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SOLVING GENERALIZED STURM-LIOUVILLE PROBLEM: NONLINEAR EIGENVALUE PROBLEM AND NEW INTEGRATION METHODS

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Abstract. We propose two methods to convert a class of generalized Sturm-Liouville problems (GSLPs) with boundary conditions that depend on the spectral parameter into a specific GSLP with homogeneous Dirichlet boundary conditions. Using sinusoidal basis functions and a collocation approach, we derive an auxiliary nonlinear eigenvalue problem to compute the eigenvalue. The corresponding GSLP eigenvector is then recovered by postprocessing the solution of this auxiliary problem. The auxiliary problem is a nonlinear matrix eigenvalue problem solved by minimizing a merit function. Eigenvalues are identified via a spectral sweep of the merit function and computed with the golden section search algorithm (GSSA). As a benchmark, we apply an analogous approach directly to the original GSLP, which yields two integration-based methods for evaluating the numerical solution.

1. Introduction

In this paper we show how to reduce a class of generalized eigenvalue problems with spectral dependence in boundary conditions to the solution of a nonlinear eigenvalue problem of the following form:

$$\mathbf{N}(\lambda)\mathbf{x} = \mathbf{0}, \tag{1.1}$$

which is an eigen-equation used to determine the eigen-pair $(\lambda, \mathbf{x})$, where $\mathbf{N}(\lambda) \in \mathbb{R}^{n\times n}$ is a nonlinear matrix function of the eigen-parameter λ, but it is independent to $\mathbf{x}$. A reader interested in further aspects of the theory of

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nonlinear eigenvalue problems should consult an excellent paper by Tisseur and Meerbergen [1]

A large class of methods that the approximate solutions of nonlinear eigenvalue problems are based on the minimization of the norm of the residual of the problem are typically formulated as a Newton type method, see [2, 3, 4, 5, 6]. As alternatives, we point out methods based on Arnoldi or rational Krylov space decompositions and preconditioned iterative methods of the Jacobi-Davidson type, see [7, 8, 9].

A lot of numerical methods have been developed in the past several decades to solve the Sturm-Liouville problem (SLP) and its generalization, namely the GSLP [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. On the other hand, the shooting methods were developed by Ghelardoni and Gheri [21], and Liu [22, 23]. Efficient and accurate codes for solving the SLP have been developed in [24, 25, 26, 27, 28, 29, 30]. Moreover, a large widespread is the application of matrix methods [31, 32, 33]. For the eigen-parameter dependent boundary conditions the numerical methods to treat the GSLP were provided in [34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Recently, Liu et al. [44] developed a powerful numerical method to precisely compute the eigenvalues of GSLP.

The integration method for the Sturm-Liouville problem (SLP) is somewhat a simple standard method of computing a logarithmic derivative and then finding the eigenvalue by solving a nonlinear first-order Riccati differential equation. In particular the standard method can be used to generate the "exact eigenvalue", which is computed to high accuracy by using the standard shooting algorithm if the right-end boundary condition can be exactly matched. Unfortunately, the standard method is difficult to be applied to the GSLP, because the boundary conditions also involve the unknown eigenparameter.

In this paper, we will transform the GSLP to a nonlinear eigenvalue problem of the form (1.1). The problem (1.1) will be solved using any of the algebraic methods for the solution of the nonlinear eigenvalue problem. As an expensive benchmark, we will use the integration method. In spirit, all of the considered methods are minimization methods, and we will use the golden section search algorithm (GSSA) as a robust way to solve these opimisation problems. The nonlinear eigenvalue problem imposes a great challenge for the researchers to develop efficient and accurate methods [7]; however, we will resolve this difficulty by using a novel minimization technique.

In Section 2, we introduce a novel detection and sweep method of the eigenvalue for the nonlinear eigenvalue problem and the numerical process is given. The GSLP is converted to a new GSLP with the homogeneous Dirichlet boundary conditions at two ends by two methods in Section 3. Then, we derive the corresponding nonlinear eigenvalue problems for these two transformed GSLPs with the sinusoidal bases to expand the new variable in Section 4.1. Two novel integration methods are introduced in Sections 4.2 and 4.3, respectively. The Chebyshev spectral collocation method is introduced in Section 4.4. Four examples solved by the proposed methods are examined in Section 5. Finally, we summarize the results of the paper in Section 6.

2. A new detection and sweep method of eigenvalue

Let $\lambda(\mathbf{N})$ be the set of all eigenvalues of Eq. (1.1), i.e. $\lambda(\mathbf{N}) = \{\lambda \mid \mathrm{Det}[\mathbf{N}(\lambda)] = 0\}$. It is known that if $\lambda \notin \lambda(\mathbf{N})$, then Eq. (1.1) has only the trivial solution $\mathbf{x} = \mathbf{0}$. If $\lambda \in \lambda(\mathbf{N})$ then Eq. (1.1) has a non-trivial solution of the vector variable with $\mathbf{x} \neq \mathbf{0}$, which is called the eigenvector. It is difficult to directly solve Eq. (1.1) to determine λ and $\mathbf{x}$ by the manual operations. Instead, numerical methods have to be employed to solve Eq. (1.1). However, without having a special treatment $\mathbf{x} = \mathbf{0}$ is always obtained by numerical method no matter which λ is specified, since the right-hand side of Eq. (1.1) is a zero vector.

