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Borut Zalar, Brigita Ferčec, Yilei Tang and Matej Mencinger

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PARTIAL QUALITATIVE ANALYSIS OF PLANAR $\mathcal{A}_Q$-RICCATI EQUATIONS

BORUT ZALAR, BRIGITA FERČEC, YILEI TANG AND MATEJ MENCINGER*

University of Maribor, Slovenia and Shanghai Jiao Tong University, China

ABSTRACT. If we view the field of complex numbers as a 2-dimensional commutative real algebra, we can consider the differential equation $z' = az^2 + bz + c$ as a particular case of $\mathcal{A}-$ Riccati equations $z' = a \cdot (z \cdot z) + b \cdot z + c$ where $\mathcal{A} = (\mathbb{R}^n, \cdot)$ is a commutative, possibly nonassociative algebra, $a, b, c \in \mathcal{A}$ and $z : I \to \mathcal{A}$ is defined on some nontrivial real interval. In the case $\mathcal{A} = \mathbb{C}$, the nature of (at most two) critical points can be described using purely algebraic conditions involving involution $*$ of $\mathbb{C}$. In the present paper we study the critical points of $\mathcal{L}(\pi)-$ Riccati equations, where $\mathcal{L}(\pi)$ is the limit case of the so-called family of planar Lyapunov algebras, which characterize 2-dimensional homogeneous systems of quadratic ODEs with stable origin. The number of possible critical points is 1, 3 or ∞, depending on coefficients. The nature of critical points is also completely described. Finally, simultaneous stability of the origin is considered for homogeneous quadratic part corresponding to algebras $\mathcal{L}(\theta)$.

1. INTRODUCTION

Let $\vec{x}' = Q(\vec{x})$ denote autonomous system of homogeneous quadratic differential equations in $\mathbb{R}^n$. One possible non-classical way of investigating possible behavior of its solutions, for example the stability of the origin, uses the theory of non-associative algebras. This was first studied by Markus in [16]. He considered $\vec{x}' = Q(\vec{x})$, and naturally associated it to a nonassociative

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*Corresponding author. matej.mencinger@um.si.

commutative algebra $\mathcal{A}_Q = (\mathbb{R}^n, \circ)$ defined by the following algebra multiplication:

$$\vec{x} \circ \vec{y} = \frac{1}{2}\left(Q\left(\vec{x}+\vec{y}\right) - Q\left(\vec{x}\right) - Q\left(\vec{y}\right)\right) \tag{1.1}$$

This approach makes sense because $\vec{x}' = Q_1(\vec{x})$ and $\vec{x}' = Q_2(\vec{x})$ are equivalent systems of ODEs if and only if algebras $\mathcal{A}_{Q_1}$ and $\mathcal{A}_{Q_2}$ are isomorphic. The algebra $\mathcal{A}_Q$ is called the real Markus algebra of the system $\vec{x}' = Q(\vec{x})$, which can be viewed as an algebraic differential equation $z' = z^2$ for $z : I \to \mathcal{A}_Q$ where $I \subset \mathbb{R}$ is some nontrivial interval. A standard monograph on the subject is [28] where details can be found, was written by S. Walcher.

Certain properties of homogeneous quadratic systems became very natural when viewed through Markus lenses. For example, it is a purely algebraic fact that every real finite-dimensional algebra contains either a nonzero element satisfying $p^2 = p$ or a nonzero element satisfying $n^2 = 0$. Since the existence of an element of the former type implies existence of solutions whose formula is given by

$$x(t) = \frac{\delta}{1-\delta t} p \text{ for any } \delta > 0$$
$$x(0) = x_0 = \delta p$$

which blow-up in finite time and whose initial conditions can be arbitrary close to the origin, a homogenous quadratic system $\vec{x}' = Q(\vec{x})$ with a stable origin corresponds to an algebra $\mathcal{A}_Q$ which contains a nonzero nilpotent element of order two. Since $(\alpha n)^2 = \alpha^2 n^2 = 0$ for all real constants α and the constant function $z : (-\infty, +\infty) \to \mathcal{A}_Q$ defined by $z(t) = n$ is obviously a solution of $z' = z^2$, it follows that in every homogenous quadratic systems of ODEs with the stable origin, the origin cannot be an isolated critical point, but in fact lies on a line which consists entirely of critical points.

In planar case, i.e. when $\dim(\mathcal{A}_Q) = 2$, some well-known classical results can be elegantly expressed using Markus approach. One such example was presented in [21] where it was proved that a nontrivial (i.e. $Q \neq 0$) system $\vec{x}' = Q(\vec{x})$ in the real plane has a stable origin if and only if $\mathcal{A}_Q$ is isomorphic to one of the planar *Lyapunov algebras* $\mathcal{L}(\theta)$, where $\theta \in (0, \pi]$. These algebras will be described in Section 2. In [2] Boujemaa, El Qotbi and Rouiouih treated the (in)stability and behavior of the solutions near the critical point away form origin. Some other recent papers concerning similar topics as well as some applications to differential and integral equations are [1], [3], [8], [9], [10], [21], [22], [23].

The purpose of studying known results in $\mathbb{R}^2$ and reformulating them into the language of Markus algebras, is to gain some structural insight and hopefully form sensible conjectures concerning possible new results in higher dimensions. One such example is [18] where 3-dimensional systems with a stable origin and a plane of critical points were successfully classified.

