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To the memory of Zvonimir Janko

ABSTRACT

Locally projective graphs in Mathieu–Conway–Monster series appear in thin–thick pairs. A possible thick extension of a thin locally projective graph associated with the fourth Janko group was questioned for a while. Such an extension could lead if not to a new sporadic simple group, to something equally exciting. This paper resolves this issue ultimately in the non-existence form confirming that the list of 26 sporadic simple groups although mysterious, is now stable. The result in fact concludes the classification project of locally projective graphs, which was running for some twenty years.

1 Locally projective graphs

The paper is devoted to the study and the classification of locally projective graphs defined in the following way.

Definition 1. *Let Φ be a connected (locally finite) graph and let F be a vertex- and edge-transitive automorphism group of Φ. Then Φ is locally projective in dimension n with respect to F if*

(a) *there is a collection of complete subgraphs in Φ, called* lines, *such that every edge is in a unique line;*

(b) *every line contains α vertices, where α is 2 (thin graph) or 3 (thick graph), and the stabiliser of a line induces on its vertices the symmetric group of degree α;*

(c) *if x is a vertex of Φ and $F(x)$ is the stabiliser of x in F, then $F(x)$ induces on the set of lines containing x the natural action of $L_n(2)$ of degree $2^n - 1$ on a projective space π_x, in particular the valency of Φ is $(\alpha - 1)(2^n - 1)$.*

Classical examples of locally projective graphs come from symplectic and orthogonal dual polar spaces over $GF(2)$ along the following construction.

Let V_{2n} be a $2n$-dimensional $GF(2)$-space, let f be a non-singular symplectic form, and let q be a quadratic form of maximal Witt index n, whose associated bilinear form is f:

$$f(u,v) = q(u) + q(v) + q(u+v) \quad \text{for all } u, v \in V_{2n}.$$

Let $Sp_{2n}(2)$ and $O^+_{2n}(2)$ be the corresponding symplectic and orthogonal groups, which are the automorphism groups of (V_{2n}, f) and (V_{2n}, f, q), respectively.

Let V_n be a maximal totally isotropic subspace in V_{2n} with respect to q (that is $q(u) = 0$ for all $u \in V_n$). Then V_n is also maximal totally singular with respect to f (that is $f(u,v) = 0$ for all $u, v \in V_n$). Notice that some of the totally singular subspaces are not totally isotropic. The geometries whose elements are

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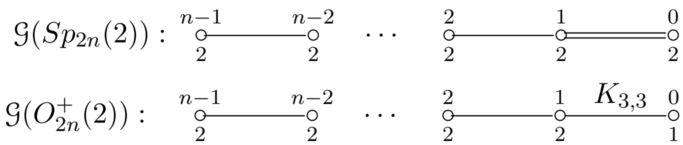
the images under $Sp_{2n}(2)$ and $O^+_{2n}(2)$ of the non-zero subspaces from V_n are the dual polar spaces with the following diagrams:



where the $(n-i)$-dimensional subspaces have type i and the incidence relation is the symmetrised inclusion. The elements of types 0 and 1 are also called points and lines, respectively. Let $\Gamma^{(n)}$ and $\Delta^{(n)}$ be the point–line graphs of $\mathcal{G}(Sp_{2n}(2))$ and $\mathcal{G}(O^+_{2n}(2))$, respectively. These are thick and thin locally projective graphs in dimension n with respect to $Sp_{2n}(2)$ and $O^+_{2n}(2)$, respectively. The orthogonal dual polar graph is *densely embedded* in the symplectic dual polar graph according to the following definition, where $G(x)$ denotes the stabiliser of a vertex x in G, $G_1(x)$ is the joint stabiliser of the vertices adjacent to x, and $G_{\frac{1}{2}}(x)$ is the stabiliser in $G(x)$ of all the lines containing x (in the thin case $G_{\frac{1}{2}}(x) = G_1(x)$, while in the thick case $G_{\frac{1}{2}}(x)/G_1(x)$ is a 2-group).

Definition 2. *Suppose that G acts locally projectively on a thick graph Γ in dimension $n \geq 2$, and let Δ be a connected subgraph in Γ. Then Δ is said to be* densely embedded *in Γ if the following conditions hold:*

(i) *Δ is thin and the subgroup H of G which stabilises Δ as a whole induces on it a locally projective action in dimension n, possibly with a non-trivial kernel;*

(ii) *if $x \in \Delta$, then $H(x)$ contains $G_1(x)$, and $H(x)/G_1(x)$ is an $L_n(2)$-complement to $G_{\frac{1}{2}}(x)/G_1(x)$ in $G(x)/G_1(x)$.*

An important role in the study of locally projective graphs is played by *geometric subgraphs* defined as follows.

Definition 3. *A connected subgraph Ξ in a locally projective graph Φ in dimension n is called* geometric *at level k, where $1 \leq k \leq n-1$ whenever together with an edge it contains the line on this edge, and the following conditions hold:*

(i) *if a vertex x is in Ξ, then the set of neighbours $\Xi(x)$ of x in Ξ is a k-dimensional subspace in the projective space π_x associated with x and the stabiliser of $\Xi(x)$ in $G(x)$ stabilises Ξ;*

(ii) *the subgraph Ξ is locally projective in dimension k with respect to the action on it of the setwise stabiliser of Ξ in G.*

In the symplectic and orthogonal graphs the geometric subgraphs at level k are those induced by the vertices and edges incident to elements of type k in the corresponding dual polar space geometry. In general, the existence of geometric subgraphs at all levels can only be guaranteed in the simply connected case (that is, when the vertex-line incidence graph is a tree) and we will see the non-existing examples. The geometric subgraphs at level 2 are called *planes* and a complete set of planes can be found in every locally projective graph of dimension at least 3 (cf. Chapter 10 in [5]). Let X be the action on a plane Ξ induced by the setwise stabiliser of Ξ. If the graph is thick, and Ξ contains a vertex x and a line l on x, then the amalgam

$$\mathcal{A} = \{X(x), X(l)\}$$

has index $(3,3)$ in the sense that $[X(x) : X(x,l)] = [X(l) : X(x,l)] = 3$. Such amalgams were classified by D. Goldschmidt in 1980 [3]. Up to isomorphism there are 15 Goldschmidt amalgams.