In order to definitely obtain a nonzero vector $\mathbf{x}$, we must create a nontrivial solution of Eq. (1.1), of which at least one component of $\mathbf{x}$ is not zero. We can normalize a nonzero eigenvector by setting the j_0-th component of $\mathbf{x}$ to be $x_{j_0} = 1$. Let n_{ij} be the components of $\mathbf{N}(\lambda)$ for each specified eigen-parameter λ, and define

$$e_i = -n_{ij_0}, \; i = 1, \ldots, n, \tag{2.1}$$

where n_{ij_0} is the j_0-th column of the matrix $\mathbf{N}$. Then, from Eq. (1.1) an $n_0 = (n-1)$-dimensional nonhomogeneous linear system can be obtained by moving the j_0-th column of the eigen-equation to the right side:

$$c_{ik} y_k = e_i, \; i, k = 1, \ldots, n_0, \tag{2.2}$$

where the coefficient matrix $[c_{ik}]$ with dimension $n_0 \times n_0$ is constructed from $[n_{ij}]$ by the following algorithm.

Algorithm: Construction of Matrix $[c_{ik}]$
1: For $i = 1$ to n_0
2: $k \leftarrow 0$
3: For $j = 1$ to n
4: If $j = j_0$ continue to next j
5: Endif
6: $k \leftarrow k + 1$
7: $c_{ik} \leftarrow n_{ij}$
8: Endfor
9: Endfor

The Gaussian elimination method is used to solve $(y_1,\dots,y_{n_0})$ in Eq. (2.2), and then $\mathbf{x}=(x_1,\dots,x_n)^{\mathrm{T}}$ is computed from $\mathbf{y}=(y_1,\dots,y_{n_0})^{\mathrm{T}}$ by the following algorithm.

Algorithm: Construction of Vector x_j
1: $k \leftarrow 0$
2: For $j=1$ to n
3: If $j=j_0$, $x_j=1$ and continue to next j
4: Endif
5: $k \leftarrow k+1$
6: $x_j \leftarrow y_k$
8: Endfor

For a given $\lambda\in\mathbb{R}$, if $\mathbf{x}$ is solved from Eqs. (2.1), (2.2), and the above two algorithms with a certain j_0 being given, then we can pick up the correct value of λ by minimizing the following merit function:

$$\min_{\lambda\in[a,b]} F(\lambda):=\|\mathbf{N}(\lambda)\mathbf{x}\|\geq 0, \tag{2.3}$$

where $\|\mathbf{N}(\lambda)\mathbf{x}\|$ denote the Euclidean norm of $\mathbf{N}(\lambda)\mathbf{x}$, and the size of $[a,b]$ must be sufficiently large to include one eigenvalue $\lambda\in[a,b]$. It is emphasized that the vector variable $\mathbf{x}$ in Eq. (2.3) is solved from Eqs. (2.1), (2.2), and the above two algorithms for each specified eigen-parameter $\lambda\in[a,b]$. The pair $(\lambda,\mathbf{x})$ obtained by this manner is an eigen-pair satisfying Eq. (1.1), such that $\|\mathbf{N}(\lambda)\mathbf{x}\|=0$ is the minimum of F. Otherwise, $\|\mathbf{N}(\lambda)\mathbf{x}\|>0$ when λ is not an eigenvalue. The above idea was first developed by Liu et al. [45] for the free vibrations of multi-degree structures using the exciting method and an iterative detection method to determine the vibration frequencies.

The numerical procedures for determining the eigenvalue of a nonlinear eigenvalue problem are summarized as follows. (i) Select $[a,b]$ and j_0 ($j_0=1$ in general). (ii) Use Algorithms Construction of Matrix $[c_{ik}]$ and Construction of Vector x_j for each required $\lambda_i\in[a,b]$. (iii) Apply the one-dimensional golden section search algorithm (GSSA) as shown in the Appendix to Eq. (2.3) for picking up the eigenvalue. The above spectral sweep method (SSM) is summarized as follows.

Algorithm: SSM
1: Give a, b, j_0, and ε
2: For each $\lambda\in[a,b]$, use Algorithms Construction of Matrix $[c_{ik}]$ and Construction of Vector x_j to obtain $\mathbf{x}$
3: Apply GSSA to Eq. (2.3)

It can be seen that the Algorithm SSM is very simple. The convergence criterion of GSSA is fixed to be 10^{-15}. For definiteness we consider a nonlinear eigenvalue problem [9]:

$$\mathbf{N}(\lambda) = \lambda^2 \mathbf{A}_2 + \lambda \mathbf{A}_1 + \mathbf{A}_0, \tag{2.4}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} -4 & 3 & 12 \\ -17 & -11 & 0 \\ 1 & -1 & 3 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 2 & -6 & 1 \\ -2 & 22 & 11 \\ 7 & -1 & 1 \end{bmatrix}, \mathbf{A}_0 = \begin{bmatrix} -16 & -4 & 7 \\ -14 & 7 & 13 \\ 6 & 8 & 7 \end{bmatrix}. \tag{2.5}$$

We apply the SSM to solve this problem with $j_0 = 1$ and $\varepsilon = 10^{-15}$. In Fig. 1 with respect to the eigen-parameter in an interval, two minimums represent two real eigenvalues. By applying the SSM we first take a large interval to sweep the values of λ. Then a rough location can be detected as shown in Fig. 1. Then a finer interval is selected to precisely obtain the eigenvalue by using the GSSA. Table 1 lists the eigenvalue, the error $\|\mathbf{N}\mathbf{x}\|$, and the number of iterations (NI) expensed in the GSSA.

