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# INEQUALITIES ASSOCIATED WITH THE BAXTER NUMBERS

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ABSTRACT. The Baxter numbers  $B_n$  enumerate a lot of algebraic and combinatorial objects such as the bases for subalgebras of the Malvenuto-Reutenauer Hopf algebra and the pairs of twin binary trees on  $n$  nodes. The Turán inequalities and higher order Turán inequalities are related to the Laguerre-Pólya ( $\mathcal{L}\text{-}\mathcal{P}$ ) class of real entire functions, and the  $\mathcal{L}\text{-}\mathcal{P}$  class has close relation with the Riemann hypothesis. The Turán type inequalities have received much attention. In this paper, we mainly concern on Turán type inequalities, more precisely, the log-behavior and the higher order Turán inequalities associated with the Baxter numbers. We prove the Turán inequalities (equivalently the log-concavity) of the sequences  $\{B_{n+1}/B_n\}_{n \geq 0}$  and  $\{\sqrt[n]{B_n}\}_{n \geq 1}$ . A monotonicity of the sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  is also obtained. Finally, we prove that the sequences  $\{B_n/n!\}_{n \geq 2}$  and  $\{B_{n+1}B_n^{-1}/n!\}_{n \geq 2}$  satisfy the higher order Turán inequalities.

## 1. INTRODUCTION

Baxter permutations were introduced by Glen Baxter [2] while studying fixed points of the composite of commuting functions. These permutations are enumerated by Baxter numbers  $B_n$ , whose formula was found by Chung, Graham, Hoggatt, and Kleiman [9] as

$$(1.1) \quad B_n = \sum_{k=1}^n \frac{2}{n(n+1)^2} \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}.$$

A combinatorial proof of (1.1) was later showed by Viennot [39]. For convenience, let  $B_0 = 1$ . The first few terms of  $\{B_n\}_{n \geq 0}$  are 1, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, see Sloane [35, A001181].

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A lot of algebraic and combinatorial objects have been found to be enumerated by the Baxter numbers since they were introduced, such as the bases for subalgebras of the Malvenuto-Reutenauer Hopf algebra [31], the pairs of twin binary trees on  $n$  nodes [22, 23], the diagonal rectangulations of an  $n \times n$  grid [26], and the bases of Baxter-Cambrian Hopf algebra [5]. Besides, many bijections related to Baxter permutations and various objects counted by Baxter numbers were given, including alternating Baxter permutations [10], non-intersecting paths [21], simple walks in Weyl chambers [11], and Baxter tree-like tableaux [1].

In this paper, we mainly study some important inequalities, more precisely, the log-behavior and *higher order Turán inequalities* associated with the Baxter numbers. Recall that a sequence  $\{a_n\}_{n \geq 0}$  with real numbers is said to satisfy the *Turán inequalities* or to be *log-concave* if

$$(1.2) \quad a_n^2 - a_{n-1}a_{n+1} \geq 0$$

for any  $n \geq 1$ . It is said to be *log-convex* if  $a_n^2 - a_{n-1}a_{n+1} \leq 0$  for any  $n \geq 1$ . A log-convex sequence  $\{a_n\}_{n \geq 0}$  is called *log-balanced* if  $\{a_n/n!\}_{i \geq 0}$  is log-concave, see Došlić [17]. Clearly, for a log-balanced sequence  $\{a_n\}_{n \geq 0}$ , there holds  $1 \leq a_{n-1}a_{n+1}/a_n^2 \leq 1 + 1/n$ .

The Turán inequalities (1.2) are also called the Newton's inequalities [13, 12, 29]. A real sequence  $\{a_n\}_{n \geq 0}$  is said to satisfy the *higher order Turán inequalities* or cubic Newton inequalities if for all  $n \geq 1$ ,

$$(1.3) \quad 4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1} a_{n+2})^2 \geq 0,$$

see [15, 33, 29]. The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions, see [38, 15]. A real entire function

$$\psi(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

is said to belong to the Laguerre-Pólya class, denoted by  $\psi \in \mathcal{L}\text{-}\mathcal{P}$ , if

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where  $c, \beta, x_k$  are real,  $\alpha \geq 0$ ,  $m$  is a nonnegative integer and  $\sum x_k^{-2} < \infty$ . If a real entire function  $\psi \in \mathcal{L}\text{-}\mathcal{P}$ , then its Maclaurin coefficients satisfy (1.2), see [12, 14]. Moreover, Dimitrov [15] proved that the higher order Turán inequalities (1.3), an extension of (1.2), is also a necessary condition for  $\psi \in \mathcal{L}\text{-}\mathcal{P}$ .

The  $\mathcal{L}\text{-}\mathcal{P}$  class has close relation with the Riemann hypothesis. Let  $\zeta$  and  $\Gamma$  denote the Riemann zeta-function and the gamma-function, respectively.

