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THE HAUSDORFF DIMENSION OF DIRECTIONAL EDGE ESCAPING POINTS SET

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Abstract. In this paper, we define the directional edge escaping points set of function iteration under a given plane partition and then prove that the upper bound of Hausdorff dimension of the directional edge escaping points set of $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, is no more than 1.

1. Introduction

The Julia sets of transcendental entire functions always have very complicated fractal structures (see [9]). We often use the Huasdorff dimension to describe them. Many profound results about the Huasdorff dimension of Julia sets of transcendental entire functions have been obtained. For example, Stallard and Bishop proved that there is a transcendental entire function such that the Huasdorff dimension of its Julia set is equal to any pre-specified number on the closed interval [1,2] (see [1, 17, 18]).

In addition to Julia set, the closely related escaping set (see [3]) is also the subject of increasing interest. In particular, there are many studies on the escaping sets of specific transcendental entire functions. Take the escaping set of exponential function for example, Schleicher and Zimmer proved that the escaping points set of λe^z with $\lambda \neq 0$ is the Cantor set of curves and has a peculiar phenomenon of "dimension paradox", which was first found by Karpińska(see [6, 7]), that is the Hausdorff dimension of the hairs without endpoints is 1, however, the Hausdorff dimension of the set of endpoints is 2 (see [16]). Further more, not only for escaping points set of exponential

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function, but for escaping parameters set of exponential functions family has been intensively studied. For example, Schleiher, Forster, Rempe, Bailesteanu and Balan proved the escaping parameters set of exponential functions family also has the properties of Cantor bundle structure and "dimension paradox" (see [10, 12, 14, 15]). Of course, there are many other entire functions have been studied deeply, such as cosine function $ae^z + be^{-z}$, where $ab \neq 0$ (see [5, 8, 10, 11, 13, 19]).

In this paper, we will combine exponential and cosine functions to study the function $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. Moreover, we will also study a kind special escaping points set, which we call it directional edge escaping points set. For a function $S(z)$, the directional edge escaping points set of it under a given plane partition is defined below.

First, We divide the complex plane with squares. Denote $S^n(z)$ as the n-fold iterate of $S(z)$, where $n \in \mathbb{N}$. Take one of the squares arbitrarily, denote it by B_0, a point z in it is called directional edge escaping point means that it satisfies

- $S^n(z) \to \infty$ as $n \to \infty$,
- $|\mathrm{ImS^n(z)}| \leq \lambda |\mathrm{S^n(z)}|$ for all $\mathrm{n} \in \mathbb{N}$,
- $B_{n+1}(z) \cap \partial S(B_n(z)) \neq \emptyset$ for all $\mathrm{n} \in \mathbb{N}$.

where $\lambda \in (0, 1)$ is a constant, $B_{n+1}(z)$ is the square $S^{n+1}(z)$ belongs to, $\partial S(B_n(z))$ is the boundary of the image of $B_n(z)$ under function $S(z)$. See the following figure 1.

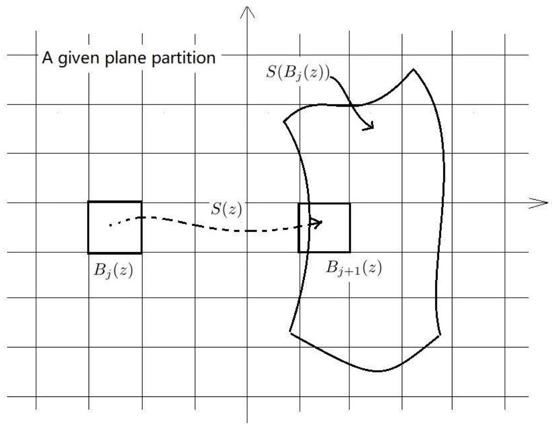

Figure 1. directional edge escaping point

As we all know, it is very important method to study the transcendental dynamics by dividing the plane(see [2]). If we imagine a series of objects connected by one rope, see the following figure 2, the concept of directional edge escaping point can emerges.

