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# A LOCAL LIMIT THEOREM FOR COEFFICIENTS OF MODIFIED BORWEIN'S METHOD

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ABSTRACT. The paper extends the study of the modified Borwein method for the calculation of the Riemann zeta-function. It presents an alternative perspective on the proof of a local limit theorem for coefficients of the method. The new approach is based on the connection with the limit theorem applied to asymptotic enumeration.

## 1. INTRODUCTION

In [1] we introduced a modification of Borwein's method for the calculation of the Riemann zeta-function and proposed an asymptotic expression for the coefficients of the method. The asymptotic modification of the algorithm proved to be more than three times faster than the original one [1]. Borwein's method for calculating Riemann zeta-function is based on the alternating series convergence [4]. It applies to complex numbers  $s = \sigma + it$  with  $\sigma \geq 1/2$ .

Let

$$d_{nk} = n \sum_{i=0}^k \frac{(n+i-1)!4^i}{(n-i)!(2i)!}, \quad n \in \mathbb{N}, \quad 0 \leq k \leq n,$$

then the Riemann zeta-function

$$\zeta(s) = \frac{1}{d_{nn}(1-2^{1-s})} \sum_{k=0}^{n-1} \frac{(-1)^k (d_{nn} - d_{nk})}{(k+1)^s} + \gamma_n(s),$$

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here

$$|\gamma_n(s)| \leq \frac{3}{(3 + \sqrt{8})^n} \frac{(1 + 2|t|)e^{\frac{\pi|t|}{2}}}{|1 - 2^{1-s}|}.$$

It is challenging to compute coefficients  $d_{nk}$  for large  $n$  directly (because of factorials in the definition). Therefore we have introduced a modification of the method. Let  $c_{nk} = 1 - d_{nk}/d_{nn}$ ,  $0 \leq k \leq n - 1$ . Now

$$\zeta(s) = \sum_{k=0}^{n-1} \frac{(-1)^k c_{nk}}{(k+1)^s} + \gamma_n(s).$$

Let

$$(1.1) \quad u_{nk} = n \frac{(n+k-1)!4^k}{(n-k)!(2k)!}, \quad n \in \mathbb{N}, \quad 0 \leq k \leq n.$$

Now we can calculate  $d_{nk}$  recurrently, i.e.  $d_{nk} = d_{n,k-1} + u_{nk}$ ,  $d_{n0} = 1$ , and

$$c_{nk} = 1 - \sum_{i=0}^k a_{ni},$$

here

$$(1.2) \quad a_{nk} = \frac{u_{nk}}{\sum_{i=0}^n u_{ni}}.$$

In [1] we received a local limit theorem for coefficients of modified Borwein's method. Note that throughout the paper, all limits, whenever unspecified, will be taken as  $n \rightarrow \infty$ .

**THEOREM 1.1.** (I. Belovas, L. Sakalauskas [1]) *Let  $\mu_n = \frac{n}{\sqrt{2}}$ ,  $\sigma_n = \frac{\sqrt{n}}{2^{3/2}}$ . Numbers  $a_{nk}$  satisfy a local limit theorem*

$$\lim_{n \rightarrow \infty} \sup_k |a_{nk} - \varphi_{\mu_n, \sigma_n}(k)| = 0,$$

where  $\varphi_{\mu, \sigma}(x)$  is the probability density function of the normal distribution with the mean  $\mu$  and the standard deviation  $\sigma$ .

The theorem was proved in a "straightforward" way, using Stirling's formula. However, alternative perspective reveals the connection with combinatorial numbers and calls for application of the results of asymptotic enumeration theory [6]. We will use a general local limit theorem by E. A. Bender, based on the nature of the generating function  $\sum u_{nk} z^n w^k$ .