In the orthogonal dual polar graph the action X on a plane is the orthogonal group $O^+_4(2) \cong S_3 \wr S_2$, while in the symplectic graph $X \cong Sp_4(2) \cong S_6$ is a completion of the Goldschmidt amalgam

$$G^1_3 \cong \{S_4 \times 2, S_4 \times 2\}.$$

2 Mathieu groups and their graphs

Most of the exceptional locally projective graphs owe their existence to the exceptional cases in the following well known [13] proposition.

Proposition 4. *Let $M \cong \bigwedge^m V_n(2) : L_n(2)$ be the semidirect product with respect to the natural action of the mth-exterior power of the natural module $V_n(2)$ of $L_n(2)$, where $n \geq 2$ and $1 \leq m \leq n-1$. Then all automorphisms of M are inner except for the following cases, where the outer automorphism group of M is of order 2:*

(i) *$n = 3$ and $m = 1$ or 2;*

(ii) *$n = 4$ and $m = 2$.*

□

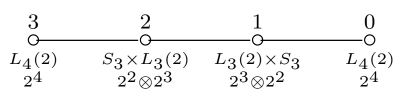
Notice that $\bigwedge^2 V_3(2)$ is the dual of $V_3(2)$. An explicit form of the outer automorphisms can be constructed as follows. There is a famous isomorphism between $L_4(2)$ and the alternating group A_8 of degree 8. This isomorphism sends $\bigwedge^2 V_4(2)$ onto the heart of the $GF(2)$-permutation module on 8 points. If V_7 is the quotient of the permutation module over the 1-dimensional submodule of constant functions, then V_7 is an indecomposable extension of $\bigwedge^2 V_4(2)$ and

$$A := V_7 : A_8 \cong \mathrm{Aut}\,(\bigwedge^2 V_4(2) : L_4(2)).$$

Further on, if $L^{(3)} \cong L_3(2)$ denotes the Levi subgroup in $L_4(2)$ (the stabiliser of a decomposition of $V_4(2)$ into the sum of 1- and 3-dimensional subspaces), then $\bigwedge^2 V_4(2)$, as a $L^{(3)}$-module, is isomorphic to the direct sum of the natural $V_3(2)$ and the dual natural $V_3(2)^*$ modules. The normalisers in A of $V_3(2) : L^{(3)}$ and of $V_3(2)^* : L^{(3)}$ are the full automorphism groups of the respective semidirect products. An automorphism will be called *special* if it acts trivially on the largest normal 2-subgroup and on the quotient over this subgroup.

To approach the Mathieu groups, we start with the locally projective action of $H \cong L_5(2)$ on the Grassmannian with the following diagram, where under the nodes we indicate the structure of the maximal parabolic subgroups.

$\mathcal{G}(L_5(2))$:



The locally projective graph is complete on 31 vertices and the structure of lines, planes etc. can only be seen through the group action.

The locally projective amalgam is

$$\mathcal{B} = \{H(x), H(l)\} \cong \{2^4 : L_4(2), (2^2 \otimes 2^3) : (S_3 \times L_3(2))\}.$$

A plane is isomorphic to the Fano plane on seven points, its stabiliser induces $L_3(2)$ on the plane, realising the Goldschmidt amalgam

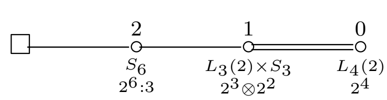
$$G_3 \cong \{S_4, S_4\}.$$

Because of Proposition 4 (ii), the intersection

$$H(x) \cap H(l) \cong (2^3 \times 2^3) : (L_3(2) \times 2)$$

possesses an outer automorphism which can be used through Goldschmidt's lemma to twist the amalgam $\mathcal{B}$ to obtain the Mathieu amalgam corresponding to a locally truncated geometry with the following diagram:

$\mathcal{H}(M_{24})$:



The details of this construction can be found in [5], where the twisted amalgam was taken as the starting point to recover the whole theory of the Mathieu groups. As indicated on the above diagram, geometric subgraphs at level 3 do not exist in the Mathieu geometry, while planes enjoy an action of S_6, realising the amalgam

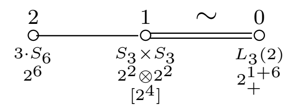
$$G_3^1 = \{S_4 \times 2, S_4 \times 2\}.$$

The non-existence of the geometric subgraphs at level 3 is due to the fact that the subamalgam

$$\mathcal{A} = \{H(x,\Pi), H(x,\Pi)\} \cong \{2_+^{1+6} : L_3(2), [2^8] : (S_3 \times S_3)\}$$

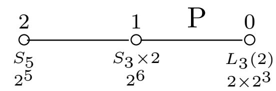
(where Π is a hyperplane in the projective space associated with x) generates the whole group M_{24}. This means that $\mathcal{A}$ is a (faithful) locally projective amalgam and the geometrisation of the corresponding locally projective graph has the following diagram:

$\mathcal{G}(M_{24})$:



Here planes are triple covers of the generalised quadrangle of order $(2,2)$ with the action of $3 \cdot S_6$ realising the same amalgam G_3^1. The graph contains a densely embedded subgraph stabilised by the smaller Mathieu group $M_{22}.2$ and corresponding to the following diagram:

$\mathcal{G}(M_{22})$:



The planes here are Petersen subgraphs with the natural action of S_5 (isomorphic to $O_4^-(2)$). In this paper the following result will prove crucial.