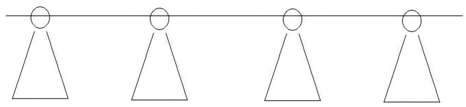

Figure 2. objects connected by one rope

It should be pointed out that the above directional edge escaping points set is very likely complicated and interesting. In order to be more intuitive, we limit the observation to the real axis and a simple linear function. Divide the real axis to partitions with integer points endpoints and consider the directional edge escaping points set of the map $y = 3x$ only in the interval $[0, 1]$. According to the concept of directional edge escaping points set, we can infer that the directional edge escaping points set of map $y = 3x$ in the interval $[0, 1]$ is a classic Cantor set without $\{0\}$, whose Hausdorff dimension is $log_3 2$. See the following figure 3.

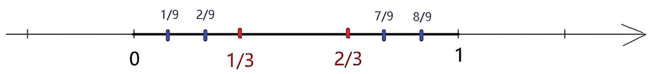

Figure 3. the directional edge escaping points set of $y = 3x$

In this paper, We will prove that the Hausdorff dimension of directional edge escaping points set of $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, is no more than 1 under one kind of complex plane partition. In order to state our conclusion, we turn to briefly introduce the concept of Hausdorff dimension(see [4]) and some notations.

For any set $U \subseteq \mathbb{C}$, denote the diameter of U by $|U| := \sup\{|z - w| : z, w \in U\}$. Let F be a set in $\mathbb{C}$, and s a positive number. Define s-dimensional measure $H^s(F)$ of F by

$$H^s(F) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : |U_i| < \delta, F \subseteq \bigcup_i U_i \right\}.$$

and define the Hausdorff dimension $\dim(F)$ of F by

$$\dim(F) := \inf \left\{ s \geq 0 : H^s(F) = 0 \right\} = \sup \left\{ s \geq 0 : H^s(F) = \infty \right\}.$$

For convenience, we might as well denote function $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, as $S(z)$.

Define $I := \{z \in \mathbb{C} : \ S^n(z) \to \infty, \text{ as } \ n \to \infty \text{ and } |\mathrm{ImS^n(z)}| \leq \lambda |\mathrm{S^n(z)}| \text{ for all } n \in \mathbb{N}\}$. Denote E_∞ as the the directional edge escaping points set, that is $E_\infty := \{z \in I : \ B_{n+1}(z) \cap \partial S(B_n(z)) \neq \emptyset \text{ for all } n \in \mathbb{N}\}$. And divide the complex plane as follow.

$$\begin{aligned}\mathbb{C} &:= \cup_{k=-\infty}^{\infty} P_k \\ &:= \cup_{k=-\infty}^{\infty} \{z \in \mathbb{C} : -\frac{\pi}{2} + k\pi \leq \mathrm{Imz} < \frac{\pi}{2} + \mathrm{k}\pi\} \\ &:= \cup_{k=-\infty}^{\infty} \cup_{j=-\infty}^{\infty} B_{j,k} \\ &:= \cup_{k=-\infty}^{\infty} \cup_{j=-\infty}^{\infty} \{z \in P_k : j\pi \leq \mathrm{Rez} < (\mathrm{j}+1)\pi\}.\end{aligned}$$

See the following figure 4.

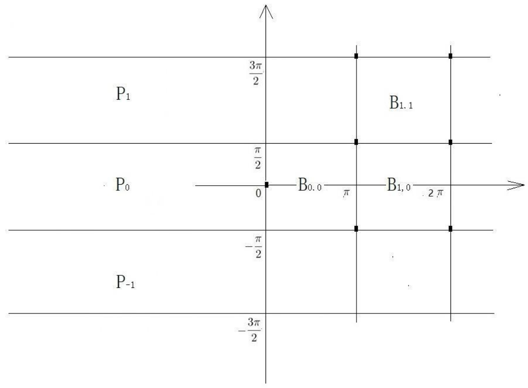

Figure 4. plane division

THEOREM 1.1. *If E_∞ be the directional edge escaping points of $S(z)$ under the foregoing division of plane, then* $\dim(E_\infty) \leq 1$.