TABLE 1. For an example of a nonlinear eigenvalue problem solved by SSM

λ	$\|\mathbf{N}\mathbf{x}\|$	NI
-0.2328574586400297	2.18×10^{-16}	30
2.355885632295363	8.33×10^{-15}	30

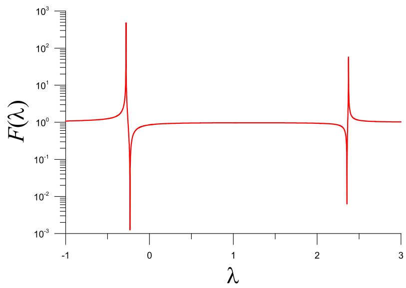

FIGURE 1. For a nonlinear eigenvalue problem solved by SSM, showing two minimums in a merit function F

3. Transforming GSLP to the one with homogeneous Dirichlet boundary conditions

A specific GSLP is given by

$$y''(x) + q(x,\lambda)y(x) = 0,\ x \in (0,1), \tag{3.1}$$

$$a_1(\lambda)y(0) + b_1(\lambda)y'(0) = 0,\ a_2(\lambda)y(1) + b_2(\lambda)y'(1) = 0. \tag{3.2}$$

It is different from the conventional SLP that the coefficients $a_1(\lambda)$, $b_1(\lambda)$, $a_2(\lambda)$ and $b_2(\lambda)$ are functions of λ. The following is an extension of the result in [46] from the SLP to the GSLP.

Theorem 3.1. *The GSLP (3.1) and (3.2) can be transformed to a nonlinear GSLP, endowed with the homogeneous Dirichlet boundary conditions on two-side:*

$$v''(x) = F_1(x,\lambda)v'(x) + F_2(x,\lambda)v(x), \tag{3.3}$$

$$v(0) = 0,\ v(1) = 0, \tag{3.4}$$

in which

$$\begin{aligned} v(x) &= [a_2(\lambda)a_3(\lambda) - (1-x)q(x,\lambda)b_2(\lambda)b_3(\lambda)]y(x) \\ &\quad +[a_3(\lambda)b_2(\lambda) + (1-x)a_2(\lambda)b_3(\lambda)]y'(x), \end{aligned} \tag{3.5}$$

$$F_1(x,\lambda) = \frac{1}{D(x,\lambda)}[a_{11}(x,\lambda)a_{22}(x,\lambda) - a_{12}(x,\lambda)a_{21}(x,\lambda)], \tag{3.6}$$

$$\begin{aligned} F_2(x,\lambda) &= \frac{1}{D(x,\lambda)}[a_{11}(x,\lambda)a_{21}(x,\lambda) + a'_{12}(x,\lambda)a_{21}(x,\lambda) \\ &\quad +a_{12}(x,\lambda)a_{22}(x,\lambda)q(x,\lambda) - a_{22}(x,\lambda)a'_{11}(x,\lambda)], \end{aligned} \tag{3.7}$$

where

$$a_3(\lambda) = a_1(\lambda)a_2(\lambda) + b_1(\lambda)b_2(\lambda)q(0,\lambda), \tag{3.8}$$

$$b_3(\lambda) = a_2(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda), \tag{3.9}$$

$$a_{11}(x,\lambda) = a_2(\lambda)a_3(\lambda) - (1-x)q(x,\lambda)b_2(\lambda)b_3(\lambda), \tag{3.10}$$

$$a_{12}(x,\lambda) = a_3(\lambda)b_2(\lambda) + (1-x)a_2(\lambda)b_3(\lambda), \tag{3.11}$$

$$\begin{aligned} a_{21}(x,\lambda) = a''_{11}(x,\lambda) &- \big[a_{11}(x,\lambda) + 2a'_{12}(x,\lambda)\big]q(x,\lambda) \\ &- a_{12}(x,\lambda)q'(x,\lambda), \end{aligned} \tag{3.12}$$

$$a_{22}(x,\lambda) = 2a'_{11}(x,\lambda) - a_{12}(x,\lambda)q(x,\lambda), \tag{3.13}$$

$$\begin{aligned} D(x,\lambda) = a_{11}(x,\lambda)&\big[a_{11}(x,\lambda) + a'_{12}(x,\lambda)\big] \\ &+ a_{12}(x,\lambda)\big[a_{12}(x,\lambda)q(x,\lambda) - a'_{11}(x,\lambda)\big]. \end{aligned} \tag{3.14}$$

Proof. Refer to [46]. □

The following result is an extension from the SLP [47] to the GSLP, which is simpler than that given in Theorem 1.