Proposition 5. *Let $\mathcal{X}$ be a locally projective amalgam corresponding to a thick action in dimension 3 and suppose that $\mathcal{X}$ contains a densely embedded subamalgam*

$$\mathcal{Y} = \{Y(x), Y(l)\} \cong \{2 \times 2^3 : L_3(2), 2^6.(S_3 \times 2)\}$$

corresponding to the action of $M_{22}.2$ on its thin locally projective graph in dimension 3. Then

(i) *$\mathcal{X}$ is isomorphic to the amalgam corresponding to the action of M_{24} on its thick locally projective graph in dimension 3 (this amalgam is also contained in the Held group He);*

(ii) *the involution in the direct factor of order 2 in $Y(x)$ is fused in $X(l)$ to an involution inside $O_2(2^3 : L_3(2))$, where $2^3 : L_3(2)$ is a direct factor of $Y(x)$.*

Proof. By Proposition 23 (i) in [7], we know that the chief $X(x)$-factors of $O_2(X(x))$ are (a) the trivial 1-dimensional, (b) the natural and (c) the dual natural modules. Then the main result of [1] applies and we obtain two possibilities for the isomorphism type of $\{X(x), X(l)\}$: the one realised in M_{24} and in the Held group, and the one realised in the alternating group A_{16} of degree 16. In [15] it was shown that in the latter amalgam the thin densely embedded subamalgam is completed in an index two subgroup of the wreath product $S_8 \wr 2$ (and not in $M_{22}.2$), which gives (i). To see (ii), let Ξ be the locally projective graph associated with $\mathcal{X}$ and let Θ be its densely embedded subgraph associated with $\mathcal{Y}$. To a vertex x of Ξ we assign the unique involution ι_x in $Z(X(x))$. Then the 14 involutions corresponding to $u \in \Xi_1(x)$ are contained in $X_1(x) \cong 2^3 \times 2$ and they are pairwise different, since the action of $O_2(X(x))$ on $X_1(x)$ is non-trivial. We have $Y_1(x) = X_1(x)$, but only 7 involutions are assigned to vertices in $\Theta_1(x)$. These involutions must be diagonal in the direct product of $L_3(2)$-modules, since ι_x projects in $M_{22} : 2$ outside the simple subgroup. Since

$$u \mapsto \iota_u$$

is bijective on the set $\{x\} \cup \Xi_1(x)$ of vertices, either y or z is contained in the 2^3-submodule (we assume that it is y). Then the element in $X(l)$ which induces the permutation $(x\,y)(z)$ conjugates ι_x onto ι_y confirming (ii). $\square$

3 Fourth Janko group

A path to the fourth Janko group J_4 lies through a twist of the locally projective amalgam of $O^+_{10}(2)$. The dual polar space of this group is described by the following diagram indicating the structure of parabolic subgroups:

$$\mathcal{G}(O^+_{10}(2)) : \quad \begin{array}{ccccccccc} 4 & & 3 & & 2 & & 1 & \mathrm{K}_{3,3} & 0 \\ \circ & \text{———} & \circ & \text{———} & \circ & \text{———} & \circ & \text{———} & \circ \\ O^+_8(2) & & S_3 \times S_8 & & L_3(2)\times(S_3 \wr S_2) & & L_4(2)\times 2 & & L_5(2) \\ 2^8 & & 2^{1+12}_+ & & 2^{3+12} & & 2^4 \times 2^4 & & 2^{10} \\ & & & & & & 2^6 & & \end{array}$$

Let $H = O^+_{10}(2) = \mathrm{Aut}\,(V^+_{10}(2), f, q)$. Let V_5 and U_5 be two disjoint maximal totally isotropic subspaces in $V^+_{10}(2)$ with bases $\{v_1, ..., v_5\}$ and $\{u_1, ..., u_5\}$, such that $f(v_i, u_j) = \delta_{ij}$. Then V_5 and U_5 are vertices in the corresponding locally projective graph at maximal distance 5 and their joint stabiliser $L^{(5)}$ in H is isomorphic to $L_5(2)$ and it acts on the subspaces as on the natural and the dual natural modules, respectively. The stabiliser H_0 of V_5 is the semidirect product of $L^{(5)}$ with the exterior square Q_{10} of V_5 generated by the Siegel transformations associated with 2-dimensional subspaces in V_5:

$$H_0 \cong 2^{10} : L_5(2)$$

as on the diagram. A 4-dimensional subspace in V_5 is an edge l containing V_5, and we choose it to be V_4 spanned by the leading four basis vectors in V_5 and denote its stabiliser by H_1. Then the second vertex on l is the subspace W_5 spanned by V_4 together with u_5. The structure of H_1 is as follows. The largest normal 2-subgroup Q_{14} in $H_0 \cap H_1$ has order 2^{14}, it is a semidirect product of Q_{10} and $Q^{(a)}_4 = O_2(L^{(5)}(V_4))$. The whole of $H_0 \cap H_1$ is the semidirect product of Q_{14} and a Levi $L_4(2)$-subgroup $L^{(4)}$ in $L^{(5)}$ which is the stabiliser of the direct sum decomposition

$$V_5 = V_4 \oplus \langle v_5 \rangle.$$

Finally, H_0 is obtained by adjoining to $H_0 \cap H_1$ the symplectic transvection τ, associated with the vector $v_5 + u_5$:

$$\tau : v \mapsto v + f(v, v_5 + u_5)\,(v_5 + u_5).$$

Notice that the vector $v_5 + u_5$ is non-isotropic, that is why τ belongs to the orthogonal group (but not to its simple index 2 subgroup). In order to describe the automorphism of $H_0 \cap H_1$ induced by τ, we need some more notation. Let Q_6 be the subgroup of order 2^6 in Q_{10} generated by the Siegel transformations associated with 2-subspaces in V_4, so that Q_6 is the exterior square of V_4. Further, let $Q^{(b)}_4$ be the subgroup of order 2^4 in Q_{10} generated by the Siegel transformations associated with 2-subspaces contained in V_5 and containing v_5. Then $Q^{(b)}_4$ is the natural module for $L^{(4)}$ and

$$Q_{10} = Q_6 \oplus Q^{(b)}_4$$

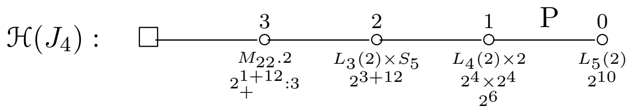
as $L^{(4)}$-modules. Finally, $Q^{(a)}_4$ is generated by the Siegel transformations associated with 2-subspace in W_5 containing u_5. The following assertion follows from the definitions.

