a,c}^{-},$$

where the inequality follows from Theorem 3.1. We start with candidates.

**Lemma 5.1.** Only possible discrete series subquotients appearing in

$$\delta([\nu^{-a} \rho, \nu^{c} \rho]) \times \delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \sigma$$

are $\sigma_{b,c,a}^{+}$, $\sigma_{b,c,a}^{-}$ and $\sigma_{a,b,c}^{-}$.

**Proof.** Consider cuspidal support of the induced representation and Mœglin Tadić classification of discrete series. Since $\nu^{1/2} \rho$ appears 3 times in the cuspidal support, possible discrete series are subrepresentations of a representation of the form

$$\delta([\nu^{-y} \rho, \nu^{c} \rho]) \times \sigma_{x},$$

for some $\frac{1}{2} \leq x, y, z \in \mathbb{Z} + \frac{1}{2}$, where either $0 < x < y < z$ or $0 < y < z < x$, and $\sigma_{x}$ is a unique irreducible subrepresentation of $\delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \sigma$. We look at the first case. Here $z$ is the largest such that $\nu^{x+z} \rho$ appears once in the cuspidal support, so we must have $z = c$. Further, $y$ is the largest such that $\nu^{x+y} \rho$ appears two times in the cuspidal support, so it must be $b$. Finally, $x$ is the largest such that $\nu^{x+z} \rho$ appears three times in the cuspidal support, so it must be $a$. Same reasoning goes for the second case, where we have $x = c$, $z = b$, and $y = a$. By Theorem 2.1 of [14], we look for subrepresentations

$$\sigma_{b,c,a}^{+} \oplus \sigma_{a,b,c}^{-} \leq \delta([\nu^{-a} \rho, \nu^{b} \rho]) \times \sigma_{c}^{-} \quad \text{and} \quad \sigma_{b,c,a}^{+} \oplus \sigma_{a,b,c}^{-} \leq \delta([\nu^{-b} \rho, \nu^{c} \rho]) \times \sigma_{a}^{-}.$$

To determine which of these discrete series do appear in (5.1), and what are their multiplicities, we need a couple of lemmas.

**Lemma 5.2.** We have in $R(G)$, with maximum multiplicity

$$\delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \sigma \geq 1 \cdot \sigma_{b,c}^{+} + 1 \cdot \sigma_{b,c}^{-} + 1 \cdot L(\delta([\nu^{-b} \rho, \nu^{c} \rho]) \times \sigma).$$

**Proof.** Check multiplicity two of $\delta([\nu^{-b} \rho, \nu^{c} \rho]) \times \sigma$ and one of $\delta([\nu^{-a} \rho, \nu^{b} \rho]) \times \sigma$ in $\mu^{*}(\delta([\nu^{1/2} \rho, \nu^{b} \rho]) \times \delta([\nu^{1/2} \rho, \nu^{c} \rho]) \times \sigma)$ and use Theorem 3.1.
Lemma 5.3. We have in $R(G)$, with maximum multiplicities:

$$
\mu^\ast(\delta([\nu^1_\rho, \nu^b_\rho]) \times \delta([\nu^{-a}_\rho, \nu^c_\rho]) \bowtie \sigma) \geq
1 \cdot \delta([\nu^2_\rho, \nu^a_\rho]) \otimes \sigma_{b,c}^+ + 1 \cdot \delta([\nu^2_\rho, \nu^a_\rho]) \otimes \sigma_{b,c}^-.
$$

Proof. The claim can be deduced from (2.1) and Lemma 5.2.

Lemma 5.4. We have in $R(G)$, with maximum multiplicities:

$$
\mu^\ast(\delta([\nu^2_\rho, \nu^b_\rho]) \times \sigma_{a,c}^+) \geq 1 \cdot \delta([\nu^{-a}_\rho, \nu^b_\rho]) \otimes \sigma_c + 1 \cdot \delta([\nu^2_\rho, \nu^a_\rho]) \otimes \sigma_{b,c}^-.
$$

Proof. By (2.1) consider $0 \leq r \leq b + \frac{1}{2}$, $\delta' \otimes \sigma' \leq \mu^\ast(\sigma_{a,c}^+)$, and

$$
\delta([\nu^{r-b}_\rho, \nu^{-\frac{1}{2}}]) \times \delta([\nu^{b+1}_\rho, \nu^b_\rho]) \times \delta' \otimes \delta([\nu^{b+1}_\rho, \nu^{b-s}_\rho]) \times \sigma'.
$$

First we look for $\delta([\nu^{-a}_\rho, \nu^b_\rho]) \otimes \sigma_c$. Observe that $\nu^c_\rho$ is not in a cuspidal support of $\delta'$, so in (2.3) we have $j = c$ and

$$
\delta' \otimes \sigma' \leq \sum_{i=-a-1}^{a} \delta([\nu^{-i}_\rho, \nu^a_\rho]) \otimes \delta([\nu^{i+1}_\rho, \nu^c_\rho]_+; \sigma).
$$

Searching for $\nu^{-a}_\rho$ in cuspidal support in (5.3), left of $\otimes$, we have options

- $r - b = -a$, so $b + 1 - s > \frac{1}{2}$, and we have $i = -\frac{1}{2}$, $s = b - a$ and $\sigma' = \delta([\nu^2_\rho, \nu^b_\rho]_+; \sigma) = \sigma_c$.
- $b + 1 - s = -a$, so $s > b + \frac{1}{2}$ and this is not possible.
- $-i = -a$, this is not possible since $\delta([\nu^{a+1}_\rho, \nu^b_\rho]_+; \sigma)$ is zero.

Looking for $\delta([\nu^2_\rho, \nu^a_\rho]) \otimes \sigma_{b,c}^-$, we have $r = b + \frac{1}{2}$ and $s = 0$. Thus we search in

$$
\delta' \otimes \delta([\nu^2_\rho, \nu^b_\rho]) \times \sigma'.
$$

Now in (2.3) we have $j = c$ and $i = -\frac{1}{2}$, so $\sigma' = \delta([\nu^2_\rho, \nu^c_\rho]_+; \sigma) = \sigma_c$. But, $\sigma_{b,c}^- \not\leq \delta([\nu^2_\rho, \nu^c_\rho]) \times \sigma_c$, and $\sigma_{b,c}^-$ appears there once, by Lemma 3.4.

Lemma 5.5. We have in $R(G)$, with maximum multiplicities:

$$
\mu^\ast(\delta([\nu^2_\rho, \nu^b_\rho]) \times \sigma_{a,c}^-) \geq 0 \cdot \delta([\nu^{-a}_\rho, \nu^b_\rho]) \otimes \sigma_c + 0 \cdot \delta([\nu^2_\rho, \nu^b_\rho]) \otimes \sigma_{b,c}^-.
$$

Proof. The proof goes as in Lemma 5.4, with the difference that one now obtains $\sigma' = \delta([\nu^2_\rho, \nu^c_\rho]_+; \sigma)$, but the term on the right-hand side is zero, by (2.2).

