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ON GROUPS WITH AVERAGE ELEMENT ORDERS EQUAL TO THE AVERAGE ORDER OF THE ALTERNATING GROUP OF DEGREE 5

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Abstract. Let G be a finite group. Denote by $\psi(G)$ the sum $\psi(G) = \sum_{x\in G}|x|$, where $|x|$ denotes the order of the element x, and by $o(G)$ the average element orders, i.e. the quotient $o(G) = \frac{\psi(G)}{|G|}$. We prove that $o(G) = o(A_5)$ if and only if $G \simeq A_5$, where A_5 is the alternating group of degree 5.

This paper is dedicated to the memory of Professor Zvonimir Janko.

1. Introduction

Let G be a finite group. Denote by $\psi(G)$ the sum

$$\psi(G) = \sum_{x\in G}|x|,$$

where $|x|$ denotes the order of the element x, and by $o(G)$ the quotient

$$o(G) = \frac{\psi(G)}{|G|}.$$

Thus $o(G)$ denotes the average element orders of G. Moreover, if $S \subseteq G$, then we define $\psi(S) = \sum_{x\in S}|x|$.

Recently many authors studied the function $\psi(G)$ and, more generally, properties of finite groups determined by their element orders (see for example [1]-[9], [11]-[18], [20]-[26], [30], [31], [33]-[37]). It is easy to see that $\psi(A_4) = 31 = \psi(D_{10})$, where A_4 is the alternating group of degree 4 and D_{10} is the dihedral group of order 10. Hence $\psi(G)$ usually does not identify the group

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G. However, it is possible to prove that if $\psi(G) = \psi(S_3)$, then $G \simeq S_3$, and that if $\psi(G) = \psi(A_5)$, then $G \simeq A_5$ (see [1], [4], [22] for more examples of groups G identified by the function $\psi(G)$). Another problem that has been recently studied by many authors is to find some bounds on $\psi(G)$ that imply that the group G belongs to some classes of groups, like the class of solvable, or nilpotent, or supersolvable groups (see for example [4], [5], [9], [14], [18], [34] and [35]).

In this paper we shall study similar problems for the function $o(G)$.

If C_n denotes the cyclic group of order n, and we consider the groups $G_1 = C_8 \times C_2$, and $G_2 = C_8 \rtimes C_2$, where $C_2 = \langle a \rangle$, $C_8 = \langle b \rangle$, $b^a = b^5$, then it is easy to prove that $\psi(G_1) = \psi(G_2) = 87$. Thus $o(G_1) = o(G_2)$ and of course G_1 and G_2 are not isomorphic. Hence usually the function $o(G)$ does not identify the group G. But again sometimes that happens, for example $o(G) = o(S_3)$ if and only if $G \simeq S_3$ (see Theorem A of [17]), and $o(G) = o(A_4)$ if and only if $G \simeq A_4$ (see [36]).

A. Jaikin-Zapirain started in his paper [27] the investigation of the function $o(G)$. He proved that if G is a finite group, then $o(G) \geq o(Z(G))$ (Lemma 2.7), and that $o(G) \leq k(G)$, the number of conjugacy classes in G (Lemma 2.9). He also posed the following question: let G be a finite (p-)group and N a normal (abelian) subgroup of G, is it true that $o(G) \geq o(N)^{\frac{1}{2}}$? Ten years later, in their paper [19], E.I. Khukhro, A. Moretó and M. Zarrin provided a negative answer to Jaikin-Zapirain's question, in fact they proved that if $c > 0$ is any real number and $p \geq \frac{3}{c}$ a prime, then there exists a finite p-group with a normal abelian subgroup N such that $o(G) < o(N)^c$.

In the same paper they posed the following conjecture.

Conjecture 1.1. *Let G be a finite group and suppose that*

$$o(G) < o(A_5).$$

Then G is solvable.

In the paper [17] we proved that the conjecture is true. In fact we proved the following theorem.