Lemma 6. *In the above terms τ commutes with Q_6 and with $L^{(4)}$, and swaps $Q^{(a)}_4$ and $Q^{(b)}_4$, permuting the Siegel transformations associated with $\langle v, v_5 \rangle$ and with $\langle v, u_5 \rangle$ for all $v \in V^{\#}_4$.* □

Now we can apply a twist. Let σ be an involutory outer automorphism of $Q_6 : L^{(4)}$, as in Proposition 4 (ii), which we extend to an automorphism of $H_0 \cap H_1$ by requesting it centralises $Q^{(a)}_4$ and $Q^{(b)}_4$. The twisted amalgam is

$$\mathcal{A}^{(1)}_5 = \{H_0, (H_0 \cap H_1) : \langle \tau\sigma \rangle\}.$$

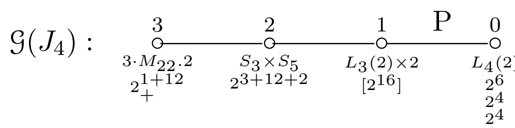
The fourth Janko group J_4 is a completion of $\mathcal{A}_5^{(1)}$, which can be characterised either as the unique completion in which Q_{10} is self-centralised [14], or as the image of the minimal (1333-dimensional) representation of the universal completion of the amalgam [11]. The corresponding geometry belongs to the following diagram:



The residue of an element of type 3 is the geometry of the Mathieu group M_{22} from the previous section, in particular the edge on the right symbolises the geometry of the Petersen graph. The geometric subgraphs at level 4 are missing, since the subamalgam (where τ and σ are assumed to be restricted to $H_0 \cap H_1 \cap H_4$)

$$\mathcal{A}_4^{(4)} := \{H_0 \cap H_4, (H_0 \cap H_1 \cap H_4) : \langle \tau\sigma \rangle\},$$

which is due to generate the stabiliser of such a subgraph, generates the whole of the Janko group J_4. Here H_4 is the stabiliser in $O_{10}^+(2)$ of a vector in V_4, say of v_1. Therefore, the constructed amalgam $\mathcal{A}_4^{(4)}$ is faithful (the members contain no nontrivial normal subgroup in their intersection) and thus corresponds to an action of J_4 on a locally projective amalgam in dimension 3. The diagram of the geometric subgraphs in that graph is the following:



The residue of an element of type 3 is the triple cover of the geometry of $M_{22}.2$ associated with the non-split extension by a normal subgroup of order 3. We follow notations for amalgams in Table 1 in [11].

The geometries $\mathcal{H}(M_{24})$ and $\mathcal{G}(M_{24})$ are subgeometries in $\mathcal{H}(J_4)$ and $\mathcal{G}(J_4)$ on elements with types 1, 2 and 3 constructed as follows. The edges of the Petersen graph are split into five antipodal triples. If we define a graph on the edges of a locally projective graph of J_4 where two edges are adjacent whenever they are antipodal in a Petersen subgraph (which is geometric at level 2), then a connected component of this graph is stabilised by a maximal 2-local subgroup in J_4 isomorphic to $2^{11} : M_{24}$, and leads to a subgeometry as described above.

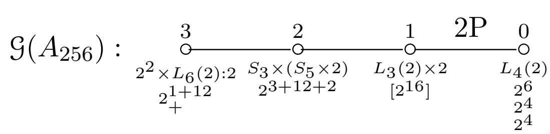
The structure of parabolics in $\mathcal{A}_4^{(4)}$ will be analysed closely, but one of the properties we state right here (cf. Section 9 in [11]).

Lemma 7. *The point-line stabiliser $H_0 \cap H_1 \cap H_4$ in $\mathcal{A}_4^{(4)}$ is a semidirect product of a group Q_{19} of order 2^{19} with centre Z of order 2^3 and a group $L^{(3)} \cong L_3(2)$ such that Z is the natural module for $L^{(3)}$.* □

The above lemma exhibits a possibility for a further twist. Indeed, by Proposition 4 and Lemma 7, $H_0 \cap H_1 \cap H_4$ possesses an involutory outer automorphism ρ which centralises Q_{19} and induces an outer automorphism of $Z : L^{(3)}$. The corresponding amalgam

$$\mathcal{A}_4^{(5)} = \{H_0 \cap H_4, (H_0 \cap H_1 \cap H_4) : \langle \tau\sigma\rho \rangle\}$$

was proved in [9] to embed in the alternating group A_{256} of degree 256. This embedding leads to the following diagram of maximal parabolics:



where the edge on the right symbolises a double cover of the Petersen graph.

It was shown in [8] that $\mathcal{A}_5^{(1)}$ does not appear as a densely embedded thin subamalgam in a thick locally projective amalgam in dimension 5. In the present paper we prove the non-existence result for the other two amalgams.