Now we determine all discrete series that appear on the left-hand side of (5.1).
**Proposition 5.6.** Writing all discrete series, with multiplicities, we have in \( \text{R}(G) \):
\[
\delta([\nu^{-a}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma \geq 1 \cdot \sigma^+_{b,c,a} + 1 \cdot \sigma^-_{b,c,a},
\]
\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^+_{a,c} \geq 1 \cdot \sigma^+_{b,c,a},
\]
\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^-_{a,c} \geq 0.
\]

**Proof.** By (5.1) and Lemma 5.1, only possible discrete series subquotients in all of these representations are \( \sigma^+_{b,c,a}, \sigma^-_{b,c,a} \) and \( \sigma^-_{a,b,c} \). By Theorem 3.7 and (4.1), \( \sigma^+_{b,c,a} \) and \( \sigma^-_{b,c,a} \) appear in \( \delta([\nu^{-a}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma \). Lemmas 3.5 and 5.3 show that they appear with multiplicity one. Now (3.1) and Lemma 5.4 show that we have only one discrete series in \( \delta \).

**Lemma 5.4** shows that we have only one discrete series in \( \delta \). Mas 3.5 and 5.3 show that they appear with multiplicity one. Now (3.1) and Lemma 5.1 show that we have only one discrete series in \( \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma \).

We have no \( \sigma^+_{a,b,c} \) in either \( \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^+_{a,c} \) nor \( \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^-_{a,c} \). Theorems 3.6 and 3.7, and (4.1), show that we have no \( \sigma^-_{a,b,c} \) in \( \delta([\nu^{-a}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma \).

\[\square\]

6. Non-tempered candidates

As noted in Section 4, we search for possible remaining non-tempered subquotients in
\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^+_{a,c} + \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^-_{a,c}.
\]

We have

**Proposition 6.1.** If \( \pi \) is a non-tempered subquotient of \( \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^+_{a,c} \), different from its Langlands quotient, then \( \pi \) is either \( L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \sigma^+_{b,c}) \) or \( L(\delta([\nu^{-a}\rho, \nu^b\rho]) \times \sigma_c) \).

**Proof.** We use Lemma 2.2 of \([14]\) (in terms of that lemma \( \pi_i \leq \delta([\nu^{1-t_i}\rho, \nu^b\rho]) \times \sigma, -l_1 = \frac{1}{2}, l_2 = b \) and \( \sigma = \sigma^+_{a,c} \)). So we look for possible embeddings
\[\pi \hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^{\beta_1}\rho]) \times \pi',\]
where \( -\alpha_1 + \beta_1 < 0 \) and \( \pi' \) is irreducible. By the lemma, there exists an irreducible representation \( \sigma_1 \) such that
\[\mu^*(\sigma^+_{a,c}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{\beta_1}\rho]) \otimes \sigma_1\]
\[\pi' \leq \delta([\nu^{\alpha_1+1}\rho, \nu^b\rho]) \times \sigma_1\]
and we must have
\[\begin{cases}
-\frac{1}{2} \leq \beta_1 \\
b \geq \alpha_1 > \beta_1, \frac{1}{2} \\
\alpha_1 \geq \frac{1}{2},
\end{cases}\]

(6.3)
We have two possible cases:

a) $\beta_1 = -\frac{1}{2}$. Now $\sigma_1 = \sigma_{a,c}^+$.
   - Assume that $\pi'$ is tempered. We may take $2\alpha_1 + 1 \in \text{Jord}_\rho(\sigma_{a,c}^+)$. Now $\alpha_1 = c$ is not possible, since that would imply $b \geq c$. Thus $\alpha_1 = a$, and
     \[
     \pi' \leq \delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+.
     \]
     Looking at the cuspidal support on the right hand side, Lemma 8.1 of [13] implies that $\pi'$ is a discrete series. Thus $\pi' = \sigma_{b,c}^+$ or $\pi' = \sigma_{b,c}^-$. By Section 8. of [22], we have $\pi' = \sigma_{b,c}^+$. So (6.1) is written as
     \[
     \pi \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \sigma_{b,c}^+.
     \]
   - If $\pi'$ is not tempered, by Lemma 2.2 of [14], there exist $2\beta_2 + 1 \in \text{Jord}_\rho(\sigma_{a,c}^+)$ and $(2\beta_2 + 1) - (2\beta_2 + 1) = (2\beta_2 + 1) - 1 \in \text{Jord}_\rho(\sigma_{a,c}^+) = \{2a + 1, 2c + 1\}$, such that (in terms of the lemma $\alpha_1 \leq (2\beta_2 + 1) - (2\beta_2 + 1)$) we have
     \[
     \alpha_1 \leq a < c < \alpha_2 \leq b.
     \]
     So $c < b$, but this is a contradiction.

b) $\beta_1 > \frac{1}{2}$. Then, by the lemma, $2\beta_1 + 1 \in \text{Jord}_\rho(\sigma_{a,c}^+) = \{2a + 1, 2c + 1\}$. Since $c > b \geq \alpha_1 > \beta_1, -\frac{1}{2}$ we have $\beta_1 = a$ and $\alpha_1 > a$. Now (6.2) gives
     \[
     \mu^*(\sigma_{a,c}^+) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_1.
     \]
     To determine $\sigma_1$, we look at $\mu^*(\sigma_{a,c}^+)$. In (2.3) it is necessary to pick $j = c$ and $i = -\frac{1}{2}$, and we have
     \[
     \sigma_1 = \delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]_+; \sigma) = \sigma_c.
     \]
     So far, we have
     \begin{align*}
     \begin{cases}
     \pi \hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^b\rho]) \rtimes \pi', \\
     \pi' \leq \delta([\nu^{\alpha_1+1}\rho, \nu^b\rho]) \rtimes \sigma_c, \\
     a < \alpha_1 \leq b.
     \end{cases}
     \end{align*}
     \tag{6.4}
     
     If $\alpha_1 = b$ then $\pi' \cong \sigma_c$ and $\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c$, so
     \[
     \pi \cong L(\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c),
     \]
     as expected. Thus, we assume
     \[
     a < \alpha_1 < b.
     \]
Since \( \text{Jordan}(\sigma_c) \cap [2\alpha_1 + 1, 2b + 1] = \{2c+1\} \cap [2\alpha_1 + 1, 2b + 1] = \emptyset \), Proposition 3.1 ii) of [14] implies that \( \delta([\mu^{\alpha_1+1} \rho, \nu^b \rho]) \times \sigma_c \) is irreducible. So
\[
\pi' \cong \delta([\nu^{\alpha_1+1} \rho, \nu^b \rho]) \times \sigma_c \cong \delta([\nu^{-b} \rho, \nu^{-\alpha_1-1} \rho]) \times \sigma_c,
\]
and finally
\[
\pi \hookrightarrow \delta([\nu^{-\alpha_1} \rho, \nu^a \rho]) \times \delta([\nu^{-b} \rho, \nu^{-\alpha_1-1} \rho]) \times \sigma_c.
\]
Now, by Lemma 5.5 of [6], either \( \pi \hookrightarrow \delta([\nu^{-b} \rho, \nu^a \rho]) \times \sigma_c \) or there exists an irreducible representation \( \pi'' \leq \delta([\nu^{-\alpha_1} \rho, \nu^a \rho]) \times \sigma_c \) such that
\[
\pi \hookrightarrow \delta([\nu^{-b} \rho, \nu^{-\alpha_1-1} \rho]) \times \pi''.
\]
The second case is not possible since, similar to (6.3), we would obtain \(-\frac{1}{2} \leq -\alpha_1 - 1(\leq -\frac{1}{2} - 1)\), a contradiction.