Theorem 1.2. *Let G be a finite group and suppose that*

$$o(G) \leq o(A_5).$$

Then either G is solvable or $G \simeq A_5$.

Notice that

$$o(A_5) = \frac{\psi(A_5)}{|A_5|} = \frac{211}{60} = 3.51666...$$

The structure of a solvable group with $o(G) \leq o(A_5)$ is still unknown.

In this paper we prove that there are no solvable groups with $o(G) = o(A_5)$. In fact we prove the following theorem.

THEOREM 1.3. *Let G be a finite group and suppose that*

$$o(G) = o(A_5) = \frac{211}{60}.$$

Then $G \simeq A_5$.

In particular the group A_5 is identified by its average order.

Notice that M. Tărnăuceanu in the paper [36] obtained a similar criterion for supersolvability, showing that if $o(G) < o(A_4)$, then G is supersolvable.

Our notation in this paper is the usual one (see for example [10] and [32]). If G is a finite group, then 1 will denote the identity element of G and sometimes also the group $\{1\}$. We shall denote by $i_2(G)$ the number of elements of G of order 2 and by $i_3(G)$ the number of elements of G of order 3. Sometimes we shall use the shorter notation i_2 and i_3, if there is no ambiguity. Moreover, if $S \subseteq G$, then we shall denote by $i_2(S)$ the number of elements of S of order 2.

In Section 2 we shall recall some useful results concerning the function $o(G)$.

In Section 3 we shall prove Theorem 1.3.

2. Some results about the function $o(G)$.

We start this section with some basic results concerning the function $o(G)$.

PROPOSITION 2.1. *Let G be a finite group and $G \neq 1$. Then the following statements hold.*
(1) *We have $o(G) \geq 2 - \frac{1}{|G|} \geq \frac{3}{2}$. In particular, if G is an elementary abelian 2-group, then $o(G) = 2 - \frac{1}{|G|}$ and if G is not an elementary abelian 2-group, then $o(G) \geq 2 + \frac{1}{|G|}$. Hence $o(G) \leq 2$ if and only if G is an elementary abelian 2-group and $o(G) = 2 - \frac{1}{|G|}$.*
(2) *If G is of odd order, then $o(G) \geq 3 - \frac{2}{|G|} \geq 3 - \frac{2}{3} = \frac{7}{3}$.*
(3) *If $G = A \times B$ with $(|A|, |B|) = 1$, then $o(G) = o(A)o(B)$. In particular, if $A \neq 1$ and $B \neq 1$, then*

$$o(G) \geq \frac{7}{2}.$$

PROOF. See [17], Lemma 1.1. □

For groups G of odd order and of exponent greater than 3, we have the following stronger result.

PROPOSITION 2.2. *Let G be a group of odd order and of exponent greater than 3. Then*

$$o(G) \geq 3.5 - \frac{2}{|G|} \geq 3.1.$$

Proof. If G is not a 3-group, then, by [28], $i_3(G) + 1 \leq \frac{3}{4}|G|$, thus there exist at least $\frac{1}{4}|G|$ elements of G of order > 3 and the rest of order ≥ 5. Then we have $\psi(G) \geq 1 + 3(|G| - 1) + 2 \cdot \frac{1}{4}|G| = -2 + 3.5|G|$, thus $o(G) \geq 3.5 - \frac{2}{|G|} \geq 3.5 - \frac{2}{5} = 3.5 - 0.4 = 3.1$.

If G is a 3-group of exponent greater than 3, then, by [29], $i_3(G) + 1 \leq \frac{7}{9}|G|$, thus there exist at least $\frac{2}{9}|G|$ elements of G of order > 3 and the rest (different from the identity) of order ≥ 9. Then we have $\psi(G) \geq 1 + 3(|G| - 1) + 6 \cdot \frac{2}{9}|G| \geq -2 + 4.3|G|$, thus $o(G) \geq 4.3 - \frac{2}{|G|} \geq 4.3 - \frac{2}{9} \geq 4.3 - 0.2 = 4.1$. □

The function $o(G)$ has a very good behavior with respect to factor groups.