Theorem 1. *Neither $\mathcal{A}_4^{(4)}$ nor $\mathcal{A}_4^{(5)}$ appear as a densely embedded subamalgam in a thick locally projective amalgam in dimension 4.*

4 An explicit form of the two amalgams

We start by deducing the structure of the amalgams $\mathcal{A}_4^{(4)}$ and $\mathcal{A}_4^{(5)}$. The vertex stabilisers are isomorphic (we denote it by $G(x)$) and can be described as a parabolic subgroup in $H \cong O_{10}^+(2)$. In fact, $G(x) = H_0 \cap H_4$ which is the stabiliser in H of a maximal isotropic subspace, say $V_5 = \langle v_1, ..., v_5\rangle$ (giving H_0) and a vector in V_5, say v_1 (giving the intersection with H_4). We have seen in the previous section that

$$H_0 \cong Q_{10} : L^{(5)} \cong \bigwedge^2 V_5 : L_5(2).$$

Let $M^{(4)}$ be the Levi subgroup in $L^{(5)}$ stabilising the decomposition

$$V_5 = \langle v_1\rangle \oplus W_4,$$

where $W_4 = \langle v_2, ..., v_5\rangle$. Of course $M^{(4)}$ is a conjugate of $L^{(4)}$, which will reappear later on. As a module for $M^{(4)}$, the subgroup Q_{10} splits into a direct sum

$$Q_{10} = R_6 \oplus R_4^{(b)},$$

where R_6 is the exterior square of W_4 generated by the Siegel transformations associated with the 2-subspaces in W_4, and $R_4^{(b)}$ is the natural module of $M^{(4)}$ generated by the Siegel transformations of the 2-subspaces $\langle v_1, v\rangle$ taken for all $v \in W_4$. The subgroup $R_4^{(a)} := O_2(L^{(5)}(v_1))$ is generated by the Siegel transformations associated with the subspaces $\langle v_1, u_j\rangle$ for $2 \le j \le 5$, where as above $U_5 = \langle u_1, ..., u_5\rangle$ is an isotropic complement to V_5 in V_{10} with $f(v_i, u_j) = \delta_{ij}$. The above description leads to the following abstract characterisation.

Lemma 8. *The group $H_0 \cap H_4$ is a semidirect product of $O_2(H_0 \cap H_4)$ and $M^{(4)} \cong L_4(2)$. Furthermore,*

(i) *$O_2(H_0 \cap H_4) = R_4^{(b)} R_6 R_4^{(a)}$ where $R_4^{(b)}$, R_6 and $R_4^{(a)}$ are the natural, the exterior square of the natural and the dual natural modules for $M^{(4)}$;*

(ii) *$R_4^{(b)}$ is the centre of $O_2(H_0 \cap H_4)$ and*

$$[R_4^{(a)}, R_6] = R_4^{(b)}$$

with $[r(W_3), r(W_2)] = r(W_3 \cap W_2)$, where W_3, W_2 are 3- and 2-subspaces in V_4 corresponding to elements in the commutator, which is non-trivial only when $W_3 \cap W_2$ is 1-dimensional and hence determines a vector from $R_4^{(b)}$.

In order to refine further to obtain the vertex-edge stabiliser

$$G(x) \cap G(l) = H_0 \cap H_1 \cap H_4,$$

along with V_5 and v_1, we stabilise a 4-subspace $V_4 = \langle v_1, v_2, v_3, v_4\rangle$. Let $L^{(3)} \cong L_3(2)$ be the Levi subgroup stabilising the direct sum decomposition

$$V_5 = \langle v_1\rangle \oplus V_3 \oplus \langle v_5\rangle,$$

where $V_3 = V_4 \cap W_4 = \langle v_2, v_3, v_4\rangle$ is the natural module of $L^{(3)}$. We summarise the structure in the following lemma.

Lemma 9. *In the above terms the following assertions hold:*

(i) *there are the following decompositions of $L^{(3)}$-modules:*

$$R_4^{(b)} = R_3^{(b)} \oplus R_1^{(b)}, \; R_4^{(a)} = R_1^{(a)} \oplus R_3^{(a)}, \; R_6 = R_3^{(c)} \oplus R_3^{(d)},$$

where $R_3^{(b)}$ is the natural module generated by the Siegel transformation associated with the 2-subspaces $\langle v_1, v\rangle$ for $v \in V_3$, $R_1^{(b)}$ is the 1-dimensional trivial module generated by the Siegel transformation of $\langle v_1, v_5\rangle$, $R_1^{(a)}$ is the trivial module generated by the Siegel transformation of $\langle v_1, u_5\rangle$, $R_3^{(a)}$ is the dual natural module generated by the Siegel transformations $\langle v_1, u_j\rangle$ for $j = 2, 3, 4$, $R_3^{(c)}$ is the dual natural module generated by Siegel transformations associated with the 2-subspace in V_3, $R_3^{(d)}$ is the dual natural module generated by Siegel transformations of the 2-subspaces $\langle v, v_5\rangle$ for $v \in V_3$;

(ii) *$R_3^{(e)} := O_2(M^{(4)}(V_3))$ is the dual natural module generated by the Siegel transformations of the subspaces $\langle v_5, u_j\rangle$ for $j = 2, 3, 4$;*

(iii) *the actions of $R_3^{(e)}$ on $R_4^{(b)}$, $R_4^{(a)}$ and R_6 can be seen by restricting the actions of $M^{(4)}$, in particular $R_3^{(e)}$ centralises $R_3^{(b)}$, $R_1^{(a)}$ and $R_3^{(c)}$;*

(iv) *$Q_6 = R_3^{(b)} \oplus R_3^{(c)}$, $Q_4^{(b)} = R_1^{(b)} \oplus R_3^{(d)}$, $Q_4^{(a)} = R_3^{(e)} \oplus R_1^{(a)}$.* □

Now the automorphisms τ, σ and ρ can be described rather explicitly.