The Riemann  $\xi$ -function is defined by

$$\xi(iz) = \frac{1}{2} \left( z^2 - \frac{1}{4} \right) \pi^{-\frac{z}{2} - \frac{1}{4}} \Gamma \left( \frac{z}{2} + \frac{1}{4} \right) \zeta \left( z + \frac{1}{2} \right),$$

see Boas [4]. It is well known that the Riemann  $\xi$ -function is an entire function of order 1 and can be rewritten as

$$(1.4) \quad \frac{1}{8} \xi \left( \frac{x}{2} \right) = \sum_{k=0}^{\infty} (-1)^k \hat{b}_k \frac{x^{2k}}{(2k)!},$$

where

$$\hat{b}_k = \int_0^{\infty} t^{2t} \Phi(t) dt \quad \text{and} \quad \Phi(t) = \sum_{n=0}^{\infty} (2n^4 \pi^2 e^{9t} - 3n^2 \pi e^{5t}) \exp(-n^2 \pi e^{4t}),$$

see Pólya [30]. Set  $z = -x^2$  in (1.4). Then one obtain an entire function of order  $1/2$ , denoted by  $\xi_1(z)$ , that is,

$$\xi_1(z) = \sum_{k=0}^{\infty} \frac{k!}{(2k)!} \hat{b}_k \frac{z^k}{k!}.$$

So the Riemann hypothesis holds if and only if  $\xi_1(z) \in \mathcal{L}\text{-}\mathcal{P}$ . See [13, 15, 8] for more details.

For the deep relation stated above, the higher order Turán inequalities have received much attention. Many combinatorial sequences have been proved to satisfy (1.3). By using the Hardy-Ramanujan-Rademacher formula, Chen, Jia, and Wang [8] proved that the partition function  $p(n)$  satisfies the higher order Turán inequalities for  $n \geq 95$ , and hence confirm a conjecture of Chen [6]. Wang [40] gave a sufficient condition to (1.3) and obtained some higher order Turán inequalities for combinatorial sequences  $\{b_n/n!\}_{n \geq 0}$  where  $b_n$  are the Motzkin numbers, the Fine number, the Franel numbers of order 3 and the Domb numbers. Hou and Li [25] presented a different sufficient condition on the higher order Turán inequalities for  $n \geq N$  and also developed an algorithm to find the number  $N$ .

The objective of this paper is to study the log-behavior and the higher order Turán inequalities associated with the Baxter numbers. It should be mentioned that the log-convexity of the sequence  $\{B_n\}_{n \geq 0}$  was first established by Došlić and Veljan [20, Theorem 4.18] in 2002 (noting the received date) with *calculus method* developed by the two authors. See [19] for more applications of the calculus method. Moreover, Došlić [17, Section 5] first showed the log-balancedness of the sequence  $\{B_n\}_{n \geq 0}$  by using a robust analytical method developed by the author, which implies the log-concavity of the sequence  $\{B_n/n!\}_{n \geq 0}$  clearly.

As noted by Liu and Wang [27, P. 459], motivated by the results of [18, 20, 19, 17] and other literature, they developed an algebraic approach for

dealing with the log-convexity of sequences  $\{z_n\}_{n \geq 0}$  satisfying a three-term recurrence, such as  $\mathbf{a}_n z_{n+1} = \mathbf{b}_n z_n \pm \mathbf{c}_n z_{n-1}$ .

By applying the result given in [27, Theorem 3.1] and the following recurrence relation

$$(1.5) \quad (n+3)(n+4)B_{n+1} = (7n^2 + 21n + 12)B_n + 8n(n-1)B_{n-1}, \quad n \geq 1,$$

which is due to Richard L. Ollerton as was recorded at OEIS [35], one can also obtain the log-convexity of the sequence  $\{B_n\}_{n \geq 0}$ . Since  $B_n$  is a sum of hypergeometric terms on  $k$ , it is natural to prove (1.5) with Zeilberger's algorithm [42]. The detailed proof is omitted here.

In this paper, we prove the log-concavity (equivalently the Turán inequalities) of the sequences  $\{B_{n+1}/B_n\}_{n \geq 0}$  and  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  in Theorems 2.1 and 3.1, respectively. A monotonicity of the sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  is also obtained. At last, in Section 4, we prove that the sequences  $\{B_n/n!\}_{n \geq 2}$  and  $\{B_{n+1}B_n^{-1}/n!\}_{n \geq 2}$  satisfy the higher order Turán inequalities. Most of our proofs depend on certain bounds of  $B_{n+1}/B_n$ .

## 2. LOG-CONCAVITY OF $\{B_{n+1}/B_n\}_{n \geq 0}$

The aim of this section is to prove the log-concavity of the ratio sequence  $\{B_{n+1}/B_n\}_{n \geq 0}$ .