THEOREM 3.2. *The GSLP (3.1) and (3.2) can be transformed to a simpler nonlinear GSLP, which is subjected to the two-side homogeneous Dirichlet boundary conditions:*

$$v''(x) = F_3(x,\lambda)v'(x) + F_4(x,\lambda)v(x), \tag{3.15}$$

$$v(0) = 0, \ \ v(1) = 0, \tag{3.16}$$

in which

$$v(x) = p_1(x,\lambda)y(x) + p_2(x,\lambda)y'(x), \tag{3.17}$$

$$\begin{aligned} F_3(x,\lambda) = \frac{1}{D(x,\lambda)}\Big\{ & \big[p_3(x,\lambda) + p_4'(x,\lambda)\big]p_1(x,\lambda) \\ & - \big[p_3'(x,\lambda) - p_4(x,\lambda)q(x,\lambda)\big]p_2(x,\lambda)\Big\}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} F_4(x,\lambda) = \frac{1}{D(x,\lambda)}\Big\{ & \big[p_3'(x,\lambda) - p_4(x,\lambda)q(x,\lambda)\big]p_4(x,\lambda) \\ & - \big[p_3(x,\lambda) + p_4'(x,\lambda)\big]p_3(x,\lambda)\Big\}, \end{aligned} \tag{3.19}$$

where

$$p_1(x,\lambda) = a_1(\lambda)[1 - s(x)] + a_2(\lambda)s(x), \tag{3.20}$$

$$p_2(x,\lambda) = b_1(\lambda)[1 - s(x)] + b_2(\lambda)s(x), \tag{3.21}$$

$$p_3(x,\lambda) = p_1'(x,\lambda) - p_2(x,\lambda)q(x,\lambda) = [a_2(\lambda) - a_1(\lambda)]s'(x) - p_2(x,\lambda)q(x,\lambda), \tag{3.22}$$

$$p_4(x,\lambda) = p_1(x,\lambda) + p_2'(x,\lambda) = [b_2(\lambda) - b_1(\lambda)]s'(x) + p_1(x,\lambda), \tag{3.23}$$

$$D(x,\lambda) = p_1(x,\lambda)p_4(x,\lambda) - p_2(x,\lambda)p_3(x,\lambda), \tag{3.24}$$

and $s(0) = 0$ and $s(1) = 1$. We take $s(x) = x$ for most cases.

PROOF. Letting

$$v(x) = [a_1(\lambda)y(x) + b_1(\lambda)y'(x)](1 - x) + [a_2(\lambda)y(x) + b_2(\lambda)y'(x)]x, \tag{3.25}$$

which can be arranged to Eq. (3.17), it is easy to check that $v(x)$ satisfies Eq. (3.16) by using the boundary conditions in Eq. (3.2).

It follows from Eqs. (3.17) and (3.1) that

$$v'(x) = p_3(x,\lambda)y(x) + p_4(x,\lambda)y'(x), \tag{3.26}$$

which together with Eq. (3.17) can generate

$$y(x) = \frac{1}{D(x,\lambda)}[p_4(x,\lambda)v(x) - p_2(x,\lambda)v'(x)], \tag{3.27}$$

$$y'(x) = \frac{1}{D(x,\lambda)}[p_1(x,\lambda)v'(x) - p_3(x,\lambda)v(x)]. \tag{3.28}$$

Taking the derivative of Eq. (3.26) and using Eq. (3.1) again yields

$$v''(x) = [p_3'(x,\lambda) - p_4(x,\lambda)q(x,\lambda)]y(x) + [p_3(x,\lambda) + p_4'(x,\lambda)]y'(x), \tag{3.29}$$

which after inserting Eqs. (3.27) and (3.28) can derive Eq. (3.15). □

Notice that for the following SLP with constant boundary conditions:

$$y''(x) + \frac{p'(x)}{p(x)}y'(x) + \frac{\lambda w(x) - q(x)}{p(x)}y(x) = 0, \ x \in (0,1), \tag{3.30}$$

$$a_1 y(0) + b_1 y'(0) = 0, \ a_2 y(1) + b_2 y'(1) = 0, \tag{3.31}$$

where $b_1 \neq 0$ and $b_2 \neq 0$, using the dependent variable transformation:

$$u(x)y(x) + y'(x) = 0, \text{ or } \ln y(x) + \int u(x)dx = 0, \tag{3.32}$$

we can transform it to

$$u'(x) = \frac{\lambda w(x) - q(x)}{p(x)} - \frac{p'(x)}{p(x)}u(x) + u^2(x), \tag{3.33}$$

which is a Riccati differential equation involving an unknown eigenvalue λ. The standard integration method can be used to integrate Eq. (3.33) with an initial condition:

$$u(0) = \frac{a_1}{b_1}. \tag{3.34}$$

Because λ is an unknown constant, it is difficult by applying the standard integration method to solve the SLP, which must exactly satisfy the following right-end boundary condition in terms of u:

$$u(1) = \frac{a_2}{b_2}.$$

Moreover, for the GSLP the standard integration method is not applicable because the boundary conditions are also dependent on the eigenvalue.