Proposition 2.3. *Let G be a finite group containing a non-trivial normal subgroup H. Then the following statements hold.*

(1) *If $x \in G \setminus H$, then the order $|xH|$ of xH in G/H divides the order of xh in G for every $h \in H$. In particular, $|xh| \geq |xH|$ for every $h \in H$.*

(2) *$o(G/H) < o(G)$.*

Proof. See [17], Lemma 3.1. □

Now we shall prove two very useful lemmas, which we shall use in our proof of Theorem 3.1.

Lemma 2.4. *Let $G = N \rtimes \langle x \rangle$, with $|x| = 2$, N of odd order and non-abelian. Then the following holds*

$$\psi(Nx) \geq 2|N| + \frac{8}{3}|N| = 4|N| + \frac{2}{3}|N|.$$

Proof. Write $I = \{n \in N \mid n^x = n^{-1}\}$. Then $i_2(Nx) = |I|$. Moreover $I \subset N$, since N is not abelian. Also $|I| = |N|/|C_N(x)|$ (see [10], Lemma 10.4.1), thus $|I|$ divides $|N|$, hence $|I| \leq \frac{|N|}{3}$, since $|N|$ is odd. Then the number of elements of Nx of order 2 is less or equal to $\frac{|N|}{3}$, hence there exist at least $\frac{2|N|}{3}$ elements of Nx of order bigger that 2 and then of order ≥ 6, by Proposition 2.3(1). Therefore we have

$$\psi(Nx) \geq 2|N| + \frac{2|N|}{3}4 = 2|N| + \frac{8}{3}|N| = 4|N| + \frac{2}{3}|N|,$$

as required. □

Lemma 2.5. *Let $G = N \rtimes \langle x \rangle$, with $|x| = 2$, N of odd order. Then the following hold:*

(1) *if 3 divides $|C_N(x)|$, then $\psi(Nx) \geq 4.66|N|$,*

(2) *if 5 divides $|C_N(x)|$, then $\psi(Nx) \geq 5.2|N|$.*

Proof. Write $I = \{n \in N \mid n^x = n^{-1}\}$. Then $i_2(Nx) = |I|$. Moreover $|I| = |N|/|C_N(x)|$, by Lemma 10.4.1 of [10].

If 3 divides $|C_N(x)|$, then $|I| \leq \frac{|N|}{3}$. Then the number of elements of Nx of order 2 is less or equal to $\frac{|N|}{3}$, hence there exist at least $\frac{2|N|}{3}$ elements of Nx of order bigger that 2 and then of order ≥ 6, by Proposition 2.3(1). Therefore we have $\psi(Nx) \geq 2|N| + \frac{2|N|}{3}4 = 2|N| + \frac{8}{3}|N| = 4|N| + \frac{2}{3}|N| \geq 4.66|N|$. That proves (1).

If 5 divides $|C_N(x)|$, then $|I| \leq \frac{|N|}{5}$. Then the number of elements of Nx of order 2 is less or equal to $\frac{|N|}{5}$, hence there exist at least $\frac{4|N|}{5}$ elements of Nx of order bigger that 2 and then of order ≥ 6, by Proposition 2.3(1). Therefore we have $\psi(Nx) \geq 2|N| + \frac{4|N|}{5}4 = 2|N| + \frac{16}{5}|N| = 5.2|N|$. Therefore (2) holds. □

3. The proof of Theorem 1.3

In this section we shall study the structure of a finite group G such that $o(G) = o(A_5)$. We start with an easy but interesting remark on the order of G.

Lemma 3.1. *Let G be a finite group with $o(G) = o(A_5)$. Then*

$$|G| = 60k,$$

where k is an odd number.

Proof. We have $\frac{\psi(G)}{|G|} = \frac{211}{60} = o(A_5)$. Moreover $\psi(G)$ is odd. Thus $211|G| = 60\psi(G)$, 60 divides $|G|$ and $|G| = 60k$, with k odd. □

By Lemma 3.1, if G is a finite group such that $o(G) = o(A_5)$, then a Sylow 2-subgroup D of G has order 4. First we show that D is not cyclic.