2. Preliminaries

LEMMA 2.1. *Let F be a subset of $\mathbb{C}$, $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, then* $\dim(F) = \dim(S(F))$.

PROOF. If $a = 0$ or $b = 0$, $S'(z) \neq 0$.
If $ab \neq 0$, let $S'(z) = ae^z - be^{-z} = 0$, then $z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}\mathrm{Arg}(\frac{\mathrm{b}}{\mathrm{a}})$. So $S'(z) \neq 0$ on $\mathbb{C} \setminus \{z : z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}\mathrm{Arg}(\frac{\mathrm{b}}{\mathrm{a}})\}$. That means $S(z)$ is locally univalent except for a countable set. And note that ignoring a countable subset has no effect on the Hausdorff dimension of the original set, we have $\dim(F) = \dim(S(F))$. □

For $z \in I$, according to $|S^n(z)| \leq |a|\exp(\mathrm{ReS^{n-1}(z)}) + |\mathrm{b}|\exp(-\mathrm{ReS^{n-1}(z)})$, we have $\{z \in I : |\mathrm{Re(S^n(z))}| \to \infty \text{ as } n \to \infty\}$. So it is no matter that we limit our discussion in

$$H^q := \{z \in \mathbb{C} : |\mathrm{Rez}| \geq \mathrm{q}\},$$

where q is large enough. Otherwise, by lemma 2.1, we consider the set $S^n(E_\infty \cap B_{j,k})$.

LEMMA 2.2. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, m, n be nonnegative integers, if $q > 0$ is sufficiently large and $z \in H^q$, then*

(a) $S^{(m)}(z) \neq 0$, where $S^{(m)}(z)$ is the m-order derivative, $S^{(0)}(z) = S(z)$;

(b) the horizontal strip domain with width smaller than 2π and the real part no less than q (or no more than $-q$) is the univalent domain of $S^{(m)}(z)$;

(c) $e < \frac{2}{3}\min\{|a|, |b|\}e^{|\mathrm{Re}z|} < |S^{(m)}(z)| < \frac{3}{2}\max\{|a|, |b|\}e^{|\mathrm{Re}z|}$;

(d) $\frac{1}{2e^\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} < 2e^\pi$, where $|\mathrm{Re}z_1 - \mathrm{Re}z_2| < \pi$ and $z_i \in H^q$,$i = 1, 2$.

PROOF. (a)If $S^{(m)}(z) = ae^z \pm be^{-z} = 0$, then $z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}Arg(\pm\frac{b}{a})$. Because $|\mathrm{Re}z| \geq \mathrm{q} > |\frac{1}{2}\log|\frac{\mathrm{b}}{\mathrm{a}}||$, then $S^{(m)}(z) \neq 0$.

(b) Note that $S^{(m)}(z) = ae^z \pm be^{-z} = \sqrt{ab}(\sqrt{\frac{a}{b}}e^z \pm \sqrt{\frac{b}{a}}e^{-z})$. If $S^{(m)}(z_1) = S^{(m)}(z_2)$, then

$$\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2} \text{ or } |\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| = 1.$$

Since q is large enough such that $|\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| \neq 1$, we have $\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2}$ and then $z_1 = z_2$ (width of strip $< 2\pi$).