4. Nonlinear eigenvalue problem and integration methods

4.1. *Nonlinear eigenvalue problem.* We can discretize Eq. (3.3), or Eq. (3.15), to a nonlinear eigenvalue problem as follows. Let

$$v(x) = \sum_{j=1}^{n} c_j \phi_j(x), \tag{4.1}$$

$$\phi_j(x) = \sin(j\pi x), \tag{4.2}$$

of which $v(x)$ automatically satisfies Eq. (3.4). Collocating Eq. (3.3) by n inner points $x_i = i/(n+1)$ yields

$$\sum_{j=1}^{n} c_j \phi_j''(x_i) - F_1(x_i,\lambda) \sum_{j=1}^{n} c_j \phi_j'(x_i) - F_2(x_i,\lambda) \sum_{j=1}^{n} c_j \phi_j(x_i) = 0, \ i = 1, \ldots, n. \tag{4.3}$$

Then, we obtain a nonlinear eigenvalue problem with dimension n:

$$\mathbf{A}(\lambda)\mathbf{c} = \mathbf{0}, \tag{4.4}$$

where $\mathbf{c} = (c_1, \ldots, c_n)^{\mathrm{T}}$ is a nonzero eigenvector to be determined, and the components a_{ij} of $\mathbf{A}$ are given by

$$a_{ij}(\lambda) := \phi_j''(x_i) - F_1(x_i,\lambda)\phi_j'(x_i) - F_2(x_i,\lambda)\phi_j(x_i), \ i,j = 1, \ldots, n. \tag{4.5}$$

In terms of $\phi_j(x) = \sin(j\pi x)$ in Eq. (4.2), a_{ij} is simply given as follows:

$$a_{ij}(\lambda) = -(j\pi)^2 \sin(j\pi x_i) - j\pi F_1(x_i,\lambda)\cos(j\pi x_i) - F_2(x_i,\lambda)\sin(j\pi x_i). \tag{4.6}$$

Now, we can apply the technique in Section 2 to solve the nonlinear eigenvalue problem (4.4) by considering the following minimization problem:

$$\min_{\lambda \in [a,b]} f(\lambda) := \|\mathbf{A}(\lambda)\mathbf{c}\| \geq 0. \tag{4.7}$$

The same procedures are applied to Eq. (3.15). For the distinct purpose they are named respectively the first method based on Eq. (3.3), and the second method based on Eq. (3.15).

As mentioned in [48] the discretization issues for nonlinear eigenvalue problems must be carefully taken into account. In general, n must be taken sufficiently large to capture the true eigenvalue of GSLP by solving the n-dimensional nonlinear eigenvalue problem.

4.2. *Integration method.* In view of Eqs. (3.15) and (3.16), the unique solution of v for each eigenvalue λ can be expected by imposing a normalization condition $v'(0) = 1$, such that we encounter an unknown constant λ in the differential equation:

$$v''(x) = F_3(x,\lambda)v'(x) + F_4(x,\lambda)v(x), \tag{4.8}$$

$$v(0) = 0,\ v'(0) = 1, \tag{4.9}$$

$$v(1) = 0. \tag{4.10}$$

We can integrate the above equation and seek the minimization of the following merit function:

$$\min_{\lambda\in[a,b]} f(\lambda) := |v(1)| \geq 0. \tag{4.11}$$

This method is named an integration method which can raise the accuracy of λ; however, the eigenfunction obtained is merely a numerical solution of $y(x)$ obtained from Eq. (3.27) by inserting λ and $v(x)$ and $v'(x)$ obtained by the integration of Eq. (4.8). Not like that in Eq. (4.1), which is a semi-analytic solution. The numerical procedures for determining the eigenvalue by using the integration method are summarized as follows. (i) Select $[a,b]$. (ii) Integrate Eqs. (4.8) and (4.9) for each required $\lambda_i \in [a,b]$. (iii) Apply the GSSA to Eq. (4.11) for picking up the eigenvalue.

4.3. *A simple integration method.* Motivated by the integration method in Section 4.2, we can also directly find the eigenvalue by the following equations:

$$y''(x) = -q(x,\lambda)y(x), \tag{4.12}$$

$$y(0) = -\frac{b_1(\lambda)}{a_1(\lambda)},\ y'(0) = 1, \tag{4.13}$$

$$\min_{\lambda\in[a,b]} f(\lambda) = |a_2(\lambda)y(1) + b_2(\lambda)y'(1)| \geq 0. \tag{4.14}$$

If $a_1 = 0$, we take $y(0) = 1$ and $y'(0) = 0$, where the former one is a normalization condition for the unique solution of $y(x)$. This method is named a simple integration method. The advantage of the simple integration method is that we do not need to transform $y(x)$ to $v(x)$.

4.4. *Chebyshev spectral collocation method.* In this occasion let us mention the Chebyshev spectral collocation method [49]. To replace that in Section 4.1, we introduce the Chebyshev function T_j as the bases of $v(x)$. To automatically satisfy the homogeneous boundary conditions in Theorem 2, we set

$$\phi_j(x) = T_j(x) - (1-x)T_j(0) - xT_j(1), \tag{4.15}$$

where

$$T_j(x) = \cos(j \arccos(2x-1)). \tag{4.16}$$

Consequently, we can derive

$$\phi_j(x) = \cos(j\arccos(2x-1)) - (1-x)\cos(j\pi) - x, \tag{4.17}$$

$$\phi_j'(x) = \frac{2j}{\sqrt{1-(2x-1)^2}}\sin(j\arccos(2x-1)) + x\cos(j\pi) - 1, \tag{4.18}$$

$$\begin{aligned}\phi_j''(x) = &\frac{4j(2x-1)}{[1-(2x-1)^2]^{3/2}}\sin(j\arccos(2x-1))\\ &-\frac{4j^2}{1-(2x-1)^2}\cos(j\arccos(2x-1)).\end{aligned} \tag{4.19}$$

In [49] the boundary conditions were implemented by the rectangular spectral technique in order to derive a square coefficient matrix. Here, because $\phi_j(x)$ satisfied the boundary conditions automatically, we no longer need to impose the boundary conditions, such that a square coefficient matrix $\mathbf{A}$ with dimension $n \times n$ can be easily deduced by collocating at n interior points as that in [49]. The new method is better than that obtained by the rectangular spectral collocation method, and is named a new Chebyshev spectral collocation method (NCSCM).

