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Accepted manuscript

TRINOMIALS $ax^8 + bx + c$ WITH GALOIS GROUPS OF ORDER 1344

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Abstract. Bruin and Elkies [7] obtained the curve of genus 2 parametrizing trinomials $ax^8 + bx + c$ whose Galois group is contained in $G_{1344} = (\mathbb{Z}/2)^3 \rtimes G_{168}$. They found some rational points of small height and computed the associated trinomials. They conjecture that the only $\mathbb{Q}$-rational points of the hyperelliptic curve

$$Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$$

are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, and $(-7, \pm 196)$. In this paper we prove that the above points are the only S-integral points with $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

1. Introduction

In the literature there are many interesting results dealing with trinomials having certain Galois group. Bremner and Spearman [3] proved that up to scaling $x^6 + 133x + 209$ is the only irreducible sextic trinomial with Galois group C_6. Brown, Spearman and Yang [5, 6] characterized rational trinomials with Galois group A_4, $A_4 \times C_2$, S_3 and $C_3 \times S_3$. Brown, Spearman and Yang [5] proved that to obtain some cyclic sextic trinomial (other than the previously mentioned $x^6 + 133x + 209$) over some number field K a rational point on the genus 2 curve $Y^2 = X^6 + 105X^4 + 2400X^2 - 19200$ should exist (other than the ones with $X = \pm 4$). Bruin and Elkies [7] determined the set of rational points on the hyperelliptic curve $Y^2 = X(81X^5 + 396X^4 + 738X^3 + 660X^2 + 269X + 48)$ via covering techniques and the so-called elliptic Chabauty's method [8, 9] and they concluded that every trinomial $ax^7 + bx + c$ over $\mathbb{Q}$ with Galois group

2000 *Mathematics Subject Classification.* Primary 11G30; Secondary 11Y50.

Key words and phrases. Trinomials,Hyperelliptic curves,S-integral points.

contained in G_{168} is equivalent to one of the following trinomials

$$\begin{aligned}
&x^7 - 7x + 3,\\
&x^7 - 154x + 99,\\
&37^2x^7 - 28x + 9,\\
&499^2x^7 - 23956x + 3^4 \cdot 113.
\end{aligned}$$

They conjecture that the only $\mathbb{Q}$-rational points of the hyperelliptic curve $Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$ are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, and $(-7, \pm 196)$. From the above list of rational points they recover the following degree-8 trinomials with Galois group contained in G_{1344}

$$\begin{aligned}
&x^8 + 16x + 28,\\
&x^8 + 576x + 1008,\\
&19^4 \cdot 53x^8 + 19x + 2,\\
&x^8 + 324x + 567.
\end{aligned}$$

They remark that the Mordell-Weil group of the Jacobian of the hyperelliptic curve $Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$ has rank 2, so classical Chabauty cannot be applied. To apply elliptic Chabauty one has to find rational points on elliptic curves over a degree 15 extension of $\mathbb{Q}$.

In this paper we provide a partial result related to the above conjecture. We prove the following statement.

Theorem 1.1. *Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$. The only S-integral points on the hyperelliptic curve*

$$\mathcal{C}_1 : Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$$

are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, and $(-7, \pm 196)$.

The proof is based on techniques developed in [10] for integral points on hyperelliptic curves and [13, 14] for S-integral points.

2. Auxiliary results

We recall some notation and results from [10, 13] related to S-integral points on hyperelliptic curves that will be used later on. Consider the hyperelliptic curve

$$\mathcal{C}: \quad ay^2 = F(x) := x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \tag{2.1}$$

where $a \neq 0, b_i \in \mathbb{Z}$. Let α be a root of F and $J(\mathbb{Q})$ be the Jacobian of the curve $\mathcal{C}$. We have that

$$x - \alpha = \kappa\xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell-Weil group of the curve $\mathcal{C}$ it is possible to provide a method to compute such a finite set. We assume that a rational point P_0 on $\mathcal{C}$ is known. Let $\epsilon_0 = 1$ if P_0 is one of the two points at infinity and $\epsilon_0 = \gamma_0 - \alpha d_0^2$, where $x(P_0) = \gamma_0/d_0^2, \gamma_0 \in \mathbb{Z}$ and $d_0 \in \mathbb{N}$. Every coset of $J(\mathbb{Q})/2J(\mathbb{Q})$ can be represented by a point of the form $\sum_{i=1}^m (P_i - P_0)$ where the set $\{P_1, \ldots, P_m\}$ is stable under the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and such that all $y(P_i)$ are non-zero. Let $x(P_i) = \gamma_i/d_i^2$, where γ_i is and algebraic integer and $d_i \in \mathbb{N}$. An algebraic number $\epsilon = \epsilon_0^{(m \mod 2)} \prod_{i=1}^m (\gamma_i - \alpha d_i^2)$ is associated to such a coset. The following result is Lemma 3.1.2 in [13].

LEMMA 2.1. *Let $\mathcal{E}$ be a set of ϵ associated as above to a complete set of coset representatives for $J(\mathbb{Q})/2J(\mathbb{Q})$. Let Δ be the discriminant of the polynomial F. For each $\epsilon \in \mathcal{E}$ let B_ϵ be the set of square-free rational integers supported only by primes dividing $a\Delta Norm_{K/\mathbb{Q}}(\epsilon)\prod_{p\in S} p$. Let $\mathcal{K} = \{\epsilon b : \epsilon \in \mathcal{E}, b \in B_\epsilon\}$. Then $\mathcal{K}$ is a finite subset of $\mathcal{O}_K$ and if (x, y) is an S-integral point on* (2.1)*, then $x - \alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}, \xi \in K$.*