**THEOREM 2.1.** *The sequence  $\{B_{n+1}/B_n\}_{n \geq 0}$  is strictly log-concave. That is,*

$$(2.1) \quad \left(\frac{B_{n+1}}{B_n}\right)^2 > \frac{B_n}{B_{n-1}} \cdot \frac{B_{n+2}}{B_{n+1}}, \quad n \geq 1.$$

In order to prove Theorem 2.1, we need a set of bounds for  $B_{n+1}/B_n$ . Based on the asymptotic expansions of the  $P$ -recursive sequences given by Birkhoff and Trjitzinsky [3] and developed by Wimp and Zeilberger [41], Hou and Zhang [24] proved a criterion for determining whether a  $P$ -recursive sequence  $\{a_n\}_{n \geq N}$  is  $r$ -log-convex for sufficiently large  $N$ . Their algorithm also gives bounds of the ratio  $a_{n+1}/a_n$  for  $n \geq N$ .

The bounds we proved in Lemma 2.2 are initially found by applying the algorithm of Hou and Zhang [24]. Then we adjust some coefficients in the bounds such that the computation in the proofs of our main results are more concise.

**LEMMA 2.2.** *Let  $r_n = B_{n+1}/B_n$ . Then for  $n \geq 1$ , it holds that*

$$(2.2) \quad \hat{u}(n) < r_n < \hat{v}(n),$$

where

$$\begin{aligned}\hat{u}(n) &= 8 - \frac{32}{n} + \frac{416}{3n^2} - \frac{15904}{27n^3} + \frac{2454}{n^4} - \frac{10450}{n^5}, \\ \hat{v}(n) &= 8 - \frac{32}{n} + \frac{416}{3n^2} - \frac{15904}{27n^3} + \frac{2463}{n^4}.\end{aligned}$$

PROOF. Note that  $\hat{u}(n) < 0$  for  $1 \leq n \leq 4$ , and

$$\hat{u}(n) = \frac{2(n-5)(108n^4 + 108n^3 + 2412n^2 + 4108n + 53669) + 254540}{27n^5} > 0$$

for  $n \geq 5$ . So we shall use mathematical induction to prove (2.2) for  $n \geq 5$ . With the help of a computer, it is easy to check that the values of  $\{r_n - \hat{u}(n)\}_{n=1}^5$  are

$$\frac{228754}{27}, \frac{96401}{432}, \frac{18553}{729}, \frac{811717}{152064}, \frac{1218857}{776250}$$

which are all positive, and the values of  $\{r_n - \hat{v}(n)\}_{n=1}^5$  are

$$-\frac{53639}{27}, -\frac{44917}{432}, -\frac{12878}{729}, -\frac{372727}{76032}, -\frac{1388101}{776250}$$

which are all negative. So (2.2) holds for  $1 \leq n \leq 5$ .

Assume that (2.2) is valid for  $n-1 \geq 5$ . We aim to prove it for  $n \geq 6$ . For this purpose, note that by (1.5) it holds that

$$(2.3) \quad r_n = \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot r_{n-1}},$$

where

$$\begin{cases} a_n = (n+3)(n+4), \\ b_n = 7n^2 + 21n + 12, \\ c_n = 8n(n-1). \end{cases}$$

Thus by (2.3) and the inductive hypothesis, we have that

$$r_n - \hat{u}(n) > \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot \hat{v}(n-1)} - \hat{u}(n) = \frac{f_1}{27n^5(n+3)(n+4)f_2},$$

where

$$\begin{cases} f_1 = 43497n^7 - 5809365n^6 + 241492004n^5 - 1235704128n^4 \\ \quad + 2920346230n^3 - 5037170424n^2 + 12024249066n + 295339948200, \\ f_2 = 216n^4 - 1728n^3 + 7632n^2 - 26848n + 87229. \end{cases}$$

We first show that  $f_1 > 0$ . For this purpose, set  $f_1 = f_{11} + f_{12}$  where

$$\begin{aligned} f_{11} &= 43497n^7 - 5809365n^6 + 241492004n^5 - 1235704128n^4 \\ &= n^4[(n-6)(43497n^2 - 5548383n + 208201706) + 13506108], \end{aligned}$$

and

$$\begin{aligned} f_{12} &= 2920346230n^3 - 5037170424n^2 + 12024249066n + 295339948200 \\ &= 2920346230(n-2)n^2 + 803522036n^2 + 12024249066n + 295339948200. \end{aligned}$$

Clearly,  $f_{11} > 0$ ,  $f_{12} > 0$  for  $n \geq 6$ . Thus  $f_1 = f_{11} + f_{12} > 0$  for  $n \geq 6$ . Notice that

$$f_2 = 72n(n-6)(3n^2 - 6n + 70) + 3392n + 87229 > 0, \quad n \geq 6.$$

It follows that

$$r_n - \hat{u}(n) > \frac{f_1}{27n^5(n+3)(n+4)f_2} > 0$$

for  $n \geq 6$ .