All 2-dimensional real commutative algebras can be classified (see [16, Theorems 6, 7 and 8] for details) in three large groups (those containing a nonzero idempotent, those containing a basis consisting of two nilpotent elements and those containing no idempotents and only one nilpotent line) and further, with respect to their multiplication tables, into 23 parametric families. Let $(\mathcal{A}, \circ)$ be one of 2-dimensional Markus algebras. Since a Markus algebra need not contain an identity element, we will define Riccati equations with respect to $\mathcal{A}$ as the following differential equation:

$$z' = a \circ z^2 + b \circ z + c \text{ for } a, b, c \in \mathcal{A} \text{ and } a \neq 0.$$

Assuming the existence of a singular point, the system can be transformed with a simple linear change of coordinates in such a way that the origin becomes one of singular points and the constant term vanish

$$z' = a \circ z^2 + b \circ z \text{ for } 0 \neq a, b \in \mathcal{A} \text{ and } a \neq 0.$$

In the sequel we call an equation of this type an $\mathcal{A}-$Riccati equation, in order to avoid any possible confusion with the classical use of the term Riccati differential equation.

If we choose any fixed algebraic basis $\{e_1, e_2\}$ of $\mathcal{A}$ and write $z(t) = x(t)e_1 + y(t)e_2$, the planar $\mathcal{A}-$Riccati equation transforms into a familiar systems of ODEs

$$\begin{aligned} \dot{x} &= \alpha_1 x + \beta_1 y + a_1 x^2 + 2b_1 xy + c_1 y^2 \\ \dot{y} &= \alpha_2 x + \beta_2 y + a_2 x^2 + 2b_2 xy + c_2 y^2 \end{aligned} \tag{1.2}$$

for some real parameters $\alpha_{1,2}$, $\beta_{1,2}$, $a_{1,2}$, $b_{1,2}$ and $c_{1,2}$, which were studied by many authors. One good recent survey concerning global behavior of (1.2) is the paper [1] by Artés, Llibre, Schlomiuk and Vulpe. The authors proved that for (1.2) there are 1765 different global geometrical configurations of singularities of quadratic differential systems in the plane. There are other 8 configurations conjectured impossible, all of them related with a single configuration of finite singularities. Another good source of general information is [29].

This system was recently considered for symmetries [8]. Note that if $b \neq 0$ the interesting dynamics occurs if the origin is a nonhyperbolic singular point of (1.2) in which case we have the system

$$\begin{aligned} x' &= -y + q_{11}x^2 + 2q_{12}xy + q_{13}y^2 \\ y' &= x + q_{21}x^2 + 2q_{22}xy + q_{23}y^2 \end{aligned} \quad , \; q_{i,j} \in \mathbb{R} \text{ for } i, j \in \{1, 3\}$$

The singular point at the origin of this system is a center (near the origin all trajectories of the system are ovals) or focus (all trajectories are spirals).

For the singular point at the origin of the planar analytic differential system in the form of a linear center perturbed by higher order terms, i.e.

$$\dot{x} = -y + F(x,y), \qquad \dot{y} = x + G(x,y), \tag{1.3}$$

where F and G are real analytic functions whose series expansions in a neighborhood of the origin start in at least second order terms Poincaré and Lyapunov [13, 22] proved that it is a center if system (1.3) admits a first integral of the form

$$\Phi = x^2 + y^2 + \sum_{k+l\geq 3} \phi_{kl} x^k y^l. \tag{1.4}$$

This is the so-called problem of distinguishing between a center and a focus, or the Poincaré center problem which was studied for the first time in 1908 by Dulac [6] where he has solved it for the case of the quadratic system. Later it was solved for the systems in the form of a linear center perturbed with homogeneous cubic nonlinearities [15], for the so-called Kukles system [14, 24], for some linear centers perturbed with homogeneous polynomials of degree five [5], and for a few other specific families of polynomial systems of ODE's. Although in the general case a first integral of the form (1.4) does not always exist we can always find series of the form (1.4) for which $\dot{\Phi} = \frac{\partial \Phi}{\partial x}(-y + F(x,y)) + \frac{\partial \Phi}{\partial y}(x + G(x,y))$ reduces to

$$\dot{\Phi} = x^2 + y^2 - g_{22} \cdot \left(x^2 + y^2\right)^2 - g_{33} \cdot \left(x^2 + y^2\right)^3 - g_{44} \cdot \left(x^2 + y^2\right)^4 - \cdots, \tag{1.5}$$

where g_{kk} is called $k-$th focus quantity and it is a polynomial in the parameters of the system. By the definition, (1.4) is a first integral of system (1.3) if $\dot{\Phi} \equiv 0$ from which it follows that all focus quantities must be zero. This is one of the tools to study the problem of distinguishing between a center and a focus in polynomial systems of the form (1.3). To find necessary conditions for the existence of a first integral of the form (1.4) for system (1.3) we look for a formal series (1.4) satisfying (1.5). To start the computational process for finding the first several conditions for integrability we write down the initial string of (1.4) up to order N

$$\Phi_N(x,y) = x^2 + y^2 + \sum_{k+l=3}^{N} \phi_{kl} x^k y^l.$$

Then for each $i = 3, \ldots, N$ we equate coefficients of terms of order i in the expression

$$\frac{\partial \Phi_N}{\partial x}(-y + F(x,y)) + \frac{\partial \Phi_N}{\partial y}(x + G(x,y)) \tag{1.6}$$

to zero obtaining systems of linear equations of unknown variables ϕ_{kl}. Then, we look for solutions of the linear systems obtained starting from system that corresponds to $i = 3$. Linear systems corresponding to odd $i = 2\ell - 1$ always have unique solutions. After solving the system we substitute the obtained values of ϕ_{kl} into the linear systems corresponding to $i > 2\ell - 1$. For systems that correspond to even $i = 2\ell$, there is one equation more than the number of variables. After dropping a suitable equation one obtains the system with the unique solution. After solving the system we assign the value 0 for the undefined ϕ_{kl} (with $k + l = 2\ell$) and substitute the obtained values of ϕ_{kl} into the linear systems corresponding to $i > 2\ell$. Next, we evaluate (1.6) with the found ϕ_{kl} ($k + l \leq 2\ell$) and find the coefficient of $(x^2 + y^2)^\ell$ which we denote by $g_{\ell-1\ell-1}$. Computing in this way we obtain a list of polynomials $g_{11}, g_{22}, g_{33}, \ldots$ in the parameters of system (1.3) (see for instance [23] for more details). We will use this approach later to prove the existence of first integral of the certain quadratic system.