Lemma 10. *Each of the automorphisms τ, σ and ρ of $G(x) \cap G(l) = H_0 \cap H_1 \cap H_4$ commutes with the action of $L^{(3)} \cong L_3(2)$; furthermore,*

(i) *τ permutes $R_1^{(b)}$ with $R_1^{(a)}$ and $R_3^{(d)}$ with $R_3^{(e)}$ and centralises the other R's and $L^{(3)}$;*

(ii) *σ acts as follows*

  (a) *it induces special outer automorphisms of $R_3^{(b)}L^{(3)}$ and $R_3^{(c)}L^{(3)}$ as in Proposition 4 (i);*

  (b) *it sends $R_3^{(a)}$ onto an $L_3(2)$-invariant diagonal of $R_3^{(a)}$ and $R_3^{(b)}$;*

(iii) *ρ induces a special outer automorphism of $R_3^{(b)} : L^{(3)}$ and centralises all the R's.*

Proof. The assertion (i) is by Lemma 6, since τ is the restriction of the symplectic transvection with respect to $v_5 + u_5$. In order to see (ii), we need to determine the action of a special outer automorphism of $Q_6 : L^{(4)}$ on the intersection of the latter group with $G(x) \cap G(l)$. This intersection J contains the whole of $Q_6 = R_3^{(b)} \oplus R_3^{(c)}$ and a maximal parabolic $R_3^{(a)}L^{(3)}$ from $L^{(4)}$. It can be seen that J is a tri-extraspecial group of plus type [12]. Now (ii) can be deduced either using the description of the automorphisms of tri-extraspecial groups and/or using the description of the special outer automorpohisms of $Q_6 : L^4$ in the paragraph after Lemma 4. Notice that the diagonal in (ii) (b) does not split over $R_3^{(b)}$ as an $L^{(3)}$-module, but splits as a module for the image of $L^{(3)}$ under a special outer automorphism of $R_3^{(c)}L^{(3)}$.
Finally, (iii) is by Lemma 7, since $R_3^{(b)}$ is the centre of $O_2(G(x) \cap G(l))$. Notice that instead of $L^{(3)}$ we can take any other $L_3(2)$-complement and that in a sense ρ partially compensates the action of σ on the classes of such complements. □

Lemma 11. *The amalgams*

$$\mathcal{A}_4^{(4)} = \{H_0 \cap H_4, (H_0 \cap H_1 \cap H_4) : \langle \tau\sigma\rangle\} \quad \text{and} \quad \mathcal{A}_4^{(5)} = \{H_0 \cap H_4, (H_0 \cap H_1 \cap H_4) : \langle \tau\sigma\rho\rangle\}$$

are faithful.

Proof. The amalgam $\{H_0 \cap H_4, (H_0 \cap H_1 \cap H_4) : \langle\tau\rangle\}$ is contained in $H \cong O^+_{10}(2)$ and generates $Q_8 : O^+_8(2)$, where Q_8 is the radical of the amalgam, which is the largest normal subgroup in the intersection of the members of the amalgam. The subgroup Q_8 is elementary abelian of order 2^8, generated by the Siegel transformations associated with the subspaces $\langle v_1, v_j\rangle$ and $\langle v_1, u_j\rangle$ for $2 \le j \le 5$. Therefore, it is sufficient to show that Q_8 is not normalised by σ. In fact, by Lemma 10 (ii) (b), σ sends the element $S(\langle v_1, u_2\rangle)$ contained in Q_8 onto the product $S(\langle v_1, u_2\rangle) \cdot S(\langle v_3, v_4\rangle)$, which is not in Q_8 (of course $S(U_2)$ is the Siegel transformation associated with a 2-subspace U_2). □

It can be seen from the structure of the parabolic subgroups in $H = O^+_{10}(2)$ indicated on a diagram of $\mathcal{G}(O^+_{10}(2))$ in Section 3 that the radical of the subamalgam

$$\{H_0 \cap H_3 \cap H_4, (H_0 \cap H_1 \cap H_3 \cap H_4) : \langle\tau\rangle\}$$

is an extraspecial group Q_{13} of order 2^{13} of plus type: $Q_{13} \cong 2^{1+12}_+$. This subgroup is generated by the Siegel transformations commuting with $S(\langle v_1, v_2\rangle)$ (which itself generates the centre of Q_{13}). The subgroup Q_{13} is normalised by σ and by ρ, and the following holds.

Lemma 12. *The subgroup $Q_{13} \cong 2^{1+12}_+$ is the vertex-wise stabiliser of a geometric subgraph at level 3 associated with the locally projective action of $\mathcal{A}^{(i)}_4$ for $i = 4$ and 5. If $I^{(i)}$ denotes the image in* $\mathrm{Out}\,(Q_{13})$ *of the stabiliser of this geometric subgraph as a whole, then*

$$I^{(4)} \cong 3 \cdot M_{22} : 2, \quad I^{(5)} \cong L_6(2) : 2.$$

Proof. The stabiliser of a geometric subgraph at level 3 is the centraliser of an involution in the fourth Janko group J_4, which is a completion of $\mathcal{A}^{(4)}_4$ [4]. In the case of $\mathcal{A}^{(5)}_4$ the action was identified in the A_{256}-completion in [9]. Notice that Q_{13} is self-centralised in J_4, while in A_{256} its centraliser is elementary abelian of order 2^3. □

5 Possibilities for thick extensions

Towards the proof of Theorem 1 we assume that Φ is a thick locally projective graph in dimension 4 with respect to a group F, and that Φ contains a densely embedded subgraph Γ with respect to G, where G is a completion of amalgam $\mathcal{A}^{(4)}_4$ or $\mathcal{A}^{(5)}_4$. Then G is a quotient of the stabiliser of Γ in F over its vertex-wise stabiliser. In order to exclude the unwanted cycles, we assume that Φ is simply connected, that is the vertex-line incidence graph is a tree. In this case Γ is just a tree and G is the universal completion of the corresponding amalgam.