It remains to prove that  $r_n - \hat{v}(n) < 0$  for  $n \geq 6$ . By the inductive hypothesis,  $r_{n-1} > \hat{u}(n-1) > 0$  for  $n-1 \geq 5$ . Therefore by (2.3) it holds that

$$r_n - \hat{v}(n) < \frac{b_n}{a_n} + \frac{c_n}{a_n \cdot \hat{u}(n-1)} - \hat{v}(n) = -\frac{f_3}{27n^4(n+3)(n+4)f_4},$$

where

$$\begin{aligned} f_3 &= (n-6)(1215n^6 + 29537847n^5 - 18982066n^4 + 397104580n^3 \\ &\quad + 1399763449n^2 + 10254978749n + 56257059186) + 190254876300, \end{aligned}$$

and

$$f_4 = (n-6)(108n^4 - 324n^3 + 2736n^2 - 824n + 51973) + 127270.$$

It is clear that  $f_3 > 0$ ,  $f_4 > 0$  for  $n \geq 6$ . Hence we obtain that

$$r_n - \hat{v}(n) < -\frac{f_3}{27n^4(n+3)(n+4)f_4} < 0$$

for  $n \geq 6$ . This completes the proof.  $\square$

We are now ready to give a proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Recall that  $r_n = B_{n+1}/B_n > 0$  for  $n \geq 0$ . So the inequality (2.1) is equivalent to

$$(2.4) \quad \frac{r_n^2}{r_{n-1}r_{n+1}} > 1$$

for  $n \geq 1$ . By Lemma 2.2,

$$0 < \hat{u}(n) < r_n, \quad n \geq 5,$$

and

$$0 < r_n < \hat{v}(n), \quad n \geq 1.$$

Hence we obtain that

$$\frac{r_n^2}{r_{n-1}r_{n+1}} > \frac{(\hat{u}(n))^2}{\hat{v}(n-1) \cdot \hat{v}(n+1)}, \quad n \geq 5.$$

For  $1 \leq n \leq 4$ , it is easy to check that the values of  $\{r_n^2/(r_{n-1}r_{n+1})-1\}_{n=1}^4$  are

$$\frac{1}{3}, \frac{5}{22}, \frac{89}{1242}, \frac{11167}{280841},$$

which means that (2.4) is true for  $1 \leq n \leq 4$ . It is sufficient to prove that

$$(2.5) \quad \frac{(\hat{u}(n))^2}{\hat{v}(n-1) \cdot \hat{v}(n+1)} > 1, \quad n \geq 5.$$

By Lemma 2.2, we get that

$$\frac{(\hat{u}(n))^2}{\hat{v}(n-1) \cdot \hat{v}(n+1)} - 1 = \frac{f_5}{n^{10}f_2f_6},$$

where  $f_2 = 216n^4 - 1728n^3 + 7632n^2 - 26848n + 87229 > 0$  for  $n \geq 6$ , which has been shown in the proof of Lemma 2.2, and

$$\begin{aligned} f_5 = & (n-23)(373248n^{14} + 2881008n^{13} - 2163024n^{12} + 24719472n^{11} + 18013968n^{10} \\ & + 4560284715n^9 + 82298410437n^8 + 1926154728903n^7 + 44427203227329n^6 \\ & + 1021582551830975n^5 + 23496190881398905n^4 + 540412815701543079n^3 \\ & + 12429494906472774753n^2 + 285878382543804079083n \\ & + 6575202798470104429509) + 151229664364892010501207, \end{aligned}$$

$$f_6 = 72(n-1)(n-2)(3n^2 + 9n + 55) + 440n + 45773,$$

which are easy to verified with the aid of a computer. Clearly,  $f_5 > 0$  for  $n \geq 23$  and  $f_6 > 0$  for  $n \geq 0$ . Combining the fact that  $f_2 > 0$  for  $n \geq 6$ , we arrive at the result that  $(\hat{u}(n))^2/(\hat{v}(n-1) \cdot \hat{v}(n+1)) > 1$  for  $n \geq 23$ , which implies that (2.4) holds for  $n \geq 23$ .

Note that although the inequality (2.5) is not true for  $5 \leq n \leq 22$ , it does not affect the validity of (2.4). It is easy to check that (2.4) holds for  $1 \leq n \leq 22$  by using a computer. Thus (2.4) is proved for all  $n \geq 1$ . This completes the proof.  $\square$

### 3. LOG-CONCAVITY AND MONOTONICITY OF $\{\sqrt[n]{B_n}\}_{n \geq 1}$

Motivated by Firoozbakht's conjecture on the strictly decreasing property of the sequence  $\{\sqrt[n]{p_n}\}_{n \geq 1}$ , where  $p_n$  is the  $n$ -th prime number (see [32, p.185]), Sun [36, 37] studied the monotonicity of many combinatorial sequences of this type, which has received much attention.

The goal of this section is to prove the log-concavity and monotonicity of the sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$ . The first main result of this section is as follows.