Using the same methods as above, we obtain

**Proposition 6.2.** If \( \pi \) is a non-tempered subquotient of \( \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c} \), different from its Langlands quotient, then \( \pi \) is \( L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma_{a,c}^\pm) \).

7. Multiplicity of \( L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma_{b,c}^\pm) \)

Here we write explicitly \( L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma_{b,c}^\pm) \) as a non-tempered subquotient of \( \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{b,c}^\pm \), different from its unique Langlands quotient, as claimed by Lemma 6.2 of [15]. We start with a couple of lemmas.

**Lemma 7.1.** Discrete series \( \sigma_{b,c}^+ \) and \( \sigma_{b,c}^- \) appear with multiplicity one in equations
\[
1 \cdot \sigma_{b,c}^+ \leq \delta([\nu^{\alpha_1+1} \rho, \nu^b \rho]) \times \sigma_{a,c}^+,
\]
\[
1 \cdot \sigma_{b,c}^- \leq \delta([\nu^{\alpha_1+1} \rho, \nu^b \rho]) \times \sigma_{a,c},
\]
\[
1 \cdot \sigma_{b,c}^+ + 1 \cdot \sigma_{b,c}^- \leq \delta([\nu^{\alpha_1+1} \rho, \nu^b \rho]) \times \delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma.
\]

**Proof.** This follows from Section 8. of [22] and (2.1).

The next two lemmas follow.

**Lemma 7.2.** Both \( \delta([\nu^{-a} \rho, \nu^{-\frac{1}{2}} \rho]) \otimes \sigma_{b,c}^+ \) and \( \delta([\nu^{-a} \rho, \nu^{-\frac{1}{2}} \rho]) \otimes \sigma_{b,c}^- \) appear with multiplicity one in \( \mu^+(\delta([\nu^{-a} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma) \).

**Lemma 7.3.** For \( \epsilon = \pm \) irreducible representation \( \delta([\nu^{-a} \rho, \nu^{-\frac{1}{2}} \rho]) \otimes \sigma_{b,c}^- \) appears with multiplicity one in \( \mu^+(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+) \).

Using Lemmas 7.2 and 7.3 we have
Proposition 7.4. With all multiplicities being one, we have in $R(G)$

$$\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_{a,c}^+ \geq 1 \cdot L(\delta([\nu^{\frac{1}{2}}p, \nu^a \rho]) \times \sigma_{b,c}^+),$$
$$\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_{a,c}^- \geq 1 \cdot L(\delta([\nu^{\frac{1}{2}}p, \nu^a \rho]) \times \sigma_{b,c}^-),$$
$$\delta([\nu^{-a}p, \nu^c \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma \geq 1 \cdot L(\delta([\nu^{\frac{1}{2}}p, \nu^a \rho]) \times \sigma_{b,c}^+) +$$
$$1 \cdot L(\delta([\nu^{\frac{1}{2}}p, \nu^a \rho]) \times \sigma_{b,c}^-).$$

8. Multiplicity of $L(\delta([\nu^{-a}p, \nu^b \rho]) \times \sigma_c)$

By Theorem 3.6 and (4.1), this subquotient does appear in $\delta([\nu^{-a}p, \nu^c \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_c$, but also as a candidate in $\delta([\nu^{\frac{1}{2}}p, \nu^a \rho]) \times \sigma^a_{a,c}$, by Proposition 6.1, so we need to check its multiplicity. We want to obtain a subquotient in some of its Jacquet module, used to identify it.

First we state a result of Proposition 3.9 of [23].

Lemma 8.1. In appropriate Grothendieck group we have

$$\nu^{\frac{1}{2}}p \times \nu^{\frac{1}{2}}p \times \sigma = \delta([\nu^{-\frac{1}{2}}p, \nu^{\frac{1}{2}}p]_{\sigma}) + \delta([\nu^{-\frac{1}{2}}p, \nu^{\frac{1}{2}}p]_{-\sigma}) + L(\nu^{\frac{1}{2}} \times \sigma_1) + L(\nu^{\frac{1}{2}}p \times \nu^{\frac{1}{2}}p \times \sigma).$$

Lemma 8.2. We have in $R(G)$, with multiplicity one

$$\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma \geq 1 \cdot L(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b).$$

Proof. By Lemma 8.1 we may assume $b \geq \frac{3}{2}$. Obviously, $L(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b)$ does appear as a subquotient, so we need to prove multiplicity one. Let us denote

$$\pi = \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \otimes L(\nu^{\frac{1}{2}}p \times \sigma_1).$$

It is enough to see that $\pi$ appears in both

i) $\mu^*(L(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b))$ and

ii) $\mu^*(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b),$ and that its multiplicity is one in ii).

We start with i). By Theorem 5.1 of [14] ii), and Lemma 5.4 there, we have in $R(G)$

$$\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b = \sigma_{\text{temp}} + L(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b),$$

where $\sigma_{\text{temp}}$ is an irreducible tempered subquotient of $\delta([\nu^{-b}p, \nu^b \rho]) \times \sigma$. By (2.3), $\pi \nleq \mu^*(\sigma_{\text{temp}})$. On the other hand, by (2.4) $\mu^*(\sigma_b) \geq \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \otimes \nu^{\frac{1}{2}}p \times \sigma_1 \geq \pi$. So from (2.1) we have

$$\mu^*(\delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \sigma_b) \geq \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \times \delta([\nu^{\frac{1}{2}}p, \nu^b \rho]) \otimes \nu^{\frac{1}{2}}p \times \sigma_1 \geq \pi.$$
To we prove ii), look for $0 \leq i \leq j \leq b + \frac{1}{2}$ and $0 \leq s \leq r \leq b + \frac{1}{2}$ such that
\[ \delta([\nu^2 \rho, \nu^b \rho]) \times \delta([\nu^2 \rho, \nu^b \rho]) \leq \delta([\nu^{r-i} \rho, \nu^{b-j} \rho]) \times \delta([\nu^{b+1-j} \rho, \nu^b \rho]) \times \delta([\nu^{b+1-s} \rho, \nu^b \rho]), \]

and
\[ L([\nu^2 \rho, \sigma]) \leq \delta([\nu^{b+1-i} \rho, \nu^{b-j} \rho]) \times \delta([\nu^{b+1-s} \rho, \nu^b \rho]) \times \sigma. \]
The first equation implies $i = r = b + \frac{1}{2}$ and $j = s = b - \frac{1}{2}$. The second is
\[ L([\nu^2 \rho, \sigma]) \leq \nu^2 \rho \times \nu^2 \rho \times \sigma. \]

Multiplicity one in the second follows from Lemma 8.1.