Our idea is to study the dynamics of 2-dimensional Riccati equations associated to commutative nonassociative algebras, using the apparatus developed by Markus, Walcher and others, which should eventually lead to elegant algebraic formulations of some known results concerning global dynamics and behavior of solutions near critical points, while our final goal is to use the obtained algebraic insight to formulate and hopefully prove some new results concerning the dynamics of quadratic systems in 3-dimensional space.

2. A simple example

The most obvious case of a planar Markus algebra is the (associative) field $\mathbb{C}$ of complex numbers, viewed as 2-dimensional real algebra. In order to illustrate what kind of algebraic formulation we are looking for in general case, we will write two simple observations in an explicit way. For the sake of reader's convenience we also include some details concerning computations.

Proposition 2.1. *Let $z' = az^2 + bz$, where $a \neq 0$, be a $\mathbb{C}-$Riccati equation. Then one of the following three possibilities must hold:*

(1) the equation has precisely one singular point, which is unstable. Moreover, every neighborhood of this singular point contains initial conditions for which the corresponding solution blows-up in finite time. This happens if and only if $b = 0$.

(2) the equation has precisely two singular points, one of which is a stable focus, while the other one is an unstable focus. This happens if and only if $b \neq 0$ and $b^2 \neq \left(b^2\right)^*$ or $b \neq 0$ and $b = b^*$.

(3) the equation has precisely two singulars points, both of which are centers. This happens if and only if $b \neq 0$ and $b = -b^*$.

Proof. (1) The polynomial $p(z) = az^2 + bz$ has a double zero if and only if $b = 0$. We can rewrite the $\mathcal{A}-$Riccati equation in the form $z' = az^2$ whose solution is

$$z(t) = \begin{cases} 0 & \text{if} \quad z_0 = 0 \\ \left(z_0^{-1} - ta\right)^{-1} & \text{if} \quad z_0 \neq 0 \end{cases}$$

In the second case the solution is defined on $(-\infty,+\infty)$ if az is not real and on $\left(-\infty, a^{-1}z_0^{-1}\right)$ otherwise. Every neighborhood of 0 contains points of the form $z_0 = \varepsilon a^{-1}$ for sufficiently small real ε, for which the solution

$$z(t) = \frac{1}{\varepsilon^{-1} - t} a^{-1}$$

goes to infinity as $t \to \varepsilon^{-1}$.

(2) If the polynomial $p(z)$ has two different zeros, we can rewrite the $\mathcal{A}-$Riccati equation in the form $z' = az(z - z_1)$ where $z_1 \neq z_2 = 0$ are distinct singular points. The solution of this equation is given by

$$z(t) = \begin{cases} z_1 & \text{if} \quad z_0 = z_1 \\ 0 & \text{if} \quad z_0 = 0 \\ z_1 \left(1 - \left(1 - z_1 z_0^{-1}\right) e^{taz_1}\right)^{-1} & \text{if} \quad z_0 \neq z_1 \text{ and } z_0 \neq 0 \end{cases}$$

The corresponding Jacobians are representing multiplications with complex numbers $az_1 = -b$ and $-az_1 = b$ whose real parts have opposite signs. If the real part of b is nonzero, we therefore have one stable and one unstable foci.

(3) If the real part of b is zero then $b = i\varphi$ for some real φ. The solution is of the form $A(1 + Be^{i\varphi t})^{-1}$ for some constants A, B, φ and is therefore periodic, which yields a center. □

3. $\mathcal{A}-$Riccati equations associated with limit Lyapunov algebra

The most natural way to describe Lyapunov algebras is using their complex envelopes. Let $\{p, p^*\}$ be some base of the complex linear space $\mathbb{C}^2$, and the product being defined by $p^2 = p$, $(p^*)^2 = p^*$ and $p \cdot p^* = p^* \cdot p = (e^{i\theta}p + e^{-i\theta}p^*)/2$, where the constant θ satisfies the condition $0 < \theta \leq \pi$. The involution is defined by $(p^*)^* = p$ and extended on $\mathbb{C}^2$ by conjugate-linearity. In this fashion we equip $\mathbb{C}^2$ with the structure of an involutive algebra $\mathcal{C}(\theta) = (\mathbb{C}^2, \cdot, *)$. The Lyapunov algebra $\mathcal{L}(\theta)$ is the self-adjoint part of $\mathcal{C}(\theta)$, i.e. $\mathcal{L}(\theta) = \{x \in \mathcal{C}(\theta) : x^* = x\}$. Since $\mathcal{C}(\theta)$ has dimension 4 over $\mathbb{R}$, the real dimension of $\mathcal{L}(\theta)$ is 2. They are interesting because of the following result (see [21]).

Theorem 3.1 ([21]). *The system of differential equations*

$$\begin{aligned} \dot{x} &= \alpha_1 x^2 + \beta_1 xy + \gamma_1 y^2 \\ \dot{y} &= \alpha_2 x^2 + \beta_2 xy + \gamma_2 y^2 \end{aligned}$$

where at least one of the above coefficients $\alpha_{1,2}$, $\beta_{1,2}$, $\gamma_{1,2}$ is nonzero has a stable origin if and only if its Markus algebra is isomorphic to one of $\mathcal{L}(\theta)$.

Because in the limit case $\theta = \pi$ the multiplication table is somewhat simpler, in the original Markus paper those algebras were listed as two families [16, Theorem 6, families 9 and 10] (for more details, see next section).