THEOREM 3.1. *The sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  is strictly log-concave. That is,*

$$(3.1) \quad \left(\sqrt[n]{B_n}\right)^2 > \sqrt[n-1]{B_{n-1}} \cdot \sqrt[n+1]{B_{n+1}}, \quad n \geq 2.$$

To prove Theorem 3.1, we shall apply a criterion given by Chen, Guo, and Wang [7].

THEOREM 3.2. [7, Theorem 3.1] *Let  $\{S_n\}_{n \geq 0}$  be a positive sequence. If the sequence  $\{S_{n+1}/S_n\}_{n \geq N}$  is log-concave and*

$$(3.2) \quad \frac{\sqrt[N+1]{S_{N+1}}}{\sqrt[N]{S_N}} > \frac{\sqrt[N+2]{S_{N+2}}}{\sqrt[N+1]{S_{N+1}}}$$

*for some positive integer  $N$ , then the sequence  $\{\sqrt[n]{S_n}\}_{n \geq N}$  is strictly log-concave.*

PROOF OF THEOREM 3.1. By Theorem 2.1, the sequence  $\{B_{n+1}/B_n\}_{n \geq 0}$  is strictly log-concave. Set  $N = 1$ . Since

$$\frac{\sqrt{B_2}}{B_1} - \frac{\sqrt[3]{B_3}}{\sqrt{B_2}} = \frac{2 - \sqrt[3]{6}}{\sqrt{2}} > 0,$$

by Theorem 3.2, it follows that the sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  is strictly log-concave as desired.  $\square$

By using the log-concavity of  $\{\sqrt[n]{B_n}\}_{n \geq 1}$ , we obtain the monotonicity of  $\{\sqrt[n]{B_n}\}_{n \geq 1}$ .

THEOREM 3.3. *The sequence  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  is strictly increasing.*

PROOF. Recall that for a real sequence  $\{S_n\}_{n \geq 0}$  with positive numbers, it was shown that

$$\liminf_{n \rightarrow \infty} \frac{S_{n+1}}{S_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{S_n}, \quad \text{and} \quad \limsup_{n \rightarrow \infty} \sqrt[n]{S_n} \leq \limsup_{n \rightarrow \infty} \frac{S_{n+1}}{S_n},$$

see Rudin [34, Subsection 3.37]. The above inequalities imply a well-known criterion, that is, if  $S_{n+1}/S_n \rightarrow C$  as  $n$  goes to infinity, then  $\sqrt[n]{S_n} \rightarrow C$  as  $n$  trends to infinity, where  $C$  is a finite real number.

By [20, Theorem 4.18], we have

$$\lim_{n \rightarrow \infty} \frac{B_{n+1}}{B_n} = 8,$$

which can also be obtained by using Lemma 2.2 together with the well-known Squeeze Theorem. Therefore

$$\lim_{n \rightarrow \infty} \sqrt[n]{B_n} = 8.$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{B_{n+1}}}{\sqrt[n]{B_n}} = 1.$$

By Theorem 3.1, we see that the sequence  $\{\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n}\}_{n \geq 1}$  is strictly decreasing. Hence we obtain that

$$\frac{\sqrt[n+1]{B_{n+1}}}{\sqrt[n]{B_n}} > 1$$

for all  $n \geq 1$ . This completes the proof.  $\square$

#### 4. HIGHER ORDER TURÁN INEQUALITIES RELATED TO $B_n$

This section is devoted to the study of higher order Turán inequalities related to the Baxter numbers. Numerical computation suggests that the sequences  $\{B_n\}_{n \geq 0}$ ,  $\{B_{n+1}/B_n\}_{n \geq 0}$ , and  $\{\sqrt[n]{B_n}\}_{n \geq 1}$  do not satisfy the higher order Turán inequalities (1.3), however the sequences  $\{B_n/n!\}_{n \geq 2}$ ,  $\{B_{n+1}B_n^{-1}/n!\}_{n \geq 2}$ , and  $\{\sqrt[n]{B_n/n!}\}_{n \geq 2}$  do. The first main result of this section is as follows.

**THEOREM 4.1.** *The sequence  $\{B_n/n!\}_{n \geq 2}$  satisfies the higher order Turán inequalities. That is, (1.3) holds for  $a_n = B_n/n!$ . More precisely, for  $n \geq 3$ ,*

$$(4.1) \quad 4 \left( \frac{B_n^2}{n!^2} - \frac{B_{n-1}}{(n-1)!} \frac{B_{n+1}}{(n+1)!} \right) \left( \frac{B_{n+1}^2}{(n+1)!^2} - \frac{B_n}{n!} \frac{B_{n+2}}{(n+2)!} \right) > \left( \frac{B_n}{n!} \frac{B_{n+1}}{(n+1)!} - \frac{B_{n-1}}{(n-1)!} \frac{B_{n+2}}{(n+2)!} \right)^2.$$

As mentioned in [6] and [8], for  $n = 1$ , the polynomial in (1.3) reduces to

$$\begin{aligned} I(a_0, a_1, a_2, a_3) &= 4(a_1^2 - a_0a_2)(a_2^2 - a_1a_3) - (a_1a_2 - a_0a_3)^2 \\ &= 3a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2 + 6a_0a_1a_2a_3 \end{aligned}$$

which is clearly an invariant of the cubic binary form

$$a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3.$$

Notice that  $27I(a_0, a_1, a_2, a_3)$  is the discriminant of the cubic polynomial  $a_3x^3 + 3a_2x^2 + 3a_1x + a_0$ . For a real cubic polynomial, the discriminant is positive if and only if the three zeros are real and distinct [28]. See also [16]. Thus, as a consequence of Theorem 4.1, we get the following result.