**Lemma 8.3.** We have in $R(G)$, with multiplicity one
\[ \delta([\nu^2 \rho, \nu^b \rho]) \times \delta([\nu^2 \rho, \nu^b \rho]) \times \sigma \geq 1 \times L(\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma). \]

**Proof.** Use Lemma 3.4, (2.1) and (2.3) to show that $\delta([\nu^{b+1} \rho, \nu^{b} \rho]) \times \delta([\nu^{b+1} \rho, \nu^{b} \rho])$ appears in $\mu^*(L(\delta([\nu^{b+1} \rho, \nu^{b} \rho]) \times \sigma))$, and Lemma 8.2 that it appears once in $\mu^*(\delta([\nu^2 \rho, \nu^b \rho]) \times \delta([\nu^2 \rho, \nu^b \rho]) \times \sigma)$.

Now we provide a subquotient that can be used to identify $L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma)$.

**Lemma 8.4.** We have in $R(G)$
\[ \mu^*(L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma)) \geq \delta([\nu^2 \rho, \nu^b a]) \times \delta([\nu^2 \rho, \nu^b a]) \times \sigma). \]

**Proof.** By Proposition 3.2, we have
\[ \delta([\nu^{-a} \rho, \nu^b a]) \times \sigma = \delta([\nu^{-a} \rho, \nu^b a]) \times \sigma = \delta([\nu^{-a} \rho, \nu^b a]) \times \sigma. \]

To prove the claim, one can use (2.1) and a square integrability criterion, see the end of Section 2 of [21].

**Lemma 8.5.** We have in $R(G)$, with maximum multiplicity
\[ \mu^*(\delta([\nu^{-a} \rho, \nu^b a]) \times \delta([\nu^2 \rho, \nu^b a]) \times \sigma \geq 1 \times \delta([\nu^2 \rho, \nu^b a]) \times \delta([\nu^2 \rho, \nu^b a]) \times \sigma). \]

**Proof.** Use (2.1) and Lemma 8.3.

**Lemma 8.6.** We have in $R(G)$, with maximum multiplicity
\[ \mu^*(\delta([\nu^2 \rho, \nu^b a]) \times \sigma) \geq 1 \times \delta([\nu^2 \rho, \nu^b a]) \times \delta([\nu^2 \rho, \nu^b a]) \times \sigma). \]

**Proof.** By (3.1) $\mu^*(\sigma) \geq \delta([\nu^2 \rho, \nu^b a]) \times \sigma$. The inequality follows by (2.1) and multiplicity one by (5.1) and Lemma 8.5.

Finally we have
Proposition 8.7. Both induced representations
\[\delta([\nu^{-a},\nu^{-b}],\nu^{c}) \times \delta([\nu^{1/2},\nu^{c}]) \rtimes \sigma \text{ and } \delta([\nu^{1/2},\nu^{b}]) \rtimes \sigma_{a,c}^{+},\]
contain \(L(\delta([\nu^{-a},\nu^{b}]) \rtimes \sigma_{b,c}) + \sigma_{b,c,a}^{+}\)
\(L(\delta([\nu^{1/2},\nu^{c}]) \rtimes \sigma_{a,c}^{+}) + L(\delta([\nu^{-a},\nu^{b}]) \rtimes \sigma_{c}).\)

Proof. Theorem 3.6 and (4.1) give existence in the first representation, and Lemmas 8.4 and 8.5 multiplicity one. For the second, use (5.1), and Lemma 8.6.

9. Multiplicity of \(L(\delta([\nu^{-b},\nu^{c}]) \rtimes \sigma_{a})\)
We observe that by Theorems 3.6 and 3.7, \(L(\delta([\nu^{-b},\nu^{c}]) \rtimes \sigma_{a})\) appears two times in (4.1). So we determine its multiplicity. First we state a result that we use later.

Lemma 9.1. We have in \(R(G)\), with multiplicity one
\[\mu^{*}(L(\delta([\nu^{-a},\nu^{c}]) \rtimes \sigma_{a})) \geq 1 \cdot \delta([\nu^{1/2},\nu^{a}]) \otimes L(\delta([\nu^{-b},\nu^{c}]) \rtimes \sigma).\]

Proof. Denote the subquotient by \(\pi\). By (2.4)
\[\mu^{*}(\sigma_{a}) \geq \delta([\nu^{1/2},\nu^{a}]) \otimes \sigma, \text{ so } \]
\[\mu^{*}(\delta([\nu^{-b},\nu^{c}]) \rtimes \sigma_{a}) \geq \pi.\]

Comparing Proposition 3.2 and Lemma 3.5, it is enough to show
\[\mu^{*}(\delta([\nu^{1/2},\nu^{a}]) \rtimes \sigma_{b,c}^{+}) \not\equiv \pi.\]

So by (2.1) consider \(0 \leq j \leq i \leq a + \frac{1}{2}\), \(\delta^{'} \otimes \sigma^{'} \leq \mu^{*}(\sigma_{b,c}^{+})\) and
\[\delta([\nu^{i-a},\nu^{1/2}]) \times \delta([\nu^{a-j+1},\nu^{b}]) \otimes \delta([\nu^{a-j},\nu^{c}]) \times \sigma.\]

As \(\delta^{'} \) should not contain neither \(\nu^{b} \) nor \(\nu^{c} \) in its cuspidal support, by (2.3), we have \(\delta^{'} \otimes \sigma^{'} = 1 \otimes \sigma_{b,c}^{+}\), and \(i = j = a + \frac{1}{2}\), giving \(\delta([\nu^{1/2},\nu^{a}]) \otimes \sigma_{b,c}^{+} \not\equiv \pi.\)

Proposition 9.2. We have in \(R(G)\), with multiplicity one
\[\delta([\nu^{-a},\nu^{c}]) \times \delta([\nu^{1/2},\nu^{b}]) \rtimes \sigma \geq 1 \cdot L(\delta([\nu^{-b},\nu^{c}]) \rtimes \sigma_{a}).\]

Proof. It is easy to check that \(\delta([\nu^{-a},\nu^{c}]) \otimes \sigma_{a}\) appears with multiplicity one in \(\mu^{*}(\delta([\nu^{-c},\nu^{b}]) \rtimes \sigma)\) appears with multiplicity one in \(\mu^{*}(\delta([\nu^{-a},\nu^{c}]) \times \delta([\nu^{1/2},\nu^{b}]) \times \sigma).\)

10. Composition factors
Here we determine composition factors of the kernel \(K_{1}\) from Section 4.

Proposition 10.1. We have in \(R(G)\)
\[\delta([\nu^{1/2},\nu^{b}]) \rtimes \sigma_{a,c}^{+} = L(\delta([\nu^{1/2},\nu^{b}]) \rtimes \sigma_{a,c}^{+}) + \sigma_{b,c,a}^{+} + \]
\[L(\delta([\nu^{1/2},\nu^{b}]) \rtimes \sigma_{b,c}^{+}) + L(\delta([\nu^{-a},\nu^{b}]) \rtimes \sigma_{c}).\]
A ONE HALF CUSPIDAL REDUCIBILITY

Proof. Discrete series subquotients are determined in Proposition 5.6. Remaining irreducible subquotients are described by Propositions 6.1, 7.4 and 8.7.

