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ON THE RAMANUJAN-NAGELL TYPE DIOPHANTINE EQUATION $Dx^2 + k^n = B$

ZHONGFENG ZHANG AND ALAIN TOGbé

Abstract. In this paper, we prove that the Ramanujan-Nagell type Diophantine equation $Dx^2 + k^n = B$ has at most three nonnegative integer solutions $(x, n)$ for $k$ a prime and $B, D$ positive integers.

1. Introduction

Studying some generalized Ramanujan-Nagell equations, Ulas [3] gave the following conjecture.

Conjecture 1.1. (Conjecture 4.4 in [3]) The Diophantine equation

$$x^2 + k^n = B$$

has at most three nonnegative integers $(x, n)$, for any given integers $k \geq 2$ and $B \geq 1$.

Meng Bai and the first author [1] confirmed Conjecture 1.1 for $k = 2$ and the authors [6] of this paper for $k$ an odd prime, i.e. they proved the following theorem.

Theorem 1.2. For any prime $p$ and any positive integer $B$, the Diophantine equation

$$x^2 + p^n = B$$

has at most three solutions $(x, n)$ in nonnegative integers. Furthermore, if $p \geq 3$ and $p^2 \nmid B$, we can replace three by two.

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Their result and our previous results (see [5]-[7]) give us the motivation to consider the following equation

\[(1.2) \quad Dx^2 + k^n = B\]

and to prove the following result.

**Theorem 1.3.** Let \( p \) be a prime, \( B \) and \( D \) be positive integers. Then, the Diophantine equation

\[(1.3) \quad Dx^2 + p^n = B\]

has at most three nonnegative integer solutions \((x,n)\). Furthermore, if \( p^2 \nmid B \), then we can replace three by two when \( p \geq 3 \) or when \( p = 2 \) with \( D \neq 1 \) when \( B \) is odd and \( D \neq 2 \) when \( B \) is even.

**Remark 1.4.** The result in Theorem 1.3 is the best possible.

1. Choose \( D \) so that \( 4D \pm 1 = p^r \), where \( p \) is a prime and \( r \geq 1 \). Then, for \( B = 64D^3 \pm 48D^2 + 13D \pm 1 \), we have \( p^2 \nmid B \) and the equation (1.3) has the solutions \((x,n) = (1,3r), (8D \pm 3, r)\), where the sign agrees with the sign in \( 4D \pm 1 \).

2. For \((p,D,B) = (2,3, \frac{3}{4}(2^{2m} + 2^{2m} + 1))\), \( m > 1 \), the equation (1.3) has the solutions

\[(x,n) = \left(\frac{1}{3}(2^{2m+1} + 1), 0\right), \left(\frac{1}{3}(2^{2m+1} - 2), 2m + 2\right), \left(\frac{2}{3}(2^{2m-1} + 1), 4m\right).\]

## 2. Preliminaries

First, we recall a result on Pell equation, which was proved by Walker [4] and a slightly improved version with a short and straightforward proof by Luo and Yuan [2].

**Lemma 2.1.** Let \((x,y)\) be a positive integer solution of the Diophantine equation

\[(2.4) \quad ux^2 - vy^2 = 1,\]

where \( u > 1 \) and \( v \) are coprime positive integers with \( uv \) nonsquare.

If every prime divisor of \( x \) divides \( u \), then either

\[x\sqrt{u} + y\sqrt{v} = \varepsilon\]

or

\[x\sqrt{u} + y\sqrt{v} = \varepsilon^3, \quad x = 3^t x_1, \quad 3 \nmid x_1, \quad 3^t + 3 = 4ux_1^2,\]

where \( \varepsilon = x_1\sqrt{u} + y_1\sqrt{v} \) is the minimal positive solution of (2.4) and \( t \) is a positive integer.

Now, we will prove a series of three results that will be useful for the proof of Theorem 1.3. The first result in this series is the following.
Lemma 2.2. Let \( D \) be a nonsquare positive integer and \( A \) a positive integer. Let \( p \) be a prime. Then, the Diophantine equation
\[
A p^{2m} - D y^2 = 1
\]
has at most one positive integer solution \((m, y)\).

Proof. Let \((m, y) = (r, a)\) be the least positive integer solution of (2.5). Consider (2.5) as an example of (2.4): letting \( u \) and \( v \) be as in Lemma 2.1, let
\[
u = A p^{2r}, \quad v = D.
\]
Let \((m, y) = (s, b)\) be any positive integer solution to (2.5). Let \( \varepsilon = \sqrt{A p^{2r}} + a \sqrt{D} \) and let \( \alpha = p^{s-r} \sqrt{A p^{2r}} + b \sqrt{D} \). By Lemma 2.1 either \( \alpha = \varepsilon \) or \( \alpha = \varepsilon^3 \).