**THEOREM 4.2.** *For any integer  $n \geq 3$ , the polynomials*

$$P_n(x) = \frac{B_{n+3}}{(n+3)!}x^3 + \frac{3B_{n+2}}{(n+2)!}x^2 + \frac{3B_{n+1}}{(n+1)!}x + \frac{B_n}{n!}$$

*have only real and distinct zeros.*

We shall apply a criterion given by Hou and Li [25] to prove Theorem 4.1.

THEOREM 4.3. [25, Theorem 5.2] *Let  $\{a_n\}_{n \geq 0}$  be a real sequence with positive numbers. Let*

$$d(x, y) = 4(1 - x)(1 - y) - (1 - xy)^2.$$

*If there exist an integer  $N$ , and two functions  $g(n)$  and  $h(n)$  such that for all  $n \geq N$ ,*

- (i)  $g(n) < a_{n-1}a_{n+1}/a_n^2 < h(n)$ ;
- (ii)  $d(g(n), g(n+1)) > 0$ ,  $d(g(n), h(n+1)) > 0$ ,  $d(h(n), g(n+1)) > 0$ ,  
 $d(h(n), h(n+1)) > 0$ ,

*then  $\{a_n\}_{n \geq N}$  satisfies the higher order Turán inequalities.*

In order to use Theorem 4.3, the first step is to find appropriate bounds for  $B_{n-1}B_{n+1}/B_n^2$ . It should be mentioned that Theorem 4.3 is applicable to any positive real sequences  $\{a_n\}_{n \geq 0}$ . In particular, for a  $P$ -recursive sequence  $\{a_n\}_{n \geq 0}$ , Hou and Li [25] gave an algorithm, called the ‘‘HT Algorithm’’, to find the number  $N$  and the bounds  $g(n)$  and  $h(n)$ . This algorithm has been implemented in a `Mathematica` package `P-rec.m` by Hou. For example, the Baxter numbers satisfy the recurrence (1.5) with initial values  $B_0 = 1, B_1 = 1, B_2 = 2$ . Running the command

$$\text{HT}[(n + 3)(n + 4)N - (7n^2 + 21n + 12) - 8n(n - 1)N^{-1}, n, N, \{1, 1, 2\}, 4]$$

outputs the result  $\{1 + \frac{3}{n^2}, 1 + \frac{5}{n^2}\}$ , which means that

$$(4.2) \quad 1 + \frac{3}{n^2} < \frac{B_{n-1}B_{n+1}}{B_n^2} < 1 + \frac{5}{n^2}$$

for  $n \geq 753$ . In fact the above bounds are valid for  $n \geq 11$ . To be rigorous, we show a simple proof.

LEMMA 4.4. *The inequalities in (4.2) hold for all  $n \geq 11$ .*

PROOF. Let  $r_n = B_{n+1}/B_n$ . By Lemma 2.2, it is easy to derive that

$$(4.3) \quad 0 < u(n) < r_n < v(n), \quad n \geq 5,$$

where

$$(4.4) \quad u(n) = 8 - \frac{32}{n} + \frac{137}{n^2} - \frac{589}{n^3} \quad \text{and} \quad v(n) = 8 - \frac{32}{n} + \frac{140}{n^2}.$$

As will be seen, the bounds given in (4.3) will make the proof more concise, although they are not sharper than those provided in Lemma 2.2.

It follows from (4.3) that

$$\frac{B_{n-1}B_{n+1}}{B_n^2} = \frac{r_n}{r_{n-1}} < \frac{v(n)}{u(n-1)}$$

for  $n \geq 6$ . Direct computation gives that

$$\frac{v(n)}{u(n-1)} - \left(1 + \frac{5}{n^2}\right) = -\frac{5n^3 - 522n^2 + 673n - 3690}{n^2(8n^3 - 56n^2 + 225n - 766)}$$

which is negative for  $n \geq 104$ . We have verified that  $r_n/r_{n-1} < 1 + 5/n^2$  for  $1 \leq n \leq 103$ . Hence,

$$(4.5) \quad \frac{B_{n-1}B_{n+1}}{B_n^2} < 1 + \frac{5}{n^2}, \quad n \geq 1.$$