(c) Suppose $\mathrm{Re}z \geq \mathrm{q} > 0$, as q is large enough, then

$$\begin{aligned}|S^{(m)}(z)| &\geq ||a|e^{\mathrm{Re}z} - |b|e^{-\mathrm{Re}z}| > |a|e^{\mathrm{Re}z} - \frac{1}{3}|a|e^{\mathrm{Re}z} \\ &= \frac{2}{3}|a|e^{\mathrm{Re}z} \geq \frac{2}{3}\min\{|a|, |b|\}e^{|\mathrm{Re}z|} > e,\end{aligned}$$

$$\begin{aligned}|S^{(m)}(z)| &\leq |a|e^{\mathrm{Re}z} + |b|e^{-\mathrm{Re}z} < |a|e^{\mathrm{Re}z} + \frac{1}{2}|a|e^{\mathrm{Re}z} \\ &= \frac{3}{2}|a|e^{|\mathrm{Re}z|} \leq \frac{3}{2}\max\{|a|, |b|\}e^{|\mathrm{Re}z|}.\end{aligned}$$

The prove is completely similar when $\mathrm{Re}z \leq -\mathrm{q} < 0$.

(d)Without losing generality, suppose $\mathrm{Re}z \geq \mathrm{q} > 0$. It can be proved similarly when $\mathrm{Re}z \leq -\mathrm{q} < 0$.

$$\frac{||a|e^{\mathrm{Re}z_1} - |b|e^{-\mathrm{Re}z_1}|}{|a|e^{\mathrm{Re}z_2} + |b|e^{-\mathrm{Re}z_2}} \leq \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \leq \frac{|a|e^{\mathrm{Re}z_1} + |b|e^{-\mathrm{Re}z_1}}{||a|e^{\mathrm{Re}z_2} - |b|e^{-\mathrm{Re}z_2}|}.$$

As $q > 0$ is large enough and $|\mathrm{Re}z_1 - \mathrm{Re}z_2| < \pi$, then $\mathrm{Re}z_1$ and $\mathrm{Re}z_2$ are positive large enough. So

$$\frac{1}{2}e^{-\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \approx e^{\mathrm{Re}z_1 - \mathrm{Re}z_2} < 2e^\pi.$$

□

According to lemma 2.2, we can further observe $S(z)$. For any given small positive number θ, as long as $q > 0$ is large enough, we have that

$$\max\{|a|, |b|\}e^{-|\mathrm{Rez}|} < \theta. \tag{2.1}$$

So $S(z) \approx ae^{z}$ or $S(z) \approx be^{-z}$ in H^q.

Take $B := B_{j,k}$ and $j > 0$ for example, $S(B)$ contains a half-annulus with inner radius of $|a|e^{j\pi} + \theta$ and outer radius of $|a|e^{(j+1)\pi} - \theta$. At the same time, $S(B)$ included in a half-annulus with inner radius of $|a|e^{j\pi} - \theta$ and outer radius of $|a|e^{(j+1)\pi} + \theta$. As the positive number θ is very small, $S(B)$ can be viewed as 'approximate-half-annulus'.

Let $R(S(B)) := \sup|S(B)|$, $r(S(B)) := \inf|S(B)|$, and

$$\widetilde{A}(r(S(B)), R(S(B))) := S(B) \cap H^q \cap \{z \in \mathbb{C} : |\mathrm{Imz}| \leq \lambda|\mathrm{z}|, \lambda \in (0,1)\},$$

which is partial approximate-annulus.

Denote $\widetilde{A}(a_0 r(S(B)) + a_1, b_0 R(S(B)) + b_1)$ as the 'approximate-half-annulus' in H^q, which is enclosed by the image of inner and outer boundary of $S(B)$ under linear transformation $a_0 z + a_1$ and $b_0 z + b_1$ respectively along radial direction, where a_0, a_1, b_0, b_1 are real number.