If \( \alpha = \varepsilon^3 \) then, by Lemma 2.1, \( p^{s-r} = 3^t \), so that \( p = 3 \). But then the equation
\[
3^t + 3 = 4 A p^{2r},
\]
which is required by Lemma 2.1, is impossible modulo 9. So by Lemma 2.1, we must have \( \alpha = \varepsilon \) and then \( s = r \), which completes the proof of Lemma 2.2.

We will now prove the second preliminary result. Here, we deal with the case where \( p \) is an odd prime with \( p^2 \nmid B \).

Lemma 2.3. Let \( B, D \) be positive integers with \( D > 1 \) and \( B \geq 4D \). Let \( p \) be an odd prime with \( p^2 \nmid B \). Then, the Diophantine equation (1.3) has at most two nonnegative integer solutions \((x, n)\).

Proof. We will consider two cases according to the divisibility of \( B \) by \( p \).

(1) \( p \nmid B \). At this level, we will also study the problem according to the divisibility of \( D \) by \( p \).

(i) If \( p \nmid D \), then \( n \) can only take the value 0 since \( p \nmid B \). So, Diophantine equation (1.3) has at most one nonnegative integer solution \((x, n)\).

(ii) If \( p \nmid D \), then here we will study the following two claims.

Claim 1: There is at most one nonnegative integer solution \((x, n)\) satisfying \( p^n < 2\sqrt{D(B-1)} - D + 1 \).

Assume that \((x_1, n_1)\) and \((x_2, n_2)\) are two distinct integer solutions of equation (1.3) satisfying \( x_1 > x_2 \geq 0 \), \( p^{n_1} < p^{n_2} < 2\sqrt{D(B-1)} - D + 1 \). Thus, we get
\[
D(x_1^2 - x_2^2) = p^{n_2} - p^{n_1} \leq p^{n_2} - 1
\]
and
\[
D(x_1^2 - x_2^2) = D(x_1 + x_2)(x_1 - x_2) \geq D(x_1 + x_2) \geq D(2x_2 + 1) \geq 2Dx_2 + D.
\]
This means that \( p^{n_2} - (D + 1) \geq 2Dx_2 \), which yields
\[
p^{2n_2} - 2(D+1)p^{n_2} + (D+1)^2 \geq 4D^2 x_2^2 = 4D(B - p^{n_2}).
\]
Therefore, we obtain
\[ p^{2n^2} + 2(D-1)p^{n^2} + (D-1)^2 + 4D \geq 4DB, \]
i.e.
\[ (p^{n^2} + D - 1)^2 \geq 4D(B-1), \]
which yields \( p^{n^2} \geq 2\sqrt{D(B-1)} - D + 1 \). This leads to a contradiction and finishes the proof of the first claim.

Claim 2: There is at most one nonnegative integer solution \((x, n)\) satisfying \( p^n \geq 2\sqrt{D(B-1)} - D + 1 \).

In this case, we have \( n > 0 \) since \( 2\sqrt{D(B-1)} - D + 1 > 1, B \geq 4D, \) and \( D > 1 \). Assume that \((x_1, n_1)\) and \((x_2, n_2)\) are two distinct integer solutions of equation (1.3) satisfying \( x_1 > x_2 \geq 0, p^{n_2} > p^{n_1} \geq 2\sqrt{D(B-1)} - D + 1 \). We have \( p \nmid x_1x_2 \) as \( p \nmid B \). So, \( p \geq 3 \) leads to \( p \nmid \gcd(x_1 + x_2, x_1 - x_2) \). Then, from
\[ D(x_1 + x_2)(x_1 - x_2) = D(x_1^2 - x_2^2) = p^{n_2} - p^{n_1} = p^{n_1}(p^{n_2-n_1} - 1) \]
and \( p \nmid D \), we deduce that \( p^{n_1} | x_1 + x_2 \) or \( p^{n_1} | x_1 - x_2 \). Therefore, we get
\[ 2x_1 - 1 \geq x_1 + x_2 \geq p^{n_1}. \]
This implies that
\[ B - p^{n_1} = Dx_1^2 \geq D \left( \frac{p^{n_1} + 1}{2} \right)^2. \]
Thus, we deduce that
\[ 4BD + 4D + 4 \geq (Dp^{n_1} + D + 2)^2, \]
which yields
\[ p^{n_1} \leq \sqrt{\frac{4B}{D} + \frac{4}{D} + \frac{4}{D^2}} - 1 - \frac{2}{D}. \]
Recall that \( D > 1 \) and \( B \geq 4D \). Thus, we have
\[ 2\sqrt{D(B-1)} - D + 1 = \sqrt{D(B-1)} + \sqrt{D(B-1)} - D + 1 \geq \sqrt{2(B-1)} + \sqrt{D(4D-1)} - D + 1 \]
\[ > \sqrt{2(B-1)} + 2D - 1 - D + 1 \]
\[ = \sqrt{2(B-1)} + D \geq \sqrt{2(B-1)} + 2 \]
and
\[ \sqrt{\frac{4B}{D} + \frac{4}{D} + \frac{4}{D^2}} - 1 - \frac{2}{D} < \sqrt{2B + 3} - 1 < \sqrt{2(B-1)} + 2. \]
This leads to a contradiction and completes the proof of the second claim.
(2) $p|B$, that is $p|B$, but $p^2 \nmid B$. At this level also, we will also study the problem according to the divisibility of $D$ by $p$.