We introduce some notation we need to provide upper bounds for the size of S-integral solutions of hyperelliptic equations. Let α be an algebraic integer of degree at least 3, and let κ be a integer belonging to K. Let α_1, α_2, α_3 be distinct conjugates of α and κ_1, κ_2, κ_3 be the corresponding conjugates of κ. Let

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2}), \quad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3}), \quad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3}),$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3}).$$

Let S be a finite set of rational primes with $|S| = s$. If $S = \emptyset$, then let $P = 1$, otherwise $P = \max S$. Let d be the degree of L. Let d_1, d_2, d_3 and r_1, r_2, r_3 be the degrees and the unit ranks of K_1, K_2, K_3 respectively. Let R be an upper bound for the regulators of K_1, K_2, K_3 and R_S an upper bound for the respective S_{K_i}-regulators of K_1, K_2, K_3. Let s_i be the number of places in S_{K_i}. Let h_{K_i} be an upper bound for the class numbers of the K_i. For a positive real number a let $\log^*(a) = \max\{1, \log a\}$. Let $c_j^* = \max_{i=1,2,3} c_j(s_i, d_i), j = 1, 2, \ldots, 5$, where

$$c_1(s_i, d_i) = \frac{((s_i - 1)!)^2}{2^{s_i-2} d_i^{s_i-1}}, \quad c_2(s_i, d_i) = 29e\sqrt{s_i - 2}\, c_1(s_i, d_i) d_i^{s_i-1} \log^*(d_i),$$

$$c_3(s_i, d_i) = \frac{((s_i - 1)!)^2}{2^{s_i-1}} \begin{cases} 2/\log 2 & \text{if } d_i = 1, \\ (\log(3d_i))^2 & \text{if } d_i \geq 2, \end{cases}$$

$$c_4(s_i, d_i) = d_i \pi^{s_i-2} c_2(s_i, d_i), \quad c_5(s_i, d_i) = 2d_i c_3(s_i, d_i).$$

Let $c_6^* = \max_{i=1,2,3} c_6(r_i, d_i)$, where

$$c_6(r_i, d_i) = \begin{cases} 0 & \text{if } r_i = 0, \\ 1/d_i & \text{if } r_i = 1, \\ 29er_i!\sqrt{r_i - 1}\log(d_i) & \text{if } r_i \geq 2. \end{cases}$$

Let

$$N = \max_{1\leq i,j\leq 3} \left|\mathrm{Norm}_{\mathbb{Q}(\alpha_i,\alpha_j)/\mathbb{Q}}(\kappa_i(\alpha_i - \alpha_j))\right|^2,$$

$$H^* = \max\left\{\pi/d, \frac{\log N}{\min_{1\leq i\leq 3} d_i} + c_6^* R + \mathrm{h}(\kappa) + h\left(\sum_{p\in S}\log p\right)\right\},$$

$$\begin{aligned}
&c_7(n,d) = \min\{1.451(30\sqrt{2})^{n+4}(n+1)^{5.5}, \pi 2^{6.5n+27}\}d^2\log(ed),\\
&c_8(n,d) = (16ed)^{2(n+1)}n^{3/2}\log(2nd)\log(2d),\\
&c_9(n,d) = (2d)^{2n+1}\log(2d)\log^3(3d),\\
&c_{10}^* = 2H^* + 2H^* d(s+1)(1 + 2(c_4^*)^2 c_7(s_1+s_2-1,d)R_S^2 \times\\
&\times\log(\sqrt{2}e\max\{(s_1+s_2-2)\pi/\sqrt{2}, c_2^* R_S\}),\\
&c_{11}^* = 4d(s+1)H^*(c_4^*)^2 c_7(s_1+s_2-1,d)R_S,\\
&c_{12}^* = 2H^* + 2H^* d(s+1) + c_{11}^*\log\left(\frac{\max\{c_5^*,1\}}{2\sqrt{2}dH^*}\right),\\
&c_{13}^* = \log 2 + 2H^* + 4(s_1+s_2-2)H^*(c_1^*)^2 c_2^* c_9(s_1+s_2-1,d)R_S^3,\\
&c_{14}^* = \frac{2H^* d^{s_1+s_2-2}P^d}{\log(2)\log^*(P^d)}(c_1^*)^2 c_8(s_1+s_2,d)R_S^2,\\
&c_{15}^* = 2H^* + 2H^* d(s+1) +\\
&+c_{14}^*\log\left(\frac{\max\{c_5^*,1\}e^{(s_1+s_2)(6(s_1+s_2)-1)}d^{3(s_1+s_2-1)}\log(2d)P^{d(s_1+s_2)}}{H^* c_9(s_1+s_2-1,d)}\right).
\end{aligned}$$

The following result is Theorem 3.7.1 in [13].

Lemma 2.2. *If $x \in \mathbb{Q}\backslash\{0\}$ is a S-integer satisfying $x - \alpha = \kappa\xi^2$ for some $\xi \in K$, then*

$$\begin{aligned}
\mathrm{h}(x) \quad \leq \quad & 20\log 2 + 13\,\mathrm{h}(\kappa) + 19\,\mathrm{h}(\alpha) + H^* +\\
& +8\max\{c_{10}^*/2, c_{13}^*/2, c_{12}^* + c_{11}^*\log c_{11}^*, c_{15}^* + c_{14}^*\log c_{14}^*\}.
\end{aligned}$$

The previous result provides an upper bound for the size of S-integral solutions, the next one gives lower bound for the size of rational solutions that is not contained in a given set W, the set of known points. This is Lemma 12.1 in [10]. Let P_0 be a fixed rational point on the curve (2.1) and let $\jmath$ be the corresponding Abel-Jacobi map given by

$$\jmath : \mathcal{C} \to J, \quad P \to [P - P_0].$$

Let $D_1, \ldots, D_r$ be generators of the free part of $J(