Since $\{z, S^1(z), S^2(z), \cdots\}$ stay in H^q, so for every $n \geq 0$ there exists a unique square $B_n(z) \subseteq H^q$ such that

$$S^n(z) \in B_n(z).$$

If necessary, we can ask q to be sufficiently large that the above lemma 2.2 holds when $|\mathrm{Rez}| > \frac{\mathrm{q}}{2}$. It follows immediately from lemma 2.2 (b) and (c) that there exists a unique holomorphic inverse branch $S_z^{-n} : B_n(z) \to H^{q-\pi} := \{z \in \mathbb{C} : |\mathrm{Rez}| \geq \mathrm{q} - \pi\}$ of S^n sending $S^n(z)$ to z. Denote

$$K_n(z) = S_z^{-n}(B_n(z)).$$

In addition, denote $R(S(B_{n-1}(z)))$ and $r(S(B_{n-1}(z)))$, i.e. $R(S^n(K_{n-1}(z)))$ and $r(S^n(K_{n-1}(z)))$, respectively, by $R_n(z)$ and $r_n(z)$. See the following figure 5 to be familiar with the above symbols.

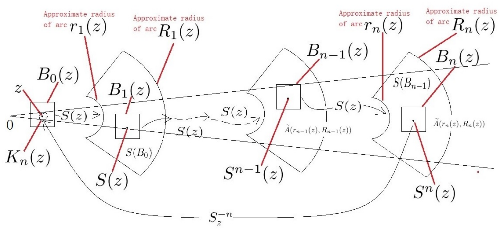

Figure 5. diagrammatic sketch of symbols

LEMMA 2.3. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. If q is large enough, then there exists constants K_1 and K_2 independent on n and z such that*

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \leq K_1$$

for all $x, y \in B_n(z)$, and

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} \leq K_2$$

for all $x, y \in K_{n-1}(z)$ i.e. $S^{n-1}(x), S^{n-1}(y) \in B_{n-1}(z)$.

PROOF. Denote $\widetilde{B_i(z)} \supset B_i(z)$ as the open square of side length 2π with sides parallel to $B_i(z)$ and center coincident with $B_i(z)$. By lemma 2.2 (b), we know that $S(z)$ is univalent on $\widetilde{B_i(z)}$ and the $S(\widetilde{B_i(z)})$ contains $\widetilde{B_{i+1}(z)}$ for $i = 0, 1, 2, \cdots$. See the following figure 6.

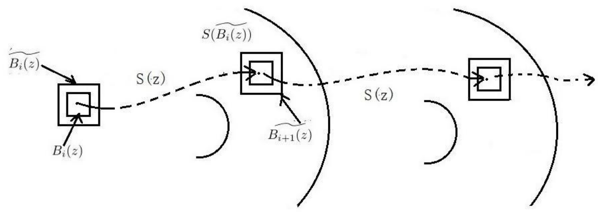

Figure 6. deviation property of $S(z)$

The module of $\widetilde{B_i(z)} \setminus B_i(z)$ is constant, by distortion theorem, for all $x, y \in B_n(z)$

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \leq K_1.$$

By lemma 2.2 (d)

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} = \frac{|S'(S^{n-1}(x))|}{|S'(S^{n-1}(y))|} \cdot \frac{|(S^{n-1})'(x)|}{|(S^{n-1})'(y)|} \leq 2e^{\pi} K_1 = K_2.$$

If let $K = \max\{\mathrm{K}_1, \mathrm{K}_2\}$, K_1, K_2 can be replaced with K at the same time. □

LEMMA 2.4. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. If $z \in I$ and q is large enough, then $|\mathrm{Re}(\mathrm{S^n(z)})|$ tends to infinity uniformly.*

PROOF. For any given $z \in I$, we have

$$\begin{aligned}|\mathrm{Re}(\mathrm{S^n(z)})| &= \sqrt{|S^n(z)|^2 - (|\mathrm{ImS^n(z)}|)^2} \\ &\geq \sqrt{|S^n(z)|^2 - (\lambda |S^n(z)|)^2} \\ &= (1-\lambda^2)^{\frac{1}{2}} |S^n(z)|.\end{aligned} \tag{2.2}$$