Proposition 10.2. We have in $R(G)$

$$\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c} = L(\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c}) + L(\delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_{b,c}).$$

Proof. Possible discrete series subquotients are discussed in Proposition 5.6. Remaining irreducible subquotients are described by Propositions 6.2 and 7.4.

Theorem 10.3. We have in $R(G)$

$$\delta([\nu^{-a} \rho, \nu^a \rho]) \times \delta([\nu^2 \rho, \nu^b \rho]) \times \sigma = L(\delta([\nu^{-a} \rho, \nu^a \rho]) \times \delta([\nu^2 \rho, \nu^b \rho]) \times \sigma) + L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_{a,c}) +$$

$$L(\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c}) + L(\delta([\nu^{-a} \rho, \nu^a \rho]) \times \sigma_b +$$

$$\sigma_{b,c,a} + L(\delta([\nu^{-a} \rho, \nu^a \rho]) \times \delta([\nu^2 \rho, \nu^a \rho]) \times \sigma) +$$

$$L(\delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_{a,c}) + L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_c) +$$

$$L(\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c}) + L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_a) +$$

$$\sigma_{a,b,c} + L(\delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_{b,c}).$$

Proof. By (4.1), Theorems 3.6 and 3.7, and Propositions 10.1 and 10.2, we listed all irreducible subquotients, up to multiplicities. Proposition 5.6 shows multiplicity one for discrete series. Propositions 7.4, 8.7 and 9.2, show multiplicity one for $L(\delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_{a,c}$, $L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_{b,c},$ and $L(\delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_{a,c}$). Remaining irreducible subquotients appear with a multiplicity one in $K_1 + K_2 + K_3 + L(\psi)$, on the right hand side of (4.1), by Theorems 3.6 and 3.7, and Propositions 10.1 and 10.2.

11. Composition series of $\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c}$ and $\delta([\nu^2 \rho, \nu^b \rho]) \times \sigma_{a,c}$.

Here we determine the composition series of the kernel $K_1$, from Section 4. For the first representation, we show that a discrete series is a subrepresentation, and use intertwining operators to position other subquotients.

The first lemma follows directly from (2.3) and (2.4).

Lemma 11.1. We have in $R(G)$, with maximum multiplicities:

$$\mu^*(\delta([\nu^2 \rho, \nu^b \rho]) \times \delta([\nu^2 \rho, \nu^a \rho]) \times \sigma_c) \geq 2 \cdot \delta([\nu^{-a} \rho, \nu^b \rho]) \otimes \sigma_c.$$

The next proposition gives positions of both $\sigma_{a,b,c}$ and $L(\delta([\nu^{-a} \rho, \nu^b \rho]) \times \sigma_c)$.
Proposition 11.2. We have embeddings
\[ \sigma_{b,c,a}^+ \hookrightarrow \delta([\nu^{-a}p, \nu^b\rho]) \rtimes \sigma_{a,b,c}^- \]
\[ \hookrightarrow \delta([\nu^b\rho, \nu^{-a}p]) \rtimes \sigma_{a,b,c}^+. \]

Proof. By Theorem 2-6. of [5] and Lemma 3.4 we have an epimorphism
\[ \delta([\nu^{-a}p, \nu^{-\frac{b}{2}}\rho]) \rtimes \sigma_{a,b,c}^- \]
Now we have a composition of an embedding and an epimorphism
\[ \delta([\nu^{-a}p, \nu^{-\frac{b}{2}}\rho]) \times \sigma_c \hookrightarrow \delta([\nu^{-a}p, \nu^{-\frac{b}{2}}\rho]) \times \sigma_c \]
\[ \hookrightarrow \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{a,b,c}^+. \]
By Proposition 3.2, and 5.6 and Lemma 11.1, all representations in (11.1),
\[ (11.1) \]
\[ \text{by Lemma 3.4, we have an epimorphism} \]
\[ \mu^\star(L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{a,b,c}^+)) \geq 1 \cdot \delta([\nu^{-b}p, \nu^c\rho]) \otimes L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma) \]
\[ \text{Lemma 11.3. We have in } R(G), \text{ with multiplicity one:} \]
\[ (11.2) \]
\[ \text{Lemma 11.4. We have in } R(G), \text{ with maximum multiplicities:} \]
\[ \mu^\star(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_c) \geq 1 \cdot \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \otimes \sigma_{b,c}^+ \]
\[ + 1 \cdot \delta([\nu^{-b}p, \nu^c\rho]) \otimes L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma) \]
The next proposition gives position of \[ L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+) \]
Proposition 11.5. There exists an embedding
\[ \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+ \hookrightarrow \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+ \]
Proof. Denote \[ \pi = \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_c \]
By Lemma 3.4, we have
\[ \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+ \hookrightarrow \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_c \cong \pi, \]
\[ \text{and by (3.2)} \]
\[ \sigma_{b,c,a}^+ \cdot L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+) \overset{R(G)}{\cong} \delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+ \hookrightarrow \pi. \]
It is enough to show that both \[ \sigma_{b,c,a}^+ \text{ and } L(\delta([\nu^\frac{b}{2}\rho, \nu^{a}p]) \rtimes \sigma_{b,c}^+) \]
appear in \[ \pi \] once. This follows by Lemmas 3.5, 11.3 and 11.4.

Finally
Proposition 11.6. Induced representation \( \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+ \) has a unique irreducible subrepresentation and unique quotient. We have an exact sequence
\[
\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+ \hookrightarrow L(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+) 
\]
\[\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+ / \sigma_{a,b,c}^+ \twoheadrightarrow L(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+).
\]

Proof. Composition factors are determined by Proposition 10.1. Position of irreducible subquotients are determined by Propositions 3.2, 11.2, 3.5 and 11.5.

By Theorem 4.1, we have

Corollary 11.7. Induced representation \( \delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^+ \) has a unique irreducible subrepresentation and a unique irreducible quotient. We have an exact sequence
\[
\delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^+ \hookrightarrow L(\delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^+) 
\]
\[\delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^+ / \sigma_{a,b,c}^+ \twoheadrightarrow L(\delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^+).
\]

Now we state the composition series of \( \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^+ \) as a direct consequence of Proposition 10.2.