Because of Theorem 3.1, from the viewpoint of our final goal, the present paper deals with two natural antipodes. If we consider planar homogeneous

$\mathcal{A}-$Riccati equations, then $\mathcal{A} = \mathbb{C}$ represents the most unstable possibility, while $\mathcal{A} = \mathcal{L}(\pi)$ is the most stable possibility.

It is not difficult to compute that the multiplication table of $\mathcal{L}(\pi)$ can be given by

$$\begin{array}{|c||c|c|} \hline \cdot & n & e \\ \hline \hline n & 0 & e \\ \hline e & e & -n \\ \hline \end{array} . \tag{3.1}$$

with the corresponding simplified $\mathcal{A}-$Riccati equation $z' = az^2 + bz$ defined by

$$\begin{aligned} \dot{x} &= -b_2 y - 2a_2 xy \\ \dot{y} &= b_2 x + b_1 y + 2a_1 xy - a_2 y^2 \end{aligned} \tag{3.2}$$

where $a = a_1 n + a_2 e$ and $b = b_1 n + b_2 e$. In the sequel we will use abbreviation $a = (a_1, a_2)$, $b = (b_1, b_2)$. The involution can be defined by $n^* = n$ and $e^* = -e$. Using the above multiplication table it is easy to compute that $(xy)^* = x^* y^*$ for all $x, y \in \mathcal{L}(\pi)$. Let $x = (x_1, x_2)$, $\mathrm{Re}\,(x) = x_1$ and $\mathrm{Im}\,(x) = x_2$. Note that e is in some sense an equivalent of the imaginary unit for $\mathcal{L}(\pi)$. More precisely, $e\,(ex) = -x$ for all $x \in \mathcal{L}(\pi)$.

In order to interpret our computations in terms of $\mathcal{L}(\pi)$ structural properties we note the following

Observation 1. *(1) Let $x = (x_1, x_2) \in \mathcal{L}(\pi)$. Then $x_1 = 0$ if and only if $x = -x^*$.*

(2) Let $x = (x_1, x_2) \in \mathcal{L}(\pi)$. Then $x_2 = 0$ if and only if $x = x^*$.

(3) Let $x = (x_1, x_2) \in \mathcal{L}(\pi)$. Then $x_2 = 0$ if and only if $x^2 = 0$.

(4) Let $a = (a_1, a_2) \in \mathcal{L}(\pi)$, $b = (b_1, b_2) \in \mathcal{L}(\pi)$. Then $a_2 b_2 = \mathrm{Re}\,(ab^*)$ and $a_1 b_2 - a_2 b_1 = -\mathrm{Im}\,(ab^*)$

(5) Let $b = (b_1, b_2) \in \mathcal{L}(\pi)$. Then $b_1^2 - 4b_2^2 > 0$ if and only if $\Big(\mathrm{Re}((eb)^2)\Big)^2 > \Big(\mathrm{Im}((eb)^2)\Big)^2$.

The proof is a straightforward computation and will be omitted.

Theorem 3.2. *The homogeneous equation $z' = az^2$, where $a \neq 0$ always has an unstable origin.*

Proof. Let $x \circ y = a \cdot (x \cdot y)$ define the multiplication in algebra $\mathcal{A}_\circ$. To check that $\mathcal{A}_\circ$ is not isomorphic to any $\mathcal{L}(\theta)$, one just has to prove that $\mathcal{A}_\circ$ has a nontrivial idempotent (yielding blow-up solutions). Therefore we seek for solutions to $az^2 = z$:

- If $a_1 a_2 \neq 0$ algebra $\mathcal{A}_\circ$ has two linearly independent idempotents $e_1 = -\frac{1}{a_2} e$ and $e_2 = -\frac{1}{8a_1 a_2} n - \frac{1}{2a_2} e$
- If $a_1 = 0$ and $a_2 \neq 0$ algebra $\mathcal{A}_\circ$ has idempotent $e_1 = -\frac{1}{a_2} e$

□

Theorem 3.3. *Let $z' = az^2+bz$, where $a \neq 0$ and $b \neq 0$, be a $\mathcal{L}(\pi)-$Riccati equation. Then one of the following possibilities must hold:*

(1a) if $a^2 \neq 0$, $(\mathrm{Im}\,(ab^*))^2 > 2\,(\mathrm{Re}\,(\ ab^*))^2$, $\mathrm{Re}\left((eb)^2\right)^2 < \mathrm{Im}\left((eb)^2\right)^2$ and $b \neq -b^*$, the equation $z' = az^2 + bz$ has three distinct singular points. Two of them are either a pair of sink and saddle or a pair of source and saddle or a pair sink-source, while the origin is a focus.

(1b) if $a^2 \neq 0$, $(\mathrm{Im}\,(ab^*))^2 > 2\,(\mathrm{Re}\,(ab^*))^2$ and $\mathrm{Re}\left((eb)^2\right)^2 > \mathrm{Im}\left((eb)^2\right)^2$, the equation $z' = az^2+bz$ has three distinct singular points. Two of them them are either a pair of sink and saddle or a pair of source and saddle or a pair sink-source, while the origin is sink (if $\mathrm{Re}\,(b) > 0$) or source (if $\mathrm{Re}\,(b) < 0$).

(1c) if $a^2 \neq 0$ and $(\mathrm{Im}\,(ab^*))^2 > 2\,(\mathrm{Re}\,(ab^*))^2$ and $b = -b^*$, $b \neq b^*$, the equation $z' = az^2 + bz$ has three distinct singular points. Two of them are either two saddles, a saddle and a source or a saddle and a sink or a pair sink-source, while the origin is a center if $a_1a_2 = 0$ and a focus otherwise.