In contrast to Eq. (4.6), a_{ij} is now very complicated given as follows:

$$\begin{aligned}a_{ij}(\lambda) = &\frac{4j(2x_i-1)}{[1-(2x_i-1)^2]^{3/2}}\sin(j\arccos(2x_i-1))\\ &-\frac{4j^2}{1-(2x_i-1)^2}\cos(j\arccos(2x_i-1))\\ &-F_3(x_i,\lambda)\left[\frac{2j}{\sqrt{1-(2x_i-1)^2}}\sin(j\arccos(2x_i-1)) + \cos(j\pi) - 1\right]\\ &-F_4(x_i,\lambda)[\cos(j\arccos(2x_i-1)) - (1-x_i)\cos(j\pi) - x_i].\end{aligned} \tag{4.20}$$

Other processes are the same to that in Section 4.1. Below, two examples are given to show that the second method outperforms NCSCM.

5. Examples

5.1. *Example 1.* We test the following SLP:

$$-y''(x) + \frac{2}{(x+1)^2}y(x) = \lambda y(x),\ x \in (0,1), \tag{5.1}$$

$$y(0) + y'(0) = 0,\ y(1) + 2y'(1) = 0, \tag{5.2}$$

with $\lambda_k = (k+1)^2\pi^2,\ k = 0,1,2,\ldots$ being the exact eigenvalues.

Take $n = 200$ and $\varepsilon = 10^{-15}$ and we determine λ iteratively. Table 2 shows for λ computed by the first and second methods, the exact λ, and the number of iterations (NI). It can be seen that the first method is more accurate than the second method. The NI is around 70.

TABLE 2. Eqs. (5.1) and (5.2) are solved by the two proposed methods, listing the computed eigenvalues and NI

First method	Second method	Exact λ	NI
9.86958508144	9.86956346316	9.86960440109	72
39.4784037345	39.4782538445	39.4784176044	64
88.8264265828	88.8260711201	88.8264396098	62

By using the same parameters in Table 2, Table 3 compares the second method with the NCSCM in Section 4.4. The eigenvalue obtained by the NCSCM is less accurate than that obtained by the second method.

TABLE 3. For Eqs. (5.1) and (5.2) solved by NCSCM and the second method, listing the computed eigenvalues and NI of NCSCM

NCSCM	Second method	Exact λ	NI
9.61802799793	9.86956346316	9.86960440109	71
39.4257328093	39.4782538445	39.4784176044	62
88.8238720492	88.8260711201	88.8264396098	61

5.2. *Example 2.* We consider [36, 38, 39]:

$$y''(x) + (\lambda - e^x)y(x) = 0, \ 0 < x < 1, \tag{5.3}$$

$$y(0) = 0, \ -\sqrt{\lambda}\sin\sqrt{\lambda}y(1) + \cos\sqrt{\lambda}y'(1) = 0. \tag{5.4}$$

In Fig. 2 the variations of the merit functions with respect to λ are plotted in a range $\lambda \in [0, 60]$, where the minimal points correspond to the eigenvalues. It can be seen that the values of $f(\lambda) = |v(1)|$ for the integration method are smaller than that of $f(\lambda) = \|\mathbf{A}(\lambda)\mathbf{c}\|$ for the second method. First, we choose a large interval of $[a, b] = [0, 60]$ to include all eigenvalues, such that the rough locations of the eigenvalues can be observed in the curve of $f(\lambda)$ as the minimal points in Fig. 2. Then, to precisely determine the individual eigenvalue, we choose a smaller interval $[a, b]$ to include that eigenvalue as an internal point. A few iterations by using the GSSA can compute very accurate eigenvalue.

We fix $n = 50$ and $\varepsilon = 10^{-15}$; Table 4 for λ was computed by the first and second methods and integration methods, and the λ in [36] was listed for

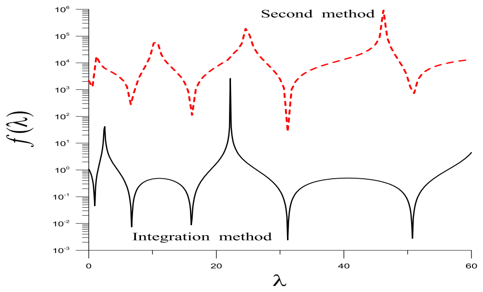

FIGURE 2. For example 2 of a GSLP, comparing the merit functions obtained by the second method and the integration method, whose local minimal points correspond to certain eigenvalues

comparison. Rather than that in Eq. (5.4), in the second method we have taken

$$-\sqrt{\lambda}\tan\sqrt{\lambda}y(1)+y'(1)=0.$$

The simple integration method is applied to this example with eigenvalues given in Table 4.