Proposition 11.8. We have a non split exact sequence
\[
L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma_{a,c}^-) \hookrightarrow \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^- \twoheadrightarrow L(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}^-).
\]

By Theorem 4.1, we have

Corollary 11.9. We have a non split exact sequence
\[
L(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{a,c}) \hookrightarrow \delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{a,c}^- \twoheadrightarrow L(\delta([\nu^{-b} \rho, \nu^{-\frac{1}{2}} \rho]) \times \sigma_{b,c}^-).
\]

12. Composition series of \( \delta([\nu^{-c} \rho, \nu^a \rho]) \times \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma \)

Here we determine the composition series of the kernel \( K_2 \) from Section 4.

Proposition 12.1. Induced representation
\[
\delta([\nu^{-c} \rho, \nu^a \rho]) \times \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma \text{ has exactly one irreducible subrepresentation, and two irreducible quotients. We have an exact sequence}
\]
\[
\sigma_{a,c}^+ \times \delta([\nu^{-b} \rho, \nu^a \rho]) \times \sigma + \sigma_{b,c,a}^- \hookrightarrow 
\]
\[\delta([\nu^{-b} \rho, \nu^a \rho]) \times \delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma / L(\delta([\nu^{-b} \rho, \nu^a \rho]) \times \sigma) \twoheadrightarrow 
\]
\[L(\delta([\nu^\frac{1}{2} \rho, \nu^a \rho]) \times \sigma_{a,c}^+) + L(\delta([\nu^\frac{1}{2} \rho, \nu^b \rho]) \times \sigma_{b,c}^-).
\]

Proof. Denote the induced representation by \( \pi \). By Theorem 3.7, \( \pi \) is a multiplicity one representation with irreducible subquotients listed in the
claim. Consider embeddings
\[
\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma_a \hookrightarrow \pi,
\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma) \hookrightarrow \pi,
\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma) \hookrightarrow \pi^+.
\]
We shall describe representations on the left and the claim will follow.

- By Proposition 3.2, \(\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma_a\) has a unique irreducible subrepresentation:
  \(L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a),\) and two irreducible quotients: \(\sigma^\pm_{b,c,a}\).
- By Corollary 4.1 of [3], \(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma)\) is a quotient of \(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma\) containing a unique irreducible subrepresentation: \(L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a)\) and a quotient:
  \(L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma)\).
- By Corollary 4.1 of [3], \(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma)\) is a quotient of \(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma\), containing a unique irreducible quotient:
  \(L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma)\)
and two irreducible subrepresentations: \(L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma^\pm_{b,c,a})\). Thus \(\pi\) has a quotient, containing two irreducible quotients: \(L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma^\pm_{b,c,a})\) and a unique irreducible subrepresentation: \(L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma)\).
- Similarly, by Lemma 3.5, for every \(\epsilon \in \{+,-\}\) \(\pi\) has a quotient, containing a unique irreducible subrepresentation \(\sigma^\epsilon_{b,c,a}\) and a unique irreducible quotient \(L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma^\epsilon_{b,c,a})\).  

By Theorem 4.1, we have

**Corollary 12.2. Induced representation**
\[
\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \delta([\nu^{a}\rho, \nu^{-1/2}\rho]) \rtimes \sigma \text{ has exactly two irreducible subrepresentations and one irreducible quotient. We have an exact sequence}
\]
\[
\sigma^+_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma) + \sigma^-_{b,c,a} \hookrightarrow \\
\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \delta([\nu^{a}\rho, \nu^{-1/2}\rho]) \rtimes \sigma \\
\langle L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma^+_{b,c,a}) + L(\delta([\nu^{1/2}\rho, \nu^a\rho]) \rtimes \sigma^-_{b,c,a}) \rangle \\
\rightarrow L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a).
\]

13. **Composition series of** \(\delta([\nu^{a}\rho, \nu^b\rho]) \rtimes \sigma_b\)

Here we determine composition series of the kernel \(K_3 \cong \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \sigma_b\) from Section 4. First we determine subrepresentations.
Proposition 13.1. We have a unique subrepresentation
\[ \sigma_{b,c,a}^{\pm} \hookrightarrow \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma_b. \]

**Proof.** Compare Theorem 3.6 and Proposition 11.2 with
\[ \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \times \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma. \]

Now we determine position of \( L(\delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a) \).

**Lemma 13.2.** We have in \( R(G) \), with maximum multiplicities
\[ \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \times \delta([\nu\rho, \nu^{\rho}]) \rtimes \sigma_a \geq 1 \times \sigma_{b,c,a}^+ + 1 \times \sigma_{b,c,a}^- + 1 \times L(\delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a), \]
\[ \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \rtimes \sigma_{a,b}^+ \geq 1 \times \sigma_{b,c,a}^+ + 0 \times \sigma_{b,c,a}^- + 1 \times L(\delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a). \]

**Proof.** By Proposition 3.2, \( \sigma_{b,c,a}^\pm \) appears in the first formula. For the multiplicity one use (2.1), (3.3) and (5.2) to search for \( \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \otimes \sigma_{b,c,a}^\pm \).

For the third summand use Lemmas 3.4 and 9.1. Similarly show the second formula.

**Lemma 13.3.** We have an embedding
\[ \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \rtimes \sigma_{a,b}^+. \]

**Proof.** By Lemma 3.4 we have composition of an embedding and an epimorphism
\[ \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \times \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_b \]
\[ \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \rtimes \sigma_{a,b}^+. \]

Lemmas 3.4 and 3.5, Theorem 3.6, and Lemma 13.2, imply that all representations here have \( \sigma_{b,c,a}^\pm \) as a subquotient, with multiplicity one. By Proposition 13.1, \( \sigma_{b,c,a}^+ \) is a unique irreducible subrepresentation of \( \delta([\nu^{-a}\rho, \nu^{\rho}]) \rtimes \sigma_b \).

The claim follows.

**Lemma 13.4.** We have an embedding
\[ \delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a / \sigma_{b,c,a}^- \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \rtimes \sigma_{a,b}^+. \]

**Proof.** Consider a composition of an embedding and an epimorphism
\[ \delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \times \delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_a \]
\[ \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\rho}]) \rtimes \sigma_{a,b}^+. \]

By Proposition 3.2 \( \delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a = \sigma_{b,c,a}^+ + \sigma_{b,c,a}^- + L(\delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a) \),
with discrete series being subrepresentations. Apply Lemma 13.2.