(i) Suppose that $p|D$. Let $D = pD_1$. If $p|D_1$, then $n = 1$ since $p^2 \nmid B$. If $p \nmid D_1$, let $B = pB_1$, then $p \nmid B_1$. It is obvious that $n \geq 1$ and the Diophantine equation (1.3) turns into $D_1x^2 + p^{n_1} = B_1$, with $n_1 = n - 1$. By the result of (1) for $D_1 > 1$ and Theorem 1.2 for $D_1 = 1$, this equation has at most two nonnegative integer solutions $(x, n_1)$, then the Diophantine equation (1.3) has at most two nonnegative integer solutions $(x, n)$.

(ii) Finally, suppose that $p \nmid D$. If $n \geq 2$, then $p|x$ and we get $p^2|B$, which is a contradiction. So we have $n \leq 1$ and then the Diophantine equation (1.3) has at most two nonnegative integer solutions $(x, n)$.

The last preliminary result deals with the case $p = 2$. The proof will follow the line of that of Lemma 2.3. But for the sake of completeness, we will give some details.

**Lemma 2.4.** Let $B, D$ be positive integers with $4 \nmid B$, $B \geq 4D$, $D \neq 1$ when $B$ is odd and $D \neq 2$ when $B$ is even. Then, the Diophantine equation

$$Dx^2 + 2^n = B$$

(2.6)

has at most two nonnegative integer solutions $(x, n)$.

**Proof.** We will also consider two cases. (1) $2 \nmid B$, then $D > 1$ since $D \neq 1$. Here will also distinguish two cases according to the parity of $D$.

(i) If $2|D$, then $n$ can only take the value 0 since $2 \nmid B$. Therefore, Diophantine equation (1.3) has at most one nonnegative integer solution $(x, n)$.

(ii) If $2 \nmid D$, then we will study the following two claims.

**Claim 1:** There is at most one nonnegative integer solution $(x, n)$ satisfying $2^n < 2\sqrt{D(B - 1)} - D + 1$.

The proof of this claim is similar to that of Lemma 2.3, Claim 1. Then, we leave it to the reader.

**Claim 2:** There is at most one nonnegative integer solution $(x, n)$ satisfying $2^n \geq 2\sqrt{D(B - 1)} - D + 1$.

In this case, we have $n > 0$ since $2\sqrt{D(B - 1)} - D + 1 > 1$, $B \geq 4D$, and $D > 1$. Assume that $(x_1, n_1)$ and $(x_2, n_2)$ are two distinct integer solutions of equation (1.3) satisfying $x_1 > x_2 \geq 0$, $2^{n_2} > 2^{n_1} \geq 2\sqrt{D(B - 1)} - D + 1$. One can see that $2 \nmid x_1x_2$ since $2 \nmid B$. So, we get $2||\gcd(x_1 + x_2, x_1 - x_2)$. Then, from

$$D(x_1 + x_2)(x_1 - x_2) = D(x_1^2 - x_2^2) = 2^n - 2^n = 2^{n_1}(2^{n_2 - n_1} - 1),$$

we deduce that $2^{n_1 - 1}|x_1 + x_2$ or $2^{n_1 - 1}|x_1 - x_2$. Hence, we obtain

$$2x_1 - 2 \geq x_1 + x_2 \geq 2^{n_1 - 1}.$$
This implies that
\[ B - 2^{n_1} = D x_1^2 \geq D (2^{n_1-2} + 1)^2. \]
Thus, we deduce that
\[ BD + 4D + 4 \geq (2^{n_1-2}D + D + 2)^2, \]
which yields
\[ 2^{n_1} \leq 4 \sqrt{\frac{B}{D} + \frac{4}{D^2} + \frac{4}{D^2}} - 4 - \frac{8}{D}. \]
Recall that \( D > 1 \) and \( 2 \nmid D \). We have \( D = 3 \) or \( D \geq 5 \). As \( B \geq 4D \), if \( D = 3 \), then a straightforward calculation shows that
\[ 4 \sqrt{\frac{B}{D} + \frac{4}{D^2} + \frac{4}{D^2}} - 4 - \frac{8}{D} < 2 \sqrt{D(B-1)} - D + 1. \]
For \( D \geq 5 \), we can make a discussion similar that of Lemma 2.3 Claim 2. This leads to a contradiction.