Proposition 3.2. *Let G be a finite group such that $o(G) = o(A_5)$. Then a Sylow 2-subgroup D of G is not cyclic.*

Proof. Suppose $o(G) = o(A_5)$ and that G has a cyclic 2-subgroup. Then G is 2-nilpotent (see, for example, 10.1.9 of [32]). Therefore $G = N \rtimes \langle y \rangle$, with $|y| = 4$ and $|N|$ odd. We have $\psi(G) = \psi(N) + \psi(Ny) + \psi(Ny^2) + \psi(Ny^3)$, and, by Proposition 2.3(1), $\psi(G) \geq \psi(N) + 4|N| + 2|N| + 4|N|$. Then $\psi(G) \geq \psi(N) + 2|G| + \frac{|G|}{2} = \psi(N) + 2.5|G|$. Then $o(G) \geq \frac{o(N)}{4} + 2.5$, and $o(N) \leq (3.52 - 2.5) \times 4 = 1.02 \times 4 = 4.08$.

If N is abelian, there exists a cyclic quotient N/V of N of order 15. Then we have $o(N/V) = \frac{21}{5}\frac{7}{3} = \frac{49}{5} = 9.8$, a contradiction, since, by Proposition 2.3(2), $o(N/V) \leq o(N) \leq 4.08$.

Then N is not abelian, therefore, by Lemma 2.4, $\psi(Ny^2) \geq 4|N| + \frac{2}{3}|N|$. Therefore we have $\psi(G) = \psi(N) + \psi(Ny) + \psi(Ny^3) + \psi(Ny^2) \geq \psi(N) + 4|N| +$

$4|N| + 4|N| + \frac{2}{3}|N| = \psi(N) + 3|G| + \frac{|G|}{6}$. Thus $o(N) \leq (o(G) - 3.166) \times 4 \leq (0.36) \times 4 = 1.44$, a contradiction with Proposition 2.2. ∎

Now we shall prove that a finite group with $o(G) = o(A_5)$ is not 2-nilpotent.

PROPOSITION 3.3. *Let $G = N \rtimes V$ be a finite group, with $|N|$ odd and $|V| = 4$. Then*

$$o(G) \neq o(A_5).$$

PROOF. Suppose $o(G) = o(A_5)$. Then V is not cyclic, by Proposition 3.2. Then V is a Klein group. Hence $G = N \cup Nx_1 \cup Nx_2 \cup Nx_3$, with $|x_1| = |x_2| = |x_3| = 2$. Thus, by Proposition 2.3(1), $\psi(G) \geq \psi(N) + 2|N| + 2|N| + 2|N| = \psi(N) + |G| + \frac{|G|}{2}$. Then $o(N) \leq (o(G) - 1.5) \times 4 \leq (2.02) \times 4 = 8.08$.

If N is abelian, then N has a cyclic quotient N/V of order 15, thus, arguing as in Proposition 3.2, $o(N/V) = 9.8$, a contradiction, since, by Proposition 2.3(2), $o(N/V) \leq o(N) \leq 8.08$.

Then N is not abelian. Hence, by Lemma 2.4, $\psi(Nx_i) \geq 4|N| + \frac{2}{3}|N|$, for every $i \in \{1, 2, 3\}$. Then $\psi(G) \geq \psi(N) + 4|N| + 4|N| + 4|N| + 2|N| = \psi(N) + 3|G| + \frac{|G|}{2} = \psi(N) + 3.5|G|$ and $o(N) \leq (o(G) - 3.5) \times 4 \leq 0.017 \times 4 = 0.068$, a contradiction with Proposition 2.2. ∎

We conclude this paper with the proof of Theorem 1.3.

PROOF. (of Theorem 1.3) Suppose that there exists a finite group G which satisfies $o(G) = o(A_5)$ and it is not isomorphic to A_5. Then G is a solvable group, by Theorem B of [17].
Moreover, $|G| = 60k$, with k odd, by Lemma 3.1.