**THEOREM 1.2.** (E. A. Bender [3]) *Let  $f(z, w)$  have a power series expansion*

$$(1.3) \quad f(z, w) = \sum_{n, k \geq 0} u_{nk} z^n w^k$$

with non-negative coefficients and let  $a < b$  be real numbers. Define

$$R(\varepsilon) = \{z : a \leq \Re z \leq b, \quad |\Im z| \leq \varepsilon\}.$$

Suppose there exists  $\varepsilon > 0, \delta > 0$ , a non-negative integer  $m$ , and functions  $A(s), r(s)$  such that

- (i) an  $A(s)$  is continuous and non-zero for  $s \in R(\varepsilon)$ ,
- (ii) an  $r(s)$  is non-zero and has a bounded third derivative for  $s \in R(\varepsilon)$ ,
- (iii) for  $s \in R(\varepsilon)$  and  $|z| \leq |r(s)|(1 + \delta)$  function

$$(1.4) \quad \left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)}$$

is analytic and bounded,

- (iv)  $(r'(\alpha)/r(\alpha))^2 - r''(\alpha)/r(\alpha) \neq 0$  for  $a \leq \alpha \leq b$ ,
- (v)  $f(z, e^s)$  is analytic and bounded for

$$|z| \leq |r(\Re s)|(1 + \delta), \quad \varepsilon \leq |\Im s| \leq \pi.$$

Then we have

$$(1.5) \quad u_{nk} \sim \frac{n^m e^{-\alpha k} A(\alpha)}{m! r^n(\alpha) \vartheta_\alpha \sqrt{2\pi n}}$$

uniformly for  $a \leq \alpha \leq b$ , where

$$(1.6) \quad \frac{k}{n} = -\frac{r'(\alpha)}{r(\alpha)}, \quad \vartheta_\alpha = \left(\frac{k}{n}\right)^2 - \frac{r''(\alpha)}{r(\alpha)}.$$

## 2. LOCAL LIMIT THEOREM FOR THE COEFFICIENTS $u_{nk}$

First, we prove an auxiliary lemma, identifying the generating function (1.3) of coefficients  $u_{nk}$  (1.1).

LEMMA 2.1. *Suppose that*

$$(2.7) \quad u_{nk} = \begin{cases} 1 & n = k = 0, \\ 0 & k > n, \\ n \frac{(n+k-1)! 4^k}{(n-k)!(2k)!} & \text{otherwise,} \end{cases}$$

then the generating function

$$(2.8) \quad \sum_{n,k \geq 0} u_{nk} x^n y^k = \frac{1}{2} \left( 1 + \frac{1}{2x^{-1}\Theta(y) - 1} - \frac{1}{2x\Theta(y) - 1} \right).$$

Here

$$\Theta(y) = y + \sqrt{y + y^2} + 1/2.$$

PROOF. By definition (2.7), we have the recurrent expression

$$(2.9) \quad u_{nk} = u_{n,k-1} \frac{4(n+k-1)(n-k+1)}{(2k-1)(2k)}.$$

Let us consider the generating function (2.8),

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} u_{nk} x^n y^k.$$

Taking into account that  $u_{n0} = 1$  and (2.9), we obtain the expression

$$(2.10) \quad \begin{aligned} f(x, y) &= \sum_{n=0}^{\infty} u_{n0} x^n + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} u_{n,k-1} \frac{4(n+k-1)(n-k+1)}{(2k-1)(2k)} x^n y^k = \\ &= \frac{1}{1-x} + 4 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} u_{n,k} \frac{(n+k)(n-k)}{(2k+1)(2k+2)} x^n y^{k+1}, \end{aligned}$$

yielding the integral equation

$$\begin{aligned} f(x, y) &= \frac{1}{1-x} - yf(x, y) + \frac{3}{2} \int_0^y f(x, t) dt + \\ &+ \int_0^{\sqrt{y}} \int_0^u 4x f_x(x, t^2) + 4x^2 f_{xx}(x, t^2) - f(x, t^2) dt du. \end{aligned}$$

It gives us the linear partial differential equation of the second order,

$$x^2 f_{xx} - (y + y^2) f_{yy} + x f_x - (1/2 + y) f_y = 0.$$

Note that, in view of (2.10), we have initial conditions

$$f(x, 0) = \frac{1}{1-x}, \quad f_y(x, 0) = 0.$$

Solving the equation (e.g., by the method of characteristics), we obtain

$$f(x, y) = \frac{1}{2} \left( 1 + \frac{1}{2x^{-1}\Theta(y) - 1} - \frac{1}{2x\Theta(y) - 1} \right),$$

which yields us the statement of the lemma.  $\square$

Now we can proceed with the local limit theorem for coefficients  $u_{nk}$  (2.7).