Let x be a vertex of Φ and let $l = \{x, y, z\}$ be a line containing x, with $l \cap \Gamma = \{x, y\}$. We start with an analysis of the stabiliser $G(x)$ in order the recover the possible structure of $F(x)$. The following lemma follows directly from Lemmas 9 and 10 (compare Section 9 in [10]).

Lemma 13. *Let $G_i(x)$ denote the joint stabiliser in G of the vertices at distance at most i from x in Γ. Then*

$$G_4(x) = 1,\ G_3(x) = R^{(b)}_4,\ G_2(x) = R^{(b)}_4 R^{(a)}_4,\ G_1(x) = R^{(b)}_4 R^{(a)}_4 R_6,\ G(x) = R^{(b)}_4 R^{(a)}_4 R_6 L^{(4)}.$$

□

Let $F_i(x)$ be the joint stabiliser in F of the vertices at distance at most i from x in Φ. Let $F_{\frac{1}{2}}(x)$ be the largest subgroup in $F(x)$ which stabilises as a whole every line containing x. We follow Section 3 in [7] for methods and results in reconstructing thick stabiliser. The next result is Lemma 13 in [7].

Lemma 14. *The following assertions hold:*

(i) $F_{\frac{1}{2}}(x) = O_2(F(x))$ *and* $F(x)/F_{\frac{1}{2}}(x) \cong L_4(2)$*;*

(ii) *the quotient* $F_{\frac{1}{2}}(x)/F_1(x)$ *is elementary abelian of order* 2^4 *isomorphic to the natural module for* $F(x)/F_{\frac{1}{2}}(x)$*.* □

Further analysis heavily relies on the structure of the geometric subgraphs in Γ and Φ as described in the next lemma.

Lemma 15. *Let* Ξ *be a geometric subgraph at level* $2 \leq m \leq 3$ *in* Γ *or* Φ *containing the flag* (x,l)*. Let* X *be an action of the stabiliser of* Ξ *in the relevant group on the subgraph, and let* $\mathcal{A} = \{X(x), X(l)\}$ *be the corresponding locally projective amalgam. Then*

(i) *if* $m = 2$*, then* $\mathcal{A}$ *is the Djoković–Miller amalgam* $\{S_3 \times 2, D_8\}$ *contained in* S_5 *in the thin case and the Goldschmidt amalgam* $G_3^1 = \{S_4 \times 2, S_4 \times 2\}$ *contained in* S_6 *in the thick case;*

(ii) *if* $m = 2$*, then* $\mathcal{A} \cong \mathcal{A}_3^{(5)}$ *contained in* $M_{22}.2$ *in the thin case and in the thick case*

$$\mathcal{A} \cong \{2_+^{1+6} : L_3(2), [2^8] : (S_3 \times S_3)\}$$

contained in M_{24}*;*

(iii) $X_{m-1}(x)$ *induces on* Ξ *an action of order* 2*.*

Proof. The level 3 case follows from the structure of the J_4-parabolic subgroups for the $\mathcal{A}_4^{(4)}$-amalgam and then also for the $\mathcal{A}_5^{(5)}$-amalgam, since the automorphism ρ does not affect the structure of the residual amalgam $\mathcal{A}$ (ρ adjusts an $L_3(2)$-complement by the centre of O_2, so that the action is unchanged). Then the thick case follows from Lemma 5 (i). The level 2 case now follows from the structure of the residues in the $M_{22}.2$- and M_{24}-geometries. Finally, (iii) is a well-known property of the relevant residual geometries. □

The assertion (iii) in the above lemma is equivalent to the validity of the crucial condition (*) (compare the paragraph prior Proposition 17 in [7] and Section 9.3 in [10]).

The next result is Lemmas 18, 19 and 20 in [7], which relies on the validity of the (*) condition we have just established.

Lemma 16. *The isomorphism*

$$F_i(x)/F_{i+1}(x) \cong G_i(x)/G_{i+1}(x)$$

holds for $1 \leq i \leq 2$*, and*

$$F_4(x) = 1.$$

□

Now it only remains to draw the connection between $F_3(x)$ and $G_3(x)$, where the latter is the dual $L_4(2)$-module by Lemma 13. The structure of $F_3(x)$ comes from Proposition 22 (iii) in [7].

Lemma 17. *One of the following holds:*

(i) $F_3(x) \cong G_3(x)$*;*

(ii) $F_3(x)$ *is elementary abelian of order* 2^5*,* $F_3(x)$ *is in the centre of* $F_1(x)$ *and* $F(x)/F_1(x) \cong 2^4 : L_4(2)$ *acts faithfully on* $F_3(x)$ *inducing the stabiliser of a hyperplane in* $GL(F_3(x))$*.* □

The following proposition is a summary of this section.

Proposition 18. *The vertex stabiliser $F(x)$ in a locally projective amalgam containing $\mathcal{A}_4^{(i)}$ as a densely embedded subamalgam for $i = 4$ or 5 has the following structure:*

(i) *$O_2(F(x))$ has order 2^{18} or 2^{19};*

(ii) *$F(x)$ possesses the normal series*

$$F(x) > F_{\frac{1}{2}}(x) > F_1(x) > F_2(x) > F_3(x) \geq [F(x), F_3(x)] > 1,$$

whose factors are $L_4(2)$, the natural, the exterior square of the natural, the natural, trivial 1- or 0-dimensional and the natural module of $L_4(2)$. □

Having Proposition 18 in hand, one can proceed to construct $\{F(x), F(l)\}$ by accomplishing the following steps:

(A) Recover the isomorphism type of $F(x)$ from the structure of chief factors in Proposition 18 and from the knowledge of the isomorphism type of its section $G(x)$;

(B) lift the automorphisms $\tau\sigma$ and/or $\tau\sigma\rho$ to an automorphism α of $F(x) \cap F(l)$ inducing on l the permutation $(x, y)(z)$;

(C) reconstruct a preimage β in $F(x)$ of an element from $F_{\frac{1}{2}}(x)$ which induces on l the permutation $(x)(yz)$ and commutes with the action of an $L_3(2)$-complement in $F(x) \cap F(l)$;

(D) check that $\langle \alpha, \beta \rangle$ maps onto an S_3-subgroup in $\mathrm{Out}\,(F(x, y, z))$.