The next proposition gives a position of \( L(\delta([\nu^{-b}\rho, \nu^{\rho}]) \rtimes \sigma_a) \).
Proposition 13.5. We have an embedding
\[ \delta([\nu^{-b}\rho, \nu^c \rho]) \rtimes \sigma_{a,b,c} \hookrightarrow \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b. \]

Proof. By Proposition 3.2 and Theorem 3.6 we have in \( R(G) \)
\[ \delta([\nu^{-b}\rho, \nu^c \rho]) \rtimes \sigma_{a,b,c} \leq \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b. \]

By Lemmas 13.3 and 13.4, these representations embed into \( \delta([\nu^x \rho, \nu^y \rho]) \rtimes \sigma_{a,b} \), which has irreducible subquotients of the first representations with multiplicity one, by Lemma 13.2.

Finally we determine position of \( L(\delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_c). \)

Proposition 13.6. We have an embedding
\[ \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_{a,b,c} \hookrightarrow \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b. \]

Proof. By Proposition 11.2 we have
\[ \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_{a,b,c} \hookrightarrow \delta([\nu^{x}\rho, \nu^y \rho]) \rtimes \sigma_{a,b,c}, \]
so
\[ \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_{a,b,c} \hookrightarrow \delta([\nu^{x}\rho, \nu^y \rho]) \rtimes \sigma, \]
and
\[ \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{x}\rho, \nu^y \rho]) \rtimes \sigma. \]
The claim follows, since by Theorem 10.3 the representation on the right is multiplicity one, and by Proposition 3.2 and Theorem 3.6 we have in \( R(G) \)
\[ \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_{a,b,c} \leq \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b. \]

Now we write composition series for \( \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b. \)

Proposition 13.7. Induced representation \( \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b \) has a unique irreducible subrepresentation. We have an exact sequence
\[ L(\delta([\nu^{-b}\rho, \nu^c \rho]) \rtimes \sigma_a) + L(\delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_c) \hookrightarrow \delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b \rtimes \sigma_{a,b,c} \rightarrow L(\delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b). \]

Proof. Composition factors are determined by Theorem 3.6. Positions of irreducible subquotients are determined by Propositions 3.2, 13.1, 13.5 and 13.6.

By Theorem 4.1, we have

Corollary 13.8. Induced representation \( \delta([\nu^{-c}\rho, \nu^a \rho]) \rtimes \sigma_b \) has a unique irreducible quotient. We have an exact sequence
\[ L(\delta([\nu^{-c}\rho, \nu^a \rho]) \rtimes \sigma_a) + L(\delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_c) \hookrightarrow \delta([\nu^{-c}\rho, \nu^a \rho]) \rtimes \sigma_b / L(\delta([\nu^{-a}\rho, \nu^b \rho]) \rtimes \sigma_b) \rightarrow \sigma_{a,b,c}. \]
14. The main result

Here we give the main result, a composition series of the representation $\psi$.

**Theorem 14.1.** Let $\psi = \delta([\nu^{-a}, \nu^b]) \times \delta([\nu^c, \nu^d]) \times \sigma$ and define representations

\[ W_1 = \sigma_{b,c,a}^+ + L(\delta([\nu^c, \nu^d]) \times \sigma_{b,c}), \]

\[ W_2 = L(\delta([\nu^c, \nu^d]) \times \sigma_{b,c,a}^+) + L(\delta([\nu^{-a}, \nu^b]) \times \sigma_c) + L(\delta([\nu^c, \nu^d]) \times \sigma_{b,c}), \]

\[ W_3 = L(\delta([\nu^c, \nu^d]) \times \sigma_{b,c,a}^+) + L(\delta([\nu^{-a}, \nu^b]) \times \sigma_c) + \sigma_{b,c,a}^+ + L(\delta([\nu^{-a}, \nu^b]) \times \delta([\nu^c, \nu^d]) \times \sigma), \]

\[ W_4 = L(\psi). \]

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \psi$, such that $V_i/V_{i-1} \cong W_i$, $i = 1, \ldots, 4$.

Further, $W_1$ is chosen to be the largest possible, then $W_2$, and so on.

**Proof.** We use the notation $K_i, H_i, f_i$ from Section 4. Composition series of $K_1, K_2$ and $K_3$ are determined by Propositions 11.6, 11.8, 12.1, and Corollary 13.8. Composition series of $H_1, H_2$ and $H_3$ are determined by Proposition 13.7 and Corollaries 12.2, 11.9 and 11.7. For all $i \geq 1$ denote

\[ k_i = K_i \cap \text{Im}(f_{i-1} \circ \cdots \circ f_0), \quad h_i = H_i \cap \text{Im}(g_{i-1} \circ \cdots \circ g_0). \]

Set $K_0 = k_0 = \{0\}$. Let $\pi$ be an irreducible subquotient of $\psi$, and $1 \leq i \leq 3$. Obviously

\[ \pi \leq \text{Im}(f_{i-1} \circ \cdots \circ f_0) \iff \forall j < i \quad \pi \not\subseteq K_j. \]

This implies that if $\pi \leq K_j$, for some $(0 <) j < i$, taking $j$ minimal, we have $\pi \in \text{Im}(f_{j-1} \circ \cdots \circ f_0)$. Thus $\pi \leq k_j$ for some $j < i$. Now we have

\[ \exists j < i \quad \pi \leq K_j \iff \exists j < i \quad \pi \leq k_j \quad \text{and so} \]

\[ \pi \leq \text{Im}(f_{i-1} \circ \cdots \circ f_0) \iff \forall j < i \quad \pi \not\subseteq k_j. \]

We conclude that $k_i$ consists of irreducible subquotients of $K_1$ that do not appear in any of $k_{i-1}, \ldots, k_0$. It is clear that we can write this in $R(G)$ as

\[ k_i = [K_i - k_{i-1} - \cdots - k_0]^{+}_{R(G)}, \]

where for $\pi \in R(G)$, $\pi = \sum m_i \pi_i$, $m_i \in \mathbb{Z}$, $\pi_i \not\cong \pi_j \in \text{Irr}G$ for $i \neq j$, we define

\[ [\pi]^{+}_{R(G)} = \sum_{\{m_i \geq 0\}} m_i \pi_i. \]
Now we have in $R(G)$$k_1 = K_1$, $k_2 = [K_2 - k_1]_{R^+_G}$, $k_3 = [K_3 - k_2 - k_1]_{R^+_G}$, $h_1 = H_1$, $h_2 = [H_2 - h_1]_{R^+_G}$, $h_3 = [H_3 - h_2 - h_1]_{R^+_G}$.