THEOREM 2.2. *Let*

$$(2.11) \quad \mu_n = \frac{n}{\sqrt{2}}, \quad \sigma_n^2 = \frac{n\sqrt{2}}{8},$$

*then for all  $k$ , such that*

$$(2.12) \quad |k - \mu_n| = o(\sigma_n^{4/3}),$$

*we have*

$$u_{nk} \sim \frac{(1 + \sqrt{2})^{2n}}{2\sqrt{2\pi}\sigma_n} \exp\left(-\frac{(k - \mu_n)^2}{2\sigma_n^2}\right).$$

PROOF. By Lemma 2.1, the generating function

$$\begin{aligned} f(z, e^s) &= \frac{1}{2} \left( 1 + \frac{1}{2z^{-1}\Theta(s) - 1} - \frac{1}{2z\Theta(s) - 1} \right) = \\ &= \frac{4\Theta^2(s) - 4z^{-1}\Theta(s) + 1}{2(2z^{-1}\Theta(s) - 1)(2z\Theta(s) - 1)}, \end{aligned}$$

where we write  $\Theta(s)$  in place of  $\Theta(e^s)$ .

Let  $r(s)$  (cf. Theorem 1.2) be a root of the function

$$h(z, e^s) = (2z^{-1}\Theta(s) - 1)(2z\Theta(s) - 1).$$

This function has two roots,  $z_1 = 2\Theta(s)$  and  $z_2 = (2\Theta(s))^{-1}$ . Let us denote

$$(2.13) \quad r_1(s) = 2\Theta(s), \quad r_2(s) = \frac{1}{2\Theta(s)}.$$

Calculating derivatives, we obtain

$$\frac{r_1'(0)}{r_1(0)} = \frac{1}{\sqrt{2}} > 0, \quad \frac{r_2'(0)}{r_2(0)} = -\frac{1}{\sqrt{2}} < 0.$$

By Theorem 1 (Bender [3]), the mean  $\mu_n = n\mu$  and  $\mu = -r'(0)/r(0)$ . Note that by definitions (1.1)-(1.2), numbers  $u_{nk}$  and  $a_{nk}$  are positive. Thus, to obtain positive  $\mu$ , we choose the root  $r_2(s)$ , corresponding the negative ratio. Hence, by (2.13), we have

$$(2.14) \quad r(s) = r_2(s) = \frac{1}{2\Theta(s)} = \frac{1}{2(e^s + \sqrt{e^s + e^{2s} + 1/2})}.$$

Thus,

$$(2.15) \quad \frac{r'(s)}{r(s)} = -\sqrt{\frac{e^s}{e^s + 1}}, \quad \frac{r'(0)}{r(0)} = -\frac{1}{\sqrt{2}},$$

and

$$(2.16) \quad \frac{r''(s)}{r(s)} = \frac{e^s}{e^s + 1} - \frac{1}{2} \sqrt{\frac{e^s}{(e^s + 1)^3}}, \quad \frac{r''(0)}{r(0)} = \frac{1}{2} - \frac{\sqrt{2}}{8}.$$

Next, consider the function  $A(s)$  (cf. (1.4) of Theorem 1.2) as the limit

$$A(s) = \lim_{z \rightarrow r(s)} f(z, e^s) \left( 1 - \frac{z}{r(s)} \right)^{m+1}.$$

Here  $m + 1$  is the order of the pole. Note that, if the pole is simple, then  $m = 0$ . Calculating  $A(s)$  we obtain

$$\begin{aligned} A(s) &= \lim_{z \rightarrow r(s)} \frac{1}{2} \left( 1 + \frac{1}{2z^{-1}\Theta(s) - 1} - \frac{1}{2z\Theta(s) - 1} \right) \left( 1 - \frac{z}{r(s)} \right) = \\ &= \lim_{z \rightarrow r(s)} \frac{1}{2} \left( 1 + \frac{1}{z^{-1}r^{-1}(s) - 1} - \frac{1}{zr^{-1}(s) - 1} \right) \left( 1 - \frac{z}{r(s)} \right) = \frac{1}{2}. \end{aligned}$$