According to lemma 2.2 (c), q is large enough, we get

$$\begin{aligned}|S^{n+1}(z)| &= |ae^{S^n(z)} + be^{-S^n(z)}| \\ &\geq \frac{2}{3}\min\{|a|,|b|\} e^{|\mathrm{ReS^n(z)}|} \\ &\geq \frac{2}{3}\min\{|a|,|b|\}\exp((1-\lambda^2)^{\frac{1}{2}}|S^n(z)|) \\ &\geq \frac{2}{(1-\lambda^2)^{\frac{1}{2}}}|S^n(z)|.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathrm{Re}(\mathrm{S^{n+1}(z)})| &\geq (1-\lambda^2)^{\frac{1}{2}}|S^{n+1}(z)| \geq 2|S^n(z)| \\ &\geq 2|\mathrm{Re}(\mathrm{S^n(z)})| \geq \cdots \geq 2^{\mathrm{n+1}}\mathrm{q}.\end{aligned}$$

□

LEMMA 2.5. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. For any given $\alpha > 0$ and $T > 0$, there exist $K_3 > 0$ and $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$|(S^{n+1})'(z)| \geq K_3 |(S^n)'(z)|^\alpha$$

for all $z \in I \cap B(0,T)$.

PROOF. By lemma 2.4, for any given $\alpha > 0$, there is $n_0 \geq 0$ such that

$$\frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\} e^{(1-\lambda^2)^{\frac{1}{2}}|S^{n+1}(z)|} \geq (2e^\pi)^\alpha |S^{n+1}(z)|^\alpha$$

for all $z \in I$ when $n \geq n_0$.

We claim that

$$\inf_{z \in \overline{I \cap B(0,T)}} \frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha} \neq 0.$$

If there no exist $j \in \{0, 1, \cdots n_0\}$ and $z_0 \in \overline{I \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$, $\frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha}$ is a positive continuous function on bounded closed sets, the claim holds. Suppose there exist $j \in \{0, 1, \cdots n_0\}$ and $z_0 \in \overline{I \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$, then exist $\{z_n\} \subseteq I \cap B(0,T)$ such that $z_n \to z_0$ or $z_n \equiv z_0$. By lemma 2.2 (d)

$$|S'(S^j(z_0))| \leftarrow |S'(S^j(z_n))| \geq \frac{1}{2e^\pi}|S^{j+1}(z_n)| \geq \frac{1}{2e^\pi} q,$$

which contradicts to $S'(S^j(z_0)) = 0$.

Let K_3 be the infimum of the function $z \mapsto |(S^{n_0+1})'(z)||(S^{n_0})'(z)|^{-\alpha}$ in $I \cap B(0,T)$, then K_3 is a positive number. Proof by induction. According to the definition of K_3, the lemma holds when $n = n_0$. Suppose it is true for $n \geq n_0$, so

$$\begin{aligned}|(S^{n+2})'(z)| &= |(S'(S^{n+1}(z))| \cdot |(S^{n+1})'(z)| \\ &\geq K_3|(S'(S^{n+1}(z))| \cdot |(S^n)'(z)|^\alpha.\end{aligned}$$

By lemma 2.2 (d) (c) and (2.2)

$$\begin{aligned}|(S'(S^{n+1}(z))| &\geq \frac{1}{2e^\pi}|S^{n+2}(z)| \geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{|\mathrm{ReS^{n+1}(z)}|} \\ &\geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{(1-\lambda^2)^{\frac{1}{2}}|S^{n+1}(z)|} \geq (2e^\pi)^\alpha|S^{n+1}(z)|^\alpha \\ &\geq (2e^\pi)^\alpha \cdot (\frac{1}{2e^\pi})^\alpha|S'(S^n(z))|^\alpha = |S'(S^n(z))|^\alpha.\end{aligned}$$

Therefore

$$|(S^{n+2})'(z)| \geq K_3|S'(S^n(z))|^\alpha \cdot |(S^n)'(z)|^\alpha = K_3|(S^{n+1})'(z)|^\alpha.$$

□

3. The proof of theorem

Based on the above preliminaries, we can begin to prove the main result of this paper.