(2a) if $a^2 \neq 0$, $(\mathrm{Im}\,(ab^*))^2 < 2\,(\mathrm{Re}\,(ab^*))^2$, $\mathrm{Re}\left((eb)^2\right)^2 < \mathrm{Im}\left((eb)^2\right)^2$ and $b \neq -b^*$, the equation $z' = az^2 + bz$ has the origin as the only singular point, which is a focus

(2b) if $a^2 \neq 0$, $(\mathrm{Im}\,(ab^*))^2 < 2\,(\mathrm{Re}\,(ab^*))^2$ and $\mathrm{Re}\left((eb)^2\right)^2 > \mathrm{Im}\left((eb)^2\right)^2$, the equation $z' = az^2 + bz$ has the origin as the only singular point, which is either a sink or a source

(2c) if $a^2 \neq 0$ and $(\mathrm{Im}\,(ab^*))^2 < 2\,(\mathrm{Re}\,(ab^*))^2$ and $b = -b^*$, $b \neq b^*$, the equation $z' = az^2 + bz$ has the origin as the only singular point, which is a center, if $a^2 = \left(a^2\right)^*$ and a focus otherwise

(3a) if $a^2 \neq 0$, $b^2 \neq 0$, $(\mathrm{Im}\,(ab^*))^2 = 2\,(\mathrm{Re}\,(ab^*))^2$ the equation $z' = az^2 + bz$ has two isolated singular points. For $\mathrm{Re}\,(b) = 0$ see (1c). For $\mathrm{Re}\,(b) \neq 0$, the origin is hyperbolic and (un)stable, if $\mathrm{Re}\,(b) < 0$ ($\mathrm{Re}\,(b) > 0$), the other one is a semi-hyperbolic saddle.

(3b) if $a^2 \neq 0$, $b^2 = 0$, $(\mathrm{Im}\,(ab^*))^2 = 2\,(\mathrm{Re}\,(ab^*))^2$ and $\mathrm{Re}\,(ab^*)^2 = 0$ then $\mathrm{Im}\,(b) = 0$ and the equation $z' = az^2 + bz$ has a line of semi-hyperbolic saddles containing the origin and an isolated singular point which is either a sink or a source.

(4) if $a^2 = 0$, $b^2 \neq 0$ and $b \neq -b^*$ the only singular point of the equation $z' = az^2 + b$ is the origin, which can be either a focus of sink or source

(5) if $a^2 = 0$, $b^2 \neq 0$ and $b = -b^*$ the only singular point of the equation $z' = az^2 + b$ is the origin. This singular point is a center.

(6a) if $a^2 = 0$, $b^2 = 0$ and $a \neq a^*$, all singular points form two perpendicular lines; one of them passing trough the origin admits infinitely many stable

and unstable singular points, while the other one admits just unstable singular points.

(6b) if $a^2 = 0$, $b^2 = 0$ and $a = a^*$, all singular points form a line passing trough the origin, which admits just stable or just unstable singular points.

Proof. Singular points of $z' = az^2 + bz$ in this case are the solutions to $-b_2y - 2a_2xy = 0$, $b_2x + b_1y + 2a_1xy - a_2y^2 = 0$. Denoting $\Delta = a_1b_2 - b_1a_2 = -\operatorname{Im}(ab^*)$ and $D = \Delta^2 - 2(a_2b_2)^2$ there are at most three possible singular points

$$x_0 = y_0 = 0,\ x_{1,2} = -\frac{b_2}{2a_2}, y_{1,2} = \frac{-\Delta \pm \sqrt{D}}{2a_2^2}. \tag{3.3}$$

The spectra of Jacobian

$$J(x,y) = \begin{bmatrix} -2a_2y & -b_2 - 2a_2x \\ b_2 + 2a_1y & b_1 + 2a_1x - 2a_2y \end{bmatrix}$$

at $(0,0)$, (x_1,y_1) and (x_2,y_2) are

$$\sigma_{J(0,0)} = \left\{ \frac{1}{2}\left(b_1 \pm \sqrt{b_1^2 - 4b_2^2}\right) \right\}, \quad \sigma_{J(x_{1,2},y_{1,2})} = \left\{ \mp\frac{\sqrt{D}}{a_2}, \frac{1}{a_2}\left(\Delta \mp \sqrt{D}\right) \right\}$$

Clearly, if $D > 0$ there are three different singular points. If $b_1^2 - 4b_2^2 < 0$ the origin is a focus. If $b_1^2 - 4b_2^2 > 0$ the origin is a sink (if $b_1 > 0$) or a source (if $b_1 < 0$).

(1a) The origin is clearly a focus. The signs of $\sigma_{J(x_{1,2},y_{1,2})} = \{\lambda_{1,2}, \mu_{1,2}\}$ depend on the sign of Δ and a_2 (see Table 1), yielding (x_1,y_1) and (x_2,y_2) to be either a pair of sink and saddle or a pair of source and saddle if $\Delta \neq 0$ and a pair sink-source, if $\Delta = 0$, as stated.

λ_1	μ_1	λ_2	μ_2	Δ	a_2
$-$	$+$	$+$	$+$	$+$	$+$
$+$	$-$	$-$	$-$	$+$	$-$
$-$	$-$	$+$	$-$	$-$	$+$
$+$	$+$	$-$	$+$	$-$	$-$

Table 1. Signs of elements of spectra of $J(x_{1,2},y_{1,2})$.

(1b) The origin is clearly a sink (if $b_1 > 0$) or a source (if $b_1 < 0$). For nature of (x_1,y_1) and (x_2,y_2) refer to Table 1.