(2) If \( 2 \mid\mid B \), that is \( 2 \mid B \), but \( 4 \nmid B \), then one can use a method similar to that of Lemma 2.3(2) for \( D > 2 \), but the case \( D = 2 \) will leads to \( D_1 = 1 \) which is not handled by Theorem 1.2. If \( D = 1 \) and \( n \geq 2 \), then \( 2 \mid x \), which leads to \( 4 \mid B \), so we have \( n \leq 1 \). We conclude that the Diophantine equation (1.3) has at most two nonnegative integer solutions \((x, n)\) in this case for \( D \neq 2 \).

3. PROOF OF THEOREM 1.3

Let us start the proof by studying some particular cases:

• If \( B < 4D \), then \( x \leq 1 \) and therefore equation (1.3) has at most two nonnegative integer solutions \((x, n)\).

• If \( D = d^2 D_1 \), we can rewrite \( D x^2 \) as \( D_1 (dx)^2 = D_1 z^2 \). If \( D_1 = 1 \), we can use Theorem 1.2, with the exceptional case \( p = 2, 2 \mid\mid B \) by Lemma 2.4.

Therefore, for the remainder of the proof, we assume that \( B \geq 4D \) and \( D > 1 \) squarefree. Moreover, we will consider two cases: \( p^2 \nmid B \) and \( p^2 \mid B \).

Case 1: \( p^2 \nmid B \). Combining Lemma 2.3 and Lemma 2.4, we see that equation (1.3) has at most two nonnegative integer solutions \((x, n)\) in this case.

Case 2: \( p^2 \mid B \). Here also, we will consider two cases according to the divisibility of \( D \) by \( p \).

(i) If \( p \nmid D \), then we will use Lemma 2.2 to prove that equation (1.3) has at most three nonnegative integer solutions \((x, n)\). Assume that \( p^{2k} \mid B \) and \( p^{2(k+1)} \nmid B \). Let \( B = p^{2k} B_0 \). We will prove that there is at most
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one nonnegative integer solution $(x, n)$ satisfying $n < 2k$ and at most two nonnegative integer solutions $(x, n)$ satisfying $n \geq 2k$.

If $(x, n)$ is a nonnegative integer solution of (1.3) with $n < 2k$, then from $Dx^2 + p^n = B = p^{2k}B_0$, we deduce that $2 \mid n$. Put $n = 2m$. Then, $p^n \mid x$. Put $x = p^m z$. Thus, we have

$$Dz^2 + 1 = B_0 p^{2(k-m)},$$

with $k - m = l \geq 1$, i.e.

$$B_0 p^{2l} - Dz^2 = 1.$$

By Lemma 2.2, the above equation has most one positive integer solution $(z, l)$. This means that equation (1.3) has at most one nonnegative integer solution $(x, n)$ satisfying $n < 2k$.

If $n \geq 2k$, then $p^k \mid x$. Put $x = p^k z$, $n = n - 2k$, $B = p^{2k}B_0$. Then, equation (1.3) becomes

$$Dz^2 + p^u = B_0,$$

with $p^2 \nmid B_0$. By Case 1, this equation has at most two nonnegative integer solution $(z, u)$, i.e. equation (1.3) has at most two nonnegative integer solutions $(x, n)$ satisfying $n \geq 2k$.

(ii) If $p \mid D$, then it is obvious that $n \geq 1$. Let $D = pD_1, n_1 = n - 1, B = pB_1$, then $p \nmid D_1$ and equation (1.3) becomes

$$D_1 x^2 + p^{n_1} = B_1.$$

If $p \mid B_1$, then $n_1 \leq 1$, and equation (1.3) has at most two nonnegative integer solutions $(x, n)$. If $p^2 \mid B_1$, then equation (1.3) has at most three nonnegative integer solutions $(x, n)$ for $D_1 = 1$ by Theorem 1.2 and for $D_1 > 1$ by Case 2 (i). This completes the proof of Theorem 1.3.

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