The function (1.4)

$$\left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)} = \frac{\Theta(s)}{2\Theta(s) - z}$$

is analytic and bounded for

$$|s| < \varepsilon, \quad |z| < |r(0)| + \delta = 3 - 2\sqrt{2} + \delta.$$

Thus, conditions (i)-(iii) and (v) of Theorem 1.2 are satisfied. To verify the condition (iv), we must calculate the expression  $(r'(\alpha)/r(\alpha))^2 - r''(\alpha)/r(\alpha)$ . By (2.15) and (2.16) we have

$$\left(\frac{r'(\alpha)}{r(\alpha)}\right)^2 - \frac{r''(\alpha)}{r(\alpha)} = \frac{1}{2} \sqrt{\frac{e^\alpha}{(e^\alpha + 1)^3}} \neq 0.$$

We obtain the parameter  $\alpha$  by solving the equation

$$\frac{k}{n} = -\frac{r'(\alpha)}{r(\alpha)}.$$

Using (2.15) we get

$$\frac{k}{n} = \frac{1}{\sqrt{1 + e^{-\alpha}}}.$$

Hence,

$$e^\alpha = \frac{k^2}{n^2 - k^2}.$$

Next (cf. (1.6) and (2.14)),

$$\vartheta_\alpha^2 = \frac{1}{2} \sqrt{\frac{e^\alpha}{(e^\alpha + 1)^3}} = \frac{1}{2e^\alpha} \left(\frac{e^\alpha}{e^\alpha + 1}\right)^{3/2},$$

$$r^n(\alpha) = (2(e^\alpha + \sqrt{e^\alpha + e^{2\alpha}} + 1/2))^{-n}.$$

Now we can calculate (1.5) of Theorem 1.2,

$$\begin{aligned} u_{nk} &\sim \frac{e^{-\alpha k \frac{1}{2}}}{r^n(\alpha) \vartheta_\alpha \sqrt{2\pi n}} = \\ &= \frac{(2(e^\alpha + \sqrt{e^\alpha + e^{2\alpha}} + 1/2))^n}{2\sqrt{\pi n} e^{\alpha(k-1/2)} \left(\frac{e^\alpha}{e^\alpha + 1}\right)^{3/4}} = \frac{\left(\frac{n+k}{n-k}\right)^n \left(\frac{k}{n}\right)^{-3/2}}{2\sqrt{\pi n} \left(\frac{k^2}{n^2 - k^2}\right)^{k-1/2}} = \\ (2.17) \quad &= \frac{(1 + \sqrt{2})^{2n} \sqrt[4]{2}}{2\sqrt{2\pi} \sigma_n} \frac{\left(\frac{1 + \frac{k}{n}}{1 - \frac{k}{n}}\right)^n}{2(1 + \sqrt{2})^{2n}} \frac{\left(\left(1 - \frac{k}{n}\right)\left(1 + \frac{k}{n}\right)\right)^{k-1/2}}{\left(\frac{k}{n}\right)^{2k+1/2}} = \\ &= \frac{(1 + \sqrt{2})^{2n}}{2\sqrt{2\pi} \sigma_n} \frac{\sqrt[4]{2}/2}{\underbrace{\sqrt{\left(1 - \frac{k}{n}\right)\left(1 + \frac{k}{n}\right)\frac{k}{n}}}_{=\theta_{nk}}} \frac{\left(1 + \frac{k}{n}\right)^{n+k} \left(1 - \frac{k}{n}\right)^{-n+k}}{\underbrace{(1 + \sqrt{2})^{2n} \left(\frac{k}{n}\right)^{2k}}_{=\delta_{nk}}}. \end{aligned}$$

Note that by (2.11) and (2.12), we have

$$(2.18) \quad \left| \frac{k}{n} - \frac{1}{\sqrt{2}} \right| = o\left(\frac{1}{\sqrt[3]{n}}\right),$$

hence  $k/n \rightarrow 1/\sqrt{2}$ , while  $n \rightarrow \infty$ . Thus,  $\theta_{nk} \rightarrow 1$ .