(1c) For (x_1,y_1) and (x_2,y_2) refer to Table 1, if $a_1 \neq 0$. If $a_1 = 0$ we have $\{\lambda_{1,2}, \mu_{1,2}\} = \left\{\mp\frac{1}{a_2}\sqrt{D}, \mp\frac{1}{a_2}\sqrt{D}\right\}$ which corresponds to a pair sink-source, as stated. The origin undergoes a Hopf bifurcation. We can perform change of time $\tau = b_2t$ and system (3.2) becomes

$$\begin{aligned} \dot{x} &= -y - 2A_2xy \\ \dot{y} &= x + 2A_1xy - A_2y^2 \end{aligned} \tag{3.4}$$

where

$$A_1 = \frac{a_1}{b_2},\ A_2 = \frac{a_2}{b_2}.$$

We compute first three focus quantities, briefly explained in the introduction [1] of this system and we obtain

$$\begin{aligned} g_{22} &= \frac{1}{2} A_1 A_2, \\ g_{33} &= \frac{1}{24}(2A_1^3 A_2 - 217 A_1 A_2^3), \\ g_{44} &= \frac{1}{2304}(88A_1^5 A_2 - 14110 A_1^3 A_2^3 + 577111 A_1 A_2^5). \end{aligned}$$

We see that there are two necessary conditions for the existence of a center: $A_1 = 0$ and $A_2 = 0$.

- **For** $A_1 = 0$ the corresponding system is $\dot{x} = -y - 2A_2xy$, $\dot{y} = x - A_2y^2$. This system has an integrating factor $\mu(x, y) = (1 + 2A_2x)^{-2}$ from which we construct first integral

$$\Phi(x,y) = \frac{1}{2(1+2A_2x)} y^2 + \frac{1}{4A_2^2(1+2A_2x)} + \frac{1}{4A_2^2} \ln(1+2A_2x) + C.$$

  Choosing $C = -1/(4A_2^2)$, expanding $\Phi(x, y)$ in a power series and then multiplying it by 2, we obtain first integral of the form (1.4). Hence, by the Poincaré-Lyapunov Theorem system (3.4) admits center at the origin.
- **For** $A_2 = 0$ system (3.2) is of the form $\dot{x} = -y$, $\dot{y} = x + 2A_1xy$. It admits first integral

$$\Phi = \frac{2A_1y - \ln(1+2A_1y)}{2A_1^2} + x^2$$

  which is analytic first integral of the form (1.4).

(2a,2b) The observation follows directly from (3.3) and the spectrum of $J(0,0)$.

(2c) The system is equivalent to (3.4). The center/focus analysis of the origin is done in the proof of (1c).

(3a) If $D = 0$ and $a_2 \neq 0$ there are two isolated singular points: $(0, 0)$ and $y_2 = y_1 = -\frac{\Delta}{2a_2^2}$ (and $x_2 = x_1 = -\frac{b_2}{2a_2}$) with $\sigma_{J(0,0)} = \left\{ \frac{1}{2}\left(b_1 \pm \sqrt{b_1^2 - 4b_2^2}\right)\right\}$ and $\sigma_{J(x_1,y_1)} = \left\{0, \frac{\Delta}{a_2}\right\}$. For case $b_1 = 0$ see (1c). If $b_1 \neq 0$, the origin is hyperbolic and (un)stable, if $b_1 < 0$ ($b_1 > 0$), while (x_1, y_1) is a semi-hyperbolic. If we apply transformation $u = x + \frac{b_2}{2a_2}$ and $v = y + \frac{\Delta}{2a_2^2}$ we arrive

[1] See for example [23] for details.

at

$$u' = \left(\tfrac{a_1 b_2}{a_2} - b_1\right) u - 2a_2 uv$$
$$v' = \left(b_2 + \tfrac{a_1 b_1}{a_2} - \tfrac{a_1^2 b_2}{a_2^2}\right) u + 2a_1 uv - a_2 v^2$$

The normal form of the above system is

$$X' = \frac{-b_1 a_2 + a_1 b_2}{a_2} X + 2a_2\left(-b_2 a_2^2 - a_1 b_1 a_2 + a_1^2 b_2\right) X^2 - 2a_2 XY$$
$$Y' = \left(b_2^2 a_2^5 - a_2 a_1^4 b_2^2 - a_1^2 b_1^2 a_2^3 + 2a_1^3 b_1 a_2^2 b_2\right) X^2 + 2a_1 a_2 \left(a_1 b_2 - b_1 a_2\right) XY - a_2 Y^2$$

and a change of coordinates $X = X$, $Y = XW$ takes it to

$$X' = \frac{(-b_1 a_2 + a_1 b_2)}{a_2} X + h.o.t$$
$$W' = -a_2\left(-b_2^2 a_2^4 + a_1^4 b_2^2 + a_1^2 b_1^2 a_2^2 - 2a_1^3 b_1 a_2 b_2\right) X + \frac{b_1 a_2 - a_1 b_2}{a_2} W + h.o.t$$

with the Jacobian

$$J_{X=0,W=0} = \begin{bmatrix} \frac{(-b_1 a_2 + a_1 b_2)}{a_2} & 0 \\ -a_2\left(-b_2^2 a_2^4 + a_1^4 b_2^2 + a_1^2 b_1^2 a_2^2 - 2a_1^3 b_1 a_2 b_2\right) & \frac{b_1 a_2 - a_1 b_2}{a_2} \end{bmatrix}$$

Eigenvalues of $J_{X=0,W=0}$ are $\lambda_1 = \frac{\Delta}{a_2}, \lambda_2 = -\frac{\Delta}{a_2}$ which means that this singular point is a (semi-hyperbolic) saddle.