Notice that

$$\frac{B_{n-1}B_{n+1}}{B_n^2} = \frac{r_n}{r_{n-1}} > \frac{u(n)}{v(n-1)}$$

for  $n \geq 2$ . Direct computation shows that

$$\frac{u(n)}{v(n-1)} - \left(1 + \frac{3}{n^2}\right) = \frac{5n^3 - 751n^2 + 775n - 589}{4n^3(2n^2 - 12n + 45)}$$

which is positive for  $n \geq 150$ . It has been checked that  $r_n/r_{n-1} > 1 + 3/n^2$  for  $11 \leq n \leq 149$ . So,

$$(4.6) \quad \frac{B_{n-1}B_{n+1}}{B_n^2} > 1 + \frac{3}{n^2}, \quad n \geq 11.$$

Combining (4.5) and (4.6) leads to the desired result that (4.2) holds for  $n \geq 11$ . This completes the proof.  $\square$

We are now ready to prove Theorem 4.1.

PROOF OF THEOREM 4.1. Let  $a_n = B_n/n!$  for  $n \geq 1$ . Then by Lemma 4.4, we have that

$$\frac{n}{n+1} \left(1 + \frac{3}{n^2}\right) < \frac{a_{n-1}a_{n+1}}{a_n^2} < \frac{n}{n+1} \left(1 + \frac{5}{n^2}\right)$$

for  $n \geq 11$ . Set

$$g(n) = \frac{n}{n+1} \left(1 + \frac{3}{n^2}\right) \quad \text{and} \quad h(n) = \frac{n}{n+1} \left(1 + \frac{5}{n^2}\right).$$

So the conditions in (i) of Theorem 4.3 are satisfied.

It remains to verify the four inequalities in (ii) of Theorem 4.3. By a direct computation, we have that

$$\begin{aligned} d(g(n), g(n+1)) &= \frac{4(n^5 - 6n^4 + 9n^3 + 4n^2 - 12n - 36)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-4)(n^4 - 2n^3 + n^2 + 8n + 20) + 44]}{n^2(n+1)^4(n+2)^2} \end{aligned}$$

which is positive for  $n \geq 4$ . Similarly, we obtain that

$$\begin{aligned} d(g(n), h(n+1)) &= \frac{4(n^5 - 11n^4 + 25n^3 + 6n^2 - 12n - 81)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-8)(n^4 - 3n^3 + n^2 + 14n + 100) + 719]}{n^2(n+1)^4(n+2)^2} \end{aligned}$$

which is positive for  $n \geq 8$ ,

$$\begin{aligned} d(h(n), g(n+1)) &= \frac{4(n^5 - 9n^4 + 11n^3 - 20n^2 - 60n - 100)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-9)(n^4 + 11n^2 + 79n + 651) + 5759]}{n^2(n+1)^4(n+2)^2} \end{aligned}$$

which is positive for  $n \geq 9$ , and

$$\begin{aligned} d(h(n), h(n+1)) &= \frac{4(n^5 - 12n^4 + 43n^3 - 24n^2 - 80n - 225)}{n^2(n+1)^4(n+2)^2} \\ &= \frac{4[(n-7)(n^4 - 5n^3 + 8n^2 + 32n + 144) + 783]}{n^2(n+1)^4(n+2)^2} \end{aligned}$$

which is positive for  $n \geq 7$ . So the conditions in (ii) of Theorem 4.3 hold for  $n \geq 11$ .

It follows from Theorem 4.3 that the sequence  $\{B_n/n!\}_{n \geq 11}$  satisfies the higher order Turán inequalities. For  $3 \leq n \leq 10$ , it is easy to check that inequality (1.3) still holds for  $a_n = B_n/n!$ . Thus we have that (4.1) holds true for  $n \geq 3$ . This completes the proof.  $\square$

Note that Theorem 4.1 and Theorem 4.5 stated below can also be proved by using a criterion given by Wang [40, Theorem 2.1], in which case one should examine the signs of eight values of four functions of  $n$ .

The second main result of this section is as follows.

**THEOREM 4.5.** *Let  $B_n$  be defined by (1.1), and  $r_n = B_{n+1}/B_n$ . Then the sequence  $\{r_n/n!\}_{n \geq 2}$  satisfies the higher order Turán inequalities. That is, for  $n \geq 3$ ,*

$$(4.7) \quad 4 \left( \frac{r_n^2}{n!^2} - \frac{r_{n-1}}{(n-1)!} \frac{r_{n+1}}{(n+1)!} \right) \left( \frac{r_{n+1}^2}{(n+1)!^2} - \frac{r_n}{n!} \frac{r_{n+2}}{(n+2)!} \right) > \left( \frac{r_n}{n!} \frac{r_{n+1}}{(n+1)!} - \frac{r_{n-1}}{(n-1)!} \frac{r_{n+2}}{(n+2)!} \right)^2.$$

In order to prove Theorem 4.5, we need the following lemma.