TABLE 4. Eqs. (5.3) and (5.4) are solved by the four proposed methods, listing the computed eigenvalues

First method	Second method	λ in [36]	Integration method	Simple integration method
0.92905224393	0.92911206156	0.92906202857	0.92906202858	0.92906202858
6.748037053	6.747829410	6.747881178	6.747881178	6.747881178
16.12494291	16.13116703	16.12454773	16.12454773	16.12454773
31.22135858	31.22128028	31.22027651	31.22027651	31.22027651
50.73533384	50.73524732	50.73392784	50.73392785	50.73392784

In Fig. 3 the fifth eigenfunction obtained by the second method and the integration method is compared. They satisfy the boundary conditions in Eq. (5.4); however, the eigenfunction obtained by the integration method is more accurate than that obtained by the second method.

By using the same parameters in Table 4, Table 5 compares the second method with NCSCM in Section 4.4. The eigenvalue obtained by the NCSCM is not accurate than that obtained by the second method. When we apply the NCSCM to find the fifth eigenfunction, it is failure as shown in Fig. 4.

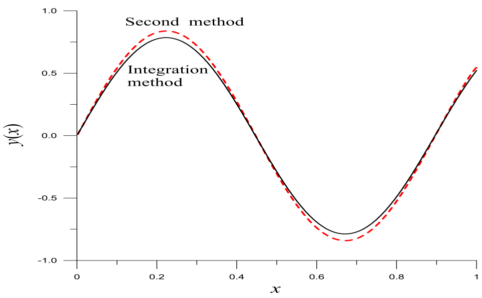

FIGURE 3. For example 2 of a GSLP, comparing the fifth eigenfunction obtained by the second method and the integration method

TABLE 5. For Eqs. (5.3) and (5.4) solved by NCSCM and the second method, listing the computed eigenvalues

NCSCM	Second method	λ in [36]
0.96393202250	0.92911206156	0.92906202857
6.575465200	6.747829410	6.747881178
16.85410202	16.13116703	16.12454773
31.96555815	31.22128028	31.22027651
50.47220206	50.73524732	50.73392784

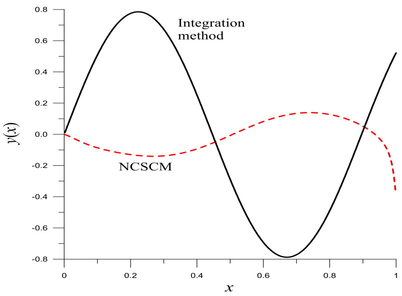

FIGURE 4. For example 2 of a GSLP, comparing the fifth eigenfunction obtained by the NCSCM and the integration method

5.3. *Example 3.* Consider [36]

$$y''(x) + \lambda y(x) = 0, \; 0 < x < 1, \tag{5.5}$$

$$y(0) - 2y'(0) = 0, \; (1 + \sqrt{\lambda})y(1) + (1 - \lambda)y'(1) = 0, \tag{5.6}$$

where the exact eigenvalues are obtained in [36].

We fix $n = 50$ and $\varepsilon = 10^{-15}$; Table 6 for λ was computed by the first and second methods and integration method, and the λ in [36] was listed for comparison. Rather than that in Eq. (5.6), in the second method we have taken

$$y(1) + \frac{1-\lambda}{1+\sqrt{\lambda}} y'(1) = 0.$$

The simple integration method is applied to this example with the eigenvalues given in Table 6.

TABLE 6. Eqs. (5.5) and (5.6) are solved by the four proposed methods, listing the computed eigenvalues

First method	Second method	Exact λ	Integration method	Simple integration method
9.938736432	9.938743578	9.938743414	9.938743414	9.938743414
40.09353738	40.09898214	40.09896973	40.09896973	40.09896973
89.55835460	89.58765316	89.58763453	89.58763453	89.58763453

In Fig. 5 the third eigenfunction obtained by the second method and the integration method is compared. They satisfy the boundary conditions in Eq. (5.6); however, the eigenfunction obtained by the integration method is more accurate than that obtained by the second method.

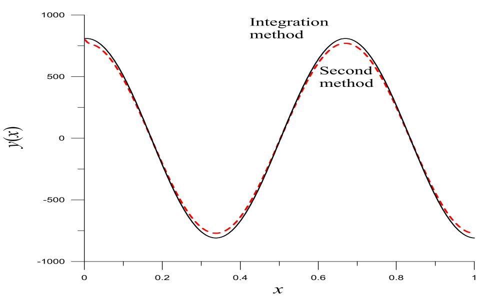

FIGURE 5. For example 3 of a GSLP, comparing the third eigenfunction obtained by the second method and the integration method