Proof. Let $E_n := \cup_{z\in I}S_z^{-n}(\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z)))$.

Then E_∞ can be covered by the set $\cup_{n\geq k}E_n$ for every $k \geq 0$ and the approximate-half-annuli $\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z))$ can be covered by $M_1r_n(z)$ squares with diameters less than 1, where M_1 is a constant. Therefore, according to lemma 2.3, $K_{n-1}(z) \cap E_n$ can be covered with no more than $M_1r_n(z)$ sets $J_{i,n}(z)$ of diameters less than $K|(S^n)'(z)|^{-1}$.

Let $T \geq 2q$. Note that any two sets $K_{n-1}(z)$ and $K_{n-1}(z')$ are either disjoint or equal, so we can find a set $Z_n \subset I$ such that $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint for $z, z' \in Z_n, z \neq z'$ and

$$E_n \cap B(0,T) \subset \cup_{z\in Z_n}K_{n-1}(z) \subset B(0,2T).$$

For the given $\epsilon > 0$, let n be large enough such that lemma 2.5 is satisfied for $\alpha = 2/\epsilon$ and $2T$. Using lemma 2.2 (d), lemma 2.5 and (2.1), we get

$$\sum_{z\in Z_n}\sum_{J_{i,n}}(\mathrm{diamJ_{i,n}(z)})^{1+\epsilon} \le \sum_{\mathrm{z\in Z_n}} \mathrm{M_1K^{1+\epsilon}r_n(z)|(S^n)'(z)|^{-(1+\epsilon)}}$$
$$\le 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||(S^n)'(z)|^{-(1+\epsilon)}$$
$$\le 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||S'(S^{n-1}(z))|^{-(1+\epsilon)}|(S^{n-1})'(z)|^{-(1+\epsilon)}$$
$$\le 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))|^{-\epsilon}|(S^{n-1})'(z)|^{-\epsilon}|(S^{n-1})'(z)|^{-1}$$
$$\le 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|(S^n)'(z)|^{-\epsilon}|(S^{n-1})'(z))|^{-1}$$
$$\le 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}K_3^{-\epsilon}|(S^{n-1})'(z)|^{-2}|(S^{n-1})'(z))|^{-1}$$
$$\le 2e^{\pi}K_3^{-\epsilon}M_1K^{1+\epsilon}e^{-(n-1)}\sum_{z\in Z_n}|(S^{n-1})'(z)|^{-2}.$$

Because $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint and the Lebesgue measure of each set of the form $K_{n-1}(z)$ is proportional to $|(S^{n-1})'(z)|^{-2}$ by lemma 2.3, we get that there exists a constant $M_2 > 0$ such that the last term in the above inequality is no more than $M_2e^{-(n-1)} \cdot \mathrm{area(B(0,2T))}$.

Hence,

$$\sum_{n=k}^{\infty}\sum_{z\in Z_n}\sum_{J_{i,n}}(\mathrm{diamJ_{i,n}(z)})^{1+\epsilon} \le M_2\cdot\mathrm{area(B(0,2T))}\sum_{\mathrm{n=k}}^{\infty}\mathrm{e^{-(n-1)}}$$
$$= 4\pi T^2M_2\frac{e^{-k+2}}{e-1}.$$

Let $k\to\infty$, then $4\pi T^2M_2\frac{e^{-k+2}}{e-1}\to 0$. That is, for any given $\epsilon > 0$, the $(1+\epsilon)$-dimensional Hausdorff measure of $E_\infty\cap B(0,T)$ is equal to zero. Hence,

$$\dim(E_\infty) \le 1.$$

□

Question: Does the same result hold for more general analytic functions?

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