**LEMMA 4.6.** *Let  $r_n = B_{n+1}/B_n$ . Then for  $n \geq 2$ , we have that*

$$(4.8) \quad \frac{n^2(n^2 + 2n + 4)}{(n^2 + 5)(n+1)^2} < \frac{r_{n-1}r_{n+1}}{r_n^2} < \frac{n^2(n^2 + 2n + 6)}{(n^2 + 3)(n+1)^2}.$$

**PROOF.** It is easy to check that (4.8) is valid for  $2 \leq n \leq 10$ . For  $n \geq 11$ , by Lemma 4.4, we have that

$$1 + \frac{3}{(n+1)^2} < \frac{r_{n+1}}{r_n} < 1 + \frac{5}{(n+1)^2},$$

and

$$\frac{n^2}{n^2 + 5} < \frac{r_{n-1}}{r_n} < \frac{n^2}{n^2 + 3}.$$

It follows that (4.8) holds true for  $n \geq 11$ . This completes the proof.  $\square$

We proceed to prove Theorem 4.5.

PROOF OF THEOREM 4.5. Let  $a_n = r_n/n!$  where  $r_n = B_{n+1}/B_n$  for  $n \geq 1$ . By Lemma 4.6, we have that

$$\frac{n}{n+1} \cdot \frac{n^2(n^2 + 2n + 4)}{(n^2 + 5)(n+1)^2} < \frac{a_{n-1}a_{n+1}}{a_n^2} < \frac{n}{n+1} \cdot \frac{n^2(n^2 + 2n + 6)}{(n^2 + 3)(n+1)^2}$$

for  $n \geq 2$ . Set

$$g(n) = \frac{n^3(n^2 + 2n + 4)}{(n^2 + 5)(n+1)^3} \quad \text{and} \quad h(n) = \frac{n^3(n^2 + 2n + 6)}{(n^2 + 3)(n+1)^3}.$$

So the conditions in (i) of Theorem 4.3 are satisfied.

It remains to verify the conditions in (ii) of Theorem 4.3. By a direct computation, we obtain that

$$\begin{aligned} d(g(n), g(n+1)) &= \frac{4}{(n^2 + 5)^2(n+1)^3(n^2 + 2n + 6)^2(n+2)^6} (n^{14} + 20n^{13} + 217n^{12} \\ &\quad + 1569n^{11} + 8343n^{10} + 34045n^9 + 109634n^8 + 282077n^7 \\ &\quad + 579185n^6 + 936561n^5 + 1156632n^4 + 1033910n^3 + 617220n^2 \\ &\quad + 217400n + 34800), \end{aligned}$$

$$\begin{aligned} d(g(n), h(n+1)) &= \frac{4}{(n^2 + 5)^2(n+1)^3(n^2 + 2n + 4)(n+2)^6} (n^{12} + 6n^{11} + 33n^{10} \\ &\quad + 143n^9 + 669n^8 + 2894n^7 + 9933n^6 + 25744n^5 + 47130n^4 \\ &\quad + 56675n^3 + 41650n^2 + 16900n + 3000), \end{aligned}$$

$$\begin{aligned} d(h(n), g(n+1)) &= \frac{4}{(n^2 + 3)^2(n+1)^3(n^2 + 2n + 6)(n+2)^6} (n^{12} + 10n^{11} + 67n^{10} \\ &\quad + 385n^9 + 1867n^8 + 6508n^7 + 16267n^6 + 31156n^5 + 46264n^4 \\ &\quad + 49173n^3 + 33126n^2 + 12348n + 2088), \end{aligned}$$

and

$$\begin{aligned} d(h(n), h(n+1)) &= \frac{4}{(n^2 + 3)^2(n+1)^3(n^2 + 2n + 4)^2(n+2)^6} (n^{14} + 8n^{13} + 61n^{12} \\ &\quad + 307n^{11} + 1295n^{10} + 4481n^9 + 12610n^8 + 29313n^7 + 57181n^6 \\ &\quad + 93487n^5 + 122784n^4 + 118356n^3 + 74376n^2 + 26496n + 4320), \end{aligned}$$

which are all positive for  $n \geq 0$ . Thus the conditions in (ii) of Theorem 4.3 are satisfied for  $n \geq 2$ .

By Theorem 4.3, the sequence  $\{r_n/n!\}_{n \geq 2}$  satisfies the higher order Turán inequalities (4.7). This completes the proof.  $\square$

At last, we propose a conjecture.

CONJECTURE 4.7. *The sequence  $\{\sqrt[n]{B_n}/n!\}_{n \geq 2}$  satisfies the higher order Turán inequalities.*

If one can find appropriate bounds for  ${}^{n+1}\sqrt{B_{n+1}}/\sqrt[n]{B_n}$  in a form of rational functions of  $n$ , then Conjecture 4.7 will be solved by applying Theorem 4.3 or Wang's criterion [40, Theorem 2.1].

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