(3b) If $a_2 \neq 0$, $b_2 = 0$ then $y = 0$ is a line of singular points. Another singular point $(x, y) = \left(0, \frac{b_1}{a_2}\right)$ is isolated. The eigenvalues of $J\left(0, \frac{b_1}{a_2}\right) = \begin{bmatrix} -2b_1 & 0 \\ 2\frac{a_1}{a_2} b_1 & -b_1 \end{bmatrix}$ are $\lambda_1 = -2b_1, \lambda_2 = -b_1$, real and of the same sign, yielding a sink or source, since $b_1 \neq 0$. The corresponding system $x' = -2a_2 xy$, $y' = b_1 y + 2a_1 xy - a_2 y^2$ contains also a line of semi-hyperbolic singular points. We can verify this by the change of coordinates $x = yX$, $y = Y$. This yields a system

$$X' = -X\left(b_1 + 2a_1 YX + a_2 Y\right)$$
$$Y' = b_1 Y + 2a_1 Y^2 X - a_2 Y^2$$

whose Jacobian at point $(X, Y) = (0, 0)$ (i.e. $(x, y) = \left(0, \frac{b_1}{a_2}\right)$) is

$$J = \begin{bmatrix} -b_1 & 0 \\ 0 & b_1 \end{bmatrix}$$

yielding a (semi-hyperbolic) saddle, as stated.

(4) System (3.2) takes the form $\dot{x} = -b_2 y$, $\dot{y} = b_2 x + b_1 y + 2a_1 xy$. The only singular point $(0, 0)$ is either a focus, if $b_1^2 < 4b_2^2$, otherwise, if $b_1 < 0$ it is a sink or a source (if $b_1 > 0$); see $\sigma_{J(0,0)}$ and (3.3).

(5) By change of time $\tau = b_2 t$ and denoting $A_1 = \frac{a_1}{b_2}$ corresponding system $\dot{x} = -b_2 y$, $\dot{y} = b_2 x + 2a_1 xy$ is equivalent to $x' = -y$, $y' = x + 2A_1 xy$, which admits a non-hyperbolic singular point at origin. The corresponding

first integral is $\Phi(x,y) = 2A_1^2x^2 + 2A_1y - \ln(1+2A_1y)$. Note that $2A_1y - \ln(1+2A_1y) = 2A_1^2y^2 - \frac{8}{3}A_1^3y^3 + O(y^4)$. Thus, expanding $\Phi(x,y)$ in a power series and dividing it by $2A_1^2$, we obtain first integral of the form (1.4). Hence, by the Poincaré-Lyapunov Theorem system $x' = -y$, $y' = x + 2A_1xy$ admits a center at the origin.

(6a) The corresponding system $\dot{x} = 0$, $\dot{y} = b_1y + 2a_1xy$ is linear. Obviously, the lines $y = 0$ and $x = -\frac{b_1}{2a_1} \neq 0$ are two (perpendicular) lines of singular points. Singular point $x_0 = -\frac{b_1}{2a_1}$, $y_0 = 0$ always splits the line $y = 0$ into two half-lines; one of them containing only stable singular point, while the other only unstable singular points. To observe that all points on the line $x = -\frac{b_1}{2a_1}$ are unstable, take (small) $\varepsilon > 0$ and consider $\dot{y} = (b_1 + 2a_1(x_0 \pm \varepsilon))y$, $y(0) = y_0$ to observe $\dot{y} = \pm\epsilon y$, $y(0) = y_0$, for some (small) $\epsilon > 0$, yielding instability, as stated.

(6b) The corresponding system $\dot{x} = 0$, $\dot{y} = b_1y$ is linear. The line $y = 0$ admits all singular points. For $(b_1 > 0)$ $b_1 < 0$ all singularities are (un)stable, as stated. □

4. Analysis of simultaneous stability of the origin

Let $\mathcal{X}$ be a quadratic dynamical system. By *simultaneous stability*, it is meant the stability of the origin for both $\mathcal{X}$ and $-\mathcal{X}$. It is known that if we write $\mathcal{X} = \mathcal{X}_L + \mathcal{X}_H$, where $\mathcal{X}_L$ and $\mathcal{X}_H$ are respectively the linear and the homogeneous quadratic part, and if $\mathcal{X}_L = 0$, the origin is stable if and only if, up to linear equivalence, $\mathcal{X}$ is

$$\text{either } \mathcal{X}_{H,1} = \begin{cases} \dot{x} = -y^2 \\ \dot{y} = 2xy \end{cases} \text{ or } \mathcal{X}_{H,k} = \begin{cases} \dot{x} = ky^2 \\ \dot{y} = 2xy + y^2 \end{cases} \quad k < -\tfrac{1}{8} \text{ is a scalar.}$$

Moreover, if $\mathcal{X}_L = 0$, the origin is stable for both $\mathcal{X}$ and $-\mathcal{X}$.

Note that algebras corresponding to systems $\mathcal{X}_{H,1}$ and $\mathcal{X}_{H,k}$ are isomorphic to algebras $\mathcal{L}(\theta)$ for $\theta \in (0,\pi]$. In the limit case $\theta = \pi$ algebra $\mathcal{L}(\theta)$ is isomorphic to algebras corresponding to $\mathcal{X}_{H,1}$, while for $\theta \in (0,\pi)$ algebras $\mathcal{L}(\theta)$ are is isomorphic to algebras corresponding to $\mathcal{X}_{H,k}$. The relation between parameters θ and k is (see [21, p. 10]) described by $k : (0,\pi) \to (-\infty, -1/8)$ defined by $k(\theta) = -1/(8\cos^2\frac{\theta}{2})$, which is clearly a bijective function. The real form systems $\dot{X} = Q_{1,\theta}(X,Y)$, $\dot{Y} = Q_{2,\theta}(X,Y)$ corresponding to algebras $\mathcal{L}(\theta)$ are obtained by taking the real basis $h_1 = p + p^*$ and $h_2 = i(p - p^*)$. The linear equivalence between $\dot{X} = Q_{1,\theta}(X,Y)$, $\dot{Y} = Q_{2,\theta}(X,Y)$ and systems $\mathcal{X}_{H,k}$ is the following (see [21, p. 10]) $X = x + \frac{1}{2}y$, $Y = -\left(\cos\frac{1}{2}\theta\right)x - \frac{1}{2}\left(\cot\frac{1}{2}\theta\right)y$.