This plan was partially realised leading to failures on step (D). Then we were returning back realising that some fancy possibilities for $F(x)$ are missed, like non-splitness, indecomposabilities and alike. Then another failure. Eventually it has been realised that the obstacle is in the impossibility to realise the amalgam of the residual locally projective action at level 3 on the vertex-wise stabiliser of the corresponding geometric subgraph. This led to the non-existence proof accomplished in the next section.

6 Acting on the kernel at level 3

We continue to use hypotheses and notations from the previous section. Let Ξ be a geometric subgraph at level 3 in Φ containing the flag (x, l) and let Θ be the intersection of Ξ with Γ, so that Θ is a geometric subgraph at level 3 in Γ. Let X and Y be the actions on Ξ and Θ of their respective stabilisers in F and G, and let N and M be the kernels of the actions. The following lemma summarises what we know about Θ, Y, N and M from Lemmas 12, 15 and 18 .

Lemma 19. *The following holds:*

(i) *$M \cong 2_+^{1+12}$;*

(ii) *$(G(x) \cap G[\Theta])/M \cong 2 \times 2^3 : L_3(2)$, $(F(x) \cap F[\Xi])/N \cong 2_+^{1+6} : L_3(2)$;*

(iii) *the action of $F^{(a)}$ of $F(x) \cap F[\Xi]$ on Ξ possesses the following normal series:*

$$F^{(a)}(x) > F^{(a)}_{\frac{1}{2}}(x) > F^{(a)}_1(x) > F^{(a)}_2(x) > F^{(a)}_3(x) = 1$$

with factors isomorphic to $L_3(2)$, the natural module $V_3(2)$, the dual natural $V_3(2)^*$ module, and the trivial 1-dimensional module for $F^{(a)}(x)/F_{\frac{1}{2}}(x) \cong L_3(2)$. □

Next we restrict the series in Proposition 18 (ii) to the intersection of $F(x)$ with the stabiliser $F[\Xi]$ of the geometric subgraph Ξ at level 3 and decide which submodules fall into the vertex-wise stabiliser N, making use of Lemma 19 (iii).

Lemma 20. *The kernel N has the following $(F(x) \cap F[\Xi])$-factors as modules for an $L_3(3) \cong (F(x) \cap F[\Xi])/O_2(F(x) \cap F[\Xi])$*

(i) *the whole of $F_3(x)$, which is the dual natural module extended by one or two trivial 1-dimensional;*

(ii) *a 3-dimensional submodule of $F_2(x)/F_3(x)$ isomorphic to the natural module;*

(iii) *a 3-dimensional submodule of $F_1(x)/F_2(x)$ isomorphic to the natural module;*

(iv) *a 1-dimensional submodule of $F_{\frac{1}{2}}(x)/F_1(x)$;*

(v) *a 3-dimensional submodule which is $O_2((F(x) \cap F[\Xi])/F_{\frac{1}{2}}(x))$.* □

Lemma 21. *The kernel N is isomorphic to the central product of $M \cong 2_+^{1+12}$ with a group of order 4 or with a groups D_8, depending on which of the possibilities is realised in Lemma 17.*

Proof. By Lemma 20, the order of N is 2^{14} or 2^{15} depending on the possibilities in Lemma 17. The subgroup M is a factor group of a subgroup of index 2 in N which misses the submodule in Lemma 20 (iv). The factor is over a subgroup of order 2 or 1. The action described in Lemma 17 (ii) gives the structure of N in case it has order 2^{15}, and the case of smaller N is also clear. □

Let a and b be elements in F stabilising x, Ξ and Θ whose actions $\bar{a}$ and $\bar{b}$ on Θ satisfy:

(1) $\bar{a}$ is the only non-trivial element in $Y_2(x)$;

(2) $\bar{b}$ is in the normal 2-subgroup of the direct factor of $Y(x) \cong 2^3 : L_3(2) \times 2$ different from $\langle \bar{a} \rangle$;

(3) $\bar{a}$ and $\bar{b}$ are conjugate in $X(l)$ as in Proposition 5 (ii).

In the next proposition we reach the final contradiction by showing that the elements a and b satisfying (1) and (2) above have centralisers in $\bar{N} := N/[N, N]$ of different orders. This is in fact not surprising, since $\bar{b}$ maps into the commutator subgroups of $I^{(4)}$ and $I^{(5)}$ in Lemma 12, while $\bar{a}$ does not. So, instead of a rather explicit calculation, below we could refer to a classification of involutions in the orthogonal groups. Notice that since the commutator subgroup of N is abelian, the orders of the centralisers do not depend on the choice of representatives.

Proposition 22. *The dimensions of $C_{\bar{N}}(a)$ and $C_{\bar{N}}(b)$ for elements satisfying (1) and (2) above are different.*

Proof. We count which part of the composition factors in Lemma 20 fall into the centralisers of a and b in N. For a we have everything from (i), (ii) and (iv) and nothing else, giving dimension of $C_{\bar{N}}(a)$ equal to 6 or 7 depending on the order of $F_3(x)$. On the other hand, for b we have everything from (i), (iii) and (iv), a 2-subspace from (ii), giving the total dimension of $C_{\bar{N}}(b)$ of dimension 8 or 9, completing the proof. □

The final contradiction, showing that elements satisfying (1) and (2) cannot possibly satisfy (3), completes the proof of Theorem 1.

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