Let us denote

$$x = \frac{k - \mu_n}{\sigma_n}.$$

By (2.11), we have

$$\frac{k}{n} = \frac{1}{\sqrt{2}} + \frac{x}{2\sqrt[4]{2}\sqrt{n}},$$

and by (2.18), we have

$$(2.19) \quad |x| = o(\sqrt[3]{n}).$$

Calculating the logarithm of  $\delta_{nk}$  (2.17), we get

$$\begin{aligned} \log \delta_{nk} &= -2n \log(1 + \sqrt{2}) - \left( n\sqrt{2} + \frac{x\sqrt{n}}{\sqrt[4]{2}} \right) \log \left( \frac{1}{\sqrt{2}} + \frac{x}{2\sqrt[4]{2}\sqrt{n}} \right) + \\ &+ \left( n + \frac{n}{\sqrt{2}} + \frac{x\sqrt{n}}{2\sqrt[4]{2}} \right) \log \left( 1 + \frac{1}{\sqrt{2}} + \frac{x}{2\sqrt[4]{2}\sqrt{n}} \right) + \\ &+ \left( -n + \frac{n}{\sqrt{2}} + \frac{x\sqrt{n}}{2\sqrt[4]{2}} \right) \log \left( 1 - \frac{1}{\sqrt{2}} - \frac{x}{2\sqrt[4]{2}\sqrt{n}} \right) = \\ &= -2n \log(1 + \sqrt{2}) - \\ &- \left( n\sqrt{2} + \frac{x\sqrt{n}}{\sqrt[4]{2}} \right) \left( \log \frac{1}{\sqrt{2}} + \log \left( 1 + \frac{x}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} \right) \right) + \\ &+ \left( \frac{1 + \sqrt{2}}{\sqrt{2}}n + \frac{x\sqrt{n}}{2\sqrt[4]{2}} \right) \left( \log \frac{\sqrt{2} + 1}{\sqrt{2}} + \log \left( 1 + \frac{x(\sqrt{2} - 1)}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} \right) \right) + \\ &+ \left( \frac{1 - \sqrt{2}}{\sqrt{2}}n + \frac{x\sqrt{n}}{2\sqrt[4]{2}} \right) \left( \log \frac{\sqrt{2} - 1}{\sqrt{2}} + \log \left( 1 - \frac{x(\sqrt{2} + 1)}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} \right) \right). \end{aligned}$$



Using Taylor series expansions for logarithms, we obtain for large enough  $n$ ,

$$\begin{aligned} \log \delta_{nk} = & -2n \log(1 + \sqrt{2}) + \left(n\sqrt{2} + \frac{x\sqrt{n}}{\sqrt[4]{2}}\right) \left(\frac{1}{2} \log 2 - \right. \\ & - \frac{x}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} + \frac{x^2}{4\sqrt{2}n} + O\left(\frac{x^3}{n\sqrt{n}}\right) \left. + \left(\frac{1 + \sqrt{2}}{\sqrt{2}}n + \right.\right. \\ & \left. \left. + \frac{x\sqrt{n}}{2\sqrt[4]{2}}\right) \left(\log \frac{\sqrt{2} + 1}{\sqrt{2}} + \frac{x(\sqrt{2} - 1)}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} - \frac{x^2(\sqrt{2} - 1)^2}{4\sqrt{2}n} + \right.\right. \\ & \left. \left. + O\left(\frac{x^3}{n\sqrt{n}}\right) + \left(\frac{1 - \sqrt{2}}{\sqrt{2}}n + \frac{x\sqrt{n}}{2\sqrt[4]{2}}\right) \left(\log \frac{\sqrt{2} - 1}{\sqrt{2}} - \right.\right. \\ & \left. \left. - \frac{x(\sqrt{2} + 1)}{\sqrt{2}\sqrt[4]{2}\sqrt{n}} - \frac{x^2(\sqrt{2} + 1)^2}{4\sqrt{2}n} + O\left(\frac{x^3}{n\sqrt{n}}\right)\right). \end{aligned}$$

By multiplying factors and combining like terms, we obtain

$$\log \delta_{nk} = -\frac{x^2}{2} + O\left(\frac{x^3}{\sqrt{n}}\right),$$

which, combined with (2.17) and (2.19), yields us the statement of the theorem.  $\square$

REMARK 2.3. Theorem 2.2 yields us the asymptotic equivalence

$$\sum_{k=0}^n u_{nk} \sim \frac{1}{2}(1 + \sqrt{2})^{2n}.$$

(cf. Lemma 2.1 of I. Belovas and L. Sakalauskas in [1])

REMARK 2.4. A central limit theorem for the coefficients of modified Borwein's method can be proved analogically, using Bender's central limit theorem applied to asymptotic enumeration (Theorem 1, [3]) [2]. However, the approach, based on Hwang's limit theorem [5], yields more strong result, enabling us to evaluate the rate of convergence to normal distribution (cf. Theorem 3.1 in [1]).

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