Using the classic results we will handle the problem of simultaneous stability for quadratic systems with $\mathcal{X}_{H,\theta}$, i.e. to $\mathcal{X}_H$ corresponding to algebras $\mathcal{L}(\theta)$. Treating (simultaneous) stability of $\mathcal{L}(\theta)-$Riccati equation is a problem for the future work.

If $\mathcal{X}_L$ is no longer the trivial linear map, classical results show that the origin is unstable for $\mathcal{X}$ or $-\mathcal{X}$ if $\mathcal{X}_L$ has a nonzero real eigenvalue or a complex eigenvalue with a nonzero real part.
When, $\mathcal{X}_L$ has zero as an eigenvalue with multiplicity two, then, in a convenient basis it can be written

$$\mathcal{X}_{L_1} = \begin{cases} \dot{x} = y \\ \dot{y} = 0 \end{cases}$$

The topological type of $\mathcal{X} = \mathcal{X}_{L_1} + \mathcal{X}_H$ has been studied for a large subfamily already in [27] (we recover the well-known cusp) and recently also in [1], and in all cases, the origin is unstable either for $\mathcal{X}$ or for $-\mathcal{X}$, yielding simultaneous instability of the origin in this case.

The remaining case occurs when $\mathcal{X}_L$ has two purely conjugate complex eigenvalues. In this case, up to an isomorphism of the corresponding algebras, $\mathcal{X}_L$ writes

$$\mathcal{X}_{L_2} = \begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

In their paper on limit cycles, Bautin and Frommer [29] gave necessary and sufficient conditions for which $\mathcal{X} = \mathcal{X}_{L_2} + \mathcal{X}_H$ is center and consequently, the origin is stable for both $\mathcal{X}$ and $-\mathcal{X}$. We recall this very useful theorem.

THEOREM 4.1 ([27]). *Suppose a quadratic dynamical system has a form*

$$\begin{aligned} \dot{x} &= -y - bx^2 - (2c+\beta)xy - dy^2 \\ \dot{y} &= x + ax^2 + (2b+\alpha)xy + cy^2 \end{aligned}$$

Then the origin is a center if and only if one of the following conditions holds:

1. $\alpha = \beta = 0$,
2. $a + c = b + d = 0$,
3. $a = c = \beta = 0$ *(or* $b = d = \alpha = 0$*)*,
4. $a + c = \beta = \alpha + 5(b+d) = bd + 2d^2 + a^2 = 0$, *with* $a + c \neq 0$,
5. $\frac{\alpha}{\beta} = \frac{b+d}{a+c} = k$, *with* $ak^3 - (3b+\alpha)k^2 + (3c+\beta)k - d = 0$.

When it is not a center, the origin is an unstable focus either for $\mathcal{X}$ or for $-\mathcal{X}$.

In what follows, we consider the following question for systems $\mathcal{X}_{H,\theta}$ corresponding to algebras $\mathcal{L}(\theta)$: If we consider the homogeneous quadratic dynamical systems $\mathcal{X}_H = \mathcal{X}_{H,1}$ and $\mathcal{X}_H = \mathcal{X}_{H,k}$, and if we add a linear part having two purely imaginary conjugate complex eigenvalues, $\mathcal{X}_L = \mathcal{X}_{L_2}$, when will we obtain a center? We partially answer this question by emphasizing two preliminary examples. The results below give an example of simultaneous stability for both the homogeneous quadratic part and the associated linear part.

Lemma 4.2. *The origin is a center for the following quadratic system*

$$\mathcal{X}_{L_2} + \mathcal{X}_{H,1} = \begin{cases} \dot{x} = -y - y^2 \\ \dot{y} = x + 2xy \end{cases}$$

Proof. The conclusion follows directly from Theorem 4.1 for $a = c = \beta = 0$ (case 3). □

Lemma 4.3. *Consider the quadratic dynamical system*

$$\mathcal{X} = \mathcal{X}_{L_2} + \mathcal{X}_{H,k} = \begin{cases} \dot{x} = -y + ky^2 \\ \dot{y} = x + 2xy + y^2 \end{cases} \quad \text{with } k < -\frac{1}{8}$$

then the origin is not simultaneously stable.

Proof. According to the classification theorem 4.1, $\mathcal{X}$ is not a center for any value of k: the origin is a focus either for $\mathcal{X}$ or for $-\mathcal{X}$, thus we have not obtained the simultaneous stability. □

A direct corollary of the above lemmas is the following

Theorem 4.4. *Let $\mathcal{X} = \mathcal{X}_{L_2} + \mathcal{X}_{H,\theta}$ then the origin is simultaneously stable if and only if $\mathcal{X}_H = \mathcal{X}_{H,1}$; i.e. in the limit case $\mathcal{L}(\pi)$.*

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B. Zalar
University of Maribor
Faculty of civil engineering, Transportation engineering and architecture
Smetanova 17, 2000 Maribor
Slovenia
E-mail: `borut.zalar@um.si`

B. Ferčec
University of Maribor
Faculty of energy technology
Hočevarjev trg 1, 8270 Krško
and
Center for applied mathematics and theoretical physics
University Of Maribor
Mladinska 3, 2000, Maribor
Slovenia.
E-mail: `brigita.fercec@um.si`

Y. Tang
School of mathematical sciences
Shanghai Jiao Tong University
800 Dongchuan Road, Minhang District Shanghai, 200240
China
E-mail: `mathtyl@sjtu.edu.cn`

M. Mencinger
University of Maribor
Faculty of civil engineering, transportation engineering and architecture
Smetanova 17, 2000 Maribor
Slovenia
and
Institute of mathematics, physics and mechanics
Jadranska 19, 1000 Ljubljana
Slovenia
and
Center for applied mathematics and theoretical physics
University of Maribor
Mladinska 3, 2000 Maribor
Slovenia
E-mail: `matej.mencinger@um.si`