5.4. *Example 4.* According to [35, 36, 37], we consider

$$y''(x) + \lambda y(x) = 0, \; 0 < x < 1, \tag{5.7}$$

$$y(0) + (\lambda - 4\pi^2)y'(0) = 0, \; y(1) - \lambda y'(1) = 0. \tag{5.8}$$

We fix $n = 50$ and $\varepsilon = 10^{-15}$. Table 7 for λ was computed by the first and second methods and integration method, and the exact λ is given. It can be seen that the first method is more accurate than the second method for the first eigenvalue, but it is not applicable to find the first eigenvalue with $j_0 = 1$, and we take $j_0 = 5$. In the integration method we take

$$\lambda y'(1) - y(1) = 0.$$

TABLE 7. Eqs. (5.7) and (5.8) are solved by the four proposed methods, listing the computed eigenvalues

First method	Second method	Exact λ	Integration method	Simple integr-ation method
9.73088720965	9.73307160092	9.73088657821	9.73088657821	9.73088657823
88.7633163807	88.7851390250	88.7633162526	88.7633162639	88.7633162643

In Fig. 6 the second eigenfunction obtained by the first method and the integration method is compared. However, the eigenfunction obtained by the integration method is more accurate than that obtained by the first method.

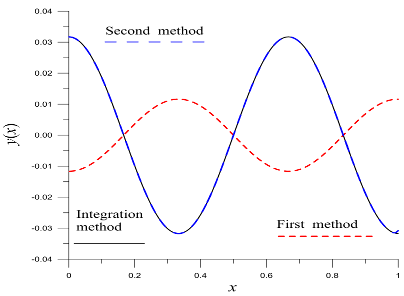

FIGURE 6. For example 4 of a GSLP, comparing the second eigenfunction obtained by the first and second methods and the integration method

In Fig. 6 the second eigenfunction obtained by the second method and the integration method is compared. They satisfy the boundary conditions in Eq. (5.8); however, in the second method we take

$$\frac{1}{4\pi^2 - \lambda} y(0) - y'(0) = 0,$$

and $s(x) = x^2$.

The simple integration method is applied to Example 4, and the eigenvalues are listed in Table 7; however, its accuracy is slightly worse than that using the integration method.

6. Conclusions

This paper was concerned with the fast iterative solution methods of the GSLP, which was recast to a specific GSLP with the homogeneous Dirichlet boundary conditions at two-side by the proposed two new transformation techniques. Then, using the sinusoidal functions as the bases and with the collocation technique, we have derived a nonlinear eigenvalue problem to determine the expansion coefficients, which are components of the eigenvector of a nonlinear eigenvalue problem. To solve the resulted nonlinear eigenvalue problem, varying the eigen-parameter in a desired range, we established the curve for eigenvalues as being the local minimums of the constructed merit functions; two examples were shown in Fig. 2 by using the spectral sweep method. In the merit function the eigenvector variable is solved from a nonhomogeneous linear system, which is available by reducing the eigen-equation with one dimension less and by moving the normalized component to the right side. We can quickly obtain the eigenvalues by using the golden section search algorithm to solve the resultant minimization problems. The proposed two methods can obtain eigenvalues with moderate accuracy, and the eigenfunctions obtained are of semi-analytic forms. Based on the GSLP with the homogeneous Dirichlet boundary conditions, we supplement a unit slope condition at the left-end for the uniqueness of the eigenfunction and then solved it by using the integration method, which together with the golden section search algorithm can expeditiously determine the eigenvalue by minimizing the right-end absolute error of the solution to satisfy the right-end boundary condition. We also directly integrated the original GSLP by using the golden section search algorithm to search the eigenvalue to satisfy the mixed right boundary condition. Very accurate eigenvalue and eigenvector can be obtained by those two integration methods merely through a few iterations, and the computations of the merit functions are quite saving. However, the eigenfunctions obtained by the integration methods are pure numerical values.

Upon comparing to the Chebyshev method the proposed second method is better in the accuracy of the eigenvalue, which is robust to find accurate eigenfunction. The Chebyshev method fails to find an accurate eigenfunction, because its resulting coefficient matrix $\mathbf{A}$ is complicated and is ill-posed to produce a correct solution.

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Appendix A.

In this appendix we give the one-dimensional golden section search algorithm (GSSA) to find the minimum of a give function $f(x),\ x \in [A, B]$ with a given stopping criterion ε, which is summarized as follows.

Algorithm: Golden-Section Search Procedure

$R \leftarrow (\sqrt{5}-1)/2$
$C \leftarrow A + (1-R)(B-A)$
$D \leftarrow A + R(B-A)$
$F_C \leftarrow f(C)$
$F_D \leftarrow f(D)$
If $|B-A| < \varepsilon$
If $F_C < F_D$
$x_{\min} = C$
$f_{\min} = F_C$
Else
$x_{\min} = D$
$f_{\min} = F_D$
Endif
Stop
Endif
If $F_C < F_D$
$B \leftarrow D$
$D \leftarrow C$
$F_D \leftarrow F_C$
$C \leftarrow A + (1-R)(B-A)$
$F_C \leftarrow f(C)$
Else
If $F_C > F_D$
$A \leftarrow C$
$C \leftarrow D$
$F_C \leftarrow F_D$
$D \leftarrow A + R(B-A)$
$F_D \leftarrow f(D)$
Else
$A \leftarrow C$
$B \leftarrow D$
$C \leftarrow A + (1-R)(B-A)$
$F_C \leftarrow f(C)$
$D \leftarrow A + R(B-A)$
$F_D \leftarrow f(D)$
Endif
Endif

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