We have $k_1 \cong K_1$ and $h_1 \cong H_1$. Further, calculating composition factors of $k_2$ and $h_2$ and comparing with composition series of $K_2$ and $H_2$, we see that $k_2$ and $h_2$ have exactly two irreducible quotients. Similarly, we determine $k_3$ and $h_3$. So we have exact sequences, and no irreducible subquotient can go on lower position:

\begin{align*}
(14.1) \quad & L(\delta([\nu^{-a} \rho, \nu^{b} \rho]) \rtimes \sigma_{b,c}^+) + L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_c) + L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_{a,c}^-) \hookrightarrow \frac{k_1}{(\sigma_{b,c}^+ + L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_{a,c}^-))} = L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_{a,c}^+), \\
(14.2) \quad & k_2/L(\delta([\nu^{b} \rho, \nu^{c} \rho]) \rtimes \sigma_a) \cong \sigma_{b,c,a}^- + L(\delta([\nu^{-b} \rho, \nu^{c} \rho]) \times \delta([\nu^{a} \rho, \nu^{b} \rho]) \times \sigma), \\
(14.3) \quad & k_3 \cong L(\delta([\nu^{-a} \rho, \nu^{c} \rho]) \rtimes \sigma_b), \\
(14.4) \quad & L(\delta([\nu^{-b} \rho, \nu^{c} \rho]) \rtimes \sigma_a) + L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_c) \hookrightarrow \frac{h_1}{(\sigma_{a,b,c}^+)} = L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_b), \\
(14.5) \quad & h_2/(L(\delta([\nu^{c} \rho, \nu^{d} \rho]) \rtimes \sigma_{b,c}^+)) = (\sigma_{b,c,a}^- + L(\delta([\nu^{-b} \rho, \nu^{d} \rho]) \rtimes \delta([\nu^{a} \rho, \nu^{b} \rho]) \times \sigma), \\
(14.6) \quad & h_3 \cong L(\delta([\nu^{c} \rho, \nu^{d} \rho]) \rtimes \sigma_{a,c}^+) + L(\delta([\nu^{a} \rho, \nu^{d} \rho]) \rtimes \sigma_{a,c}^-).
\end{align*}

We define representations $V_i$, $i = 1, \ldots, 4$ as follows. By (14.1) $V_1 \hookrightarrow \psi$. Let $V_1$ be its image. Further, (14.1) and (14.4) show $W_2 \hookrightarrow \psi/V_1$. Let $V_2$ be the preimage of $W_2$ in $\psi$. Denote representations 
\begin{align*}
\zeta &= L(\delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma_{a,c}^+), \\
\nu &= L(\delta([\nu^{-a} \rho, \nu^{b} \rho]) \rtimes \sigma_c), \\
M &= \sigma_{b,c,a}^- + L(\delta([\nu^{-b} \rho, \nu^{d} \rho]) \rtimes \delta([\nu^{a} \rho, \nu^{b} \rho]) \rtimes \sigma), \\
\tau &= \psi/V_2.
\end{align*}

By (14.1) and (14.4) we have embeddings

\begin{align*}
(14.7) \quad & \zeta \hookrightarrow \tau \hookrightarrow \nu.
\end{align*}
By (14.1) and (14.2) we have an embedding and an epimorphism
\[ M \hookrightarrow \tau / \zeta \overset{P_{\zeta}}{\twoheadrightarrow} \tau, \text{ thus } P_{\zeta}^{-1}(M) / \zeta \cong M. \]

By (14.4) and (14.5) we have an embedding and an epimorphism
\[ M \hookrightarrow \tau / \nu \overset{P_{\nu}}{\twoheadrightarrow} \tau, \text{ thus } P_{\nu}^{-1}(M) / \nu \cong M. \]

We have in \( R(G) \)
(14.8)
\[ P_{\zeta}^{-1}(M) = M + \zeta \text{ and } P_{\nu}^{-1}(M) = M + \nu. \]

We claim that in \( R(G) \)
\[ P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) = M. \]

Inclusions \( P_{\zeta}^{-1}(M) \hookrightarrow \tau \) and \( P_{\nu}^{-1}(M) \hookrightarrow \tau \) induce an embedding
\[ P_{\zeta}^{-1}(M) / (P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M)) \hookrightarrow \tau / P_{\nu}^{-1}(M). \]

This shows that in \( R(G) \)
\[ P_{\zeta}^{-1}(M) - P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \leq \tau - P_{\nu}^{-1}(M), \text{ and by (14.8)} \]
\[ M + \zeta - P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \leq \tau - M - \nu. \]

So, we consider in \( R(G) \)
(14.9)
\[ M + \zeta + \nu \leq P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) + (\tau - M) \]

By Proposition 10.2, \( \psi \) is a multiplicity one, and so is \( \tau \). Also, in \( R(G) \)
\[ \tau = \psi - V_2 = \psi - W_1 - W_2 \geq M. \]

Now, any irreducible subquotient of \( M \) does not appear in \( \tau - M \), so by (14.9),

it must appear in \( P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \). We conclude that in \( R(G) \)
\[ P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \geq M. \]

Thus (14.8) shows that in \( R(G) \)
\[ P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) = M. \]

Now, since \( \tau / \zeta \) is a multiplicity one, and \( P_{\zeta} \) is injective on \( P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \),

both compositions
\[ M \hookrightarrow \tau / \zeta \rightarrow (\tau / \zeta) / P_{\zeta}(P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M)) \text{ and} \]
\[ P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \overset{P_{\zeta}}{\twoheadrightarrow} \tau / \zeta \rightarrow (\tau / \zeta) / i(M) \]

are zero. Thus \( i(M) = P_{\zeta}(P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M)). \)

We conclude \( M \cong P_{\zeta}^{-1}(M) \cap P_{\nu}^{-1}(M) \), and have an embedding
(14.10)
\[ M \hookrightarrow \tau. \]
Combining (14.7) and (14.10) we have an embedding
\[ W_3 \hookrightarrow \psi/V_2. \]
Let \( V_3 \) be the preimage of \( W_3 \) in \( \psi \). We see that in \( R(G) \):
\[ \psi^{R(G)} = W_1 + W_2 + W_3 + L(\psi). \]
We proved the filtration formula. Now we show the last claim, about maximality. Decompositions of \( k_1 \) and \( h_1 \), (14.1) and (14.4), show that no irreducible subquotient of \( W_2 \), can be a subrepresentation of \( \psi \). They also show that \( L(\delta([\nu^{-a}, \nu^c, \rho]) \rtimes \sigma^{+}_{a,c}) \) and \( L(\delta([\nu^{-a}, \nu^c, \rho]) \rtimes \sigma_{b,c,a}) \) can not be embedded into \( \psi/V_1 \). To see the same for factors of \( M \), first assume that \( \sigma_{b,c,a} \hookrightarrow \psi/V_1 \). Since \( k_1/V_1 \hookrightarrow \psi/V_1 \), and \( k_1 \) doesn’t contain \( \sigma_{b,c,a} \), we obtain \( \sigma_{b,c,a} \hookrightarrow \psi/k_1 \). On the other hand \( k_2 \hookrightarrow \psi/k_2 \), and \( k_2 \) contains \( \sigma_{b,c,a} \) but not as a subrepresentation. Since \( \psi \) is a multiplicity one, we got a contradiction. Similarly for the other factor of \( M \).

Acknowledgements.
The author would like to thank Ivan Matić for pointing to a result important for this paper.

References
A ONE HALF CUSPIDAL REDUCIBILITY 29


I. Ciganović
Department of Mathematics
University of Zagreb
10 000 Zagreb
Croatia
E-mail: igor.ciganovic@math.hr