We shall reach a contradiction, which will indicate that if a finite group G satisfies $o(G) = o(A_5)$, then $G \simeq A_5$, as required.

By Hall's theorem, there exists a subgroup H of G of index 4. Write $M = H_G$, the core of H in G. Then M is normal in G and G/M is a subgroup of S_4. Also $|M|$ is odd, 4 divides $|G/M|$ and 8 does not divide $|G/M|$. Moreover $|G/M|$ is not 4, by Proposition 3.3. Therefore $|G/M| = 12$ and $G/M \simeq A_4$. Then there exists a normal subgroup N/M of G/M, with $|G/N| = 3$, and $N = M \rtimes V$ where V is a Klein group. Write $G = N\langle y\rangle$. If $|yn| > 3$, for every $n \in N$, then $|yn| \geq 6$, for every $n \in N$, by Proposition 2.3(1). Then we have $\psi(Ny) \geq 6|N|$ and $\psi(Ny^2) \geq 6|N|$. Hence $\psi(G) \geq \psi(N) + 6|N| + 6|N| = \psi(N) + 4|G|$, and $o(G) \geq o(N)/3 + 4$, a contradiction since $o(G) = o(A_5) \leq 3.52$.

Therefore we can suppose that $|y| = 3$.

Then $G = N \rtimes \langle y\rangle$.

Now we prove that $o(N) \leq 4.56$.

In fact, we have $\psi(G) \geq \psi(N) + 3|N| + 3|N| = \psi(N) + 2|G|$, and $o(G) \geq o(N)/3 + 2$. Hence $o(N) \leq (3.52 - 2) \times 3 = 1.52 \times 3 = 4.56$, as required.

Recall that $N = M \rtimes V$, where V is a Klein group and $|M|$ is odd.

Then 5 divides the order of M, since 5 divides the order of G.

We claim that there exists a non-trivial element $a \in V$ such that 5 divides $|C_M(a)|$.

Suppose not. Write $V = \{1, x_1, x_2, x_3\}$. By Theorem 6.2.2 of [10] there exists a non-trivial V-invariant Sylow 5-subgroup P of M. Then $C_P(x_i) = \{1\}$, otherwise there exists an element of order 5 in $C_M(x_i)$, and 5 divides $C_M(x_i)$. Write $J_i = \{x \in P \mid x^{x_i} = x^{-1}\}$. Then $P = J_1 = J_2$, since $|J_i| = |P|/|C_P(x_i)|$, by Lemma 10.4.1 of [10]. But then x_1 inverts all elements of P and x_2 inverts all elements of P and then $x_3 = x_1x_2$ centralizes all elements of P, i.e. $C_P(x_3) = P$, a contradiction since 5 does not divides $|C_M(x_3)|$.

Let a be a non-trivial element of V such that 5 divides $|C_M(a)|$. Then 5 divides also $|(C_M(a))^y| = |C_M(a^y)|$, and $|(C_M(a))^{y^2}| = |C_M(a^{y^2})|$, since M is normal in G.

Also $N/M = \{M, aM, a^yM, a^{y^2}M\}$, since G/M is isomorphic to A_4.

Then we have $\psi(N) = \psi(M) + \psi(aM) + \psi(a^yM) + \psi(a^{y^2}M)$. By Lemma 2.5(2), $\psi(aM) \geq 5.2|M|$, $\psi(a^yM) \geq 5.2|M|$ and $\psi(a^{y^2})M \geq 5.2|M|$. Hence $\psi(N) \geq \psi(M) + 15.6|M| = \psi(M) + 3.9|N|$, hence $o(N) \geq o(M)/4 + 3.9$ and $o(M) \leq (4.56 - 3.9) \times 4 = 0.66 \times 4 = 2.64$, contradicting Proposition 2.2, since M is a group of odd order and 5 divides $|M|$.

The proof of Theorem 1.3 is now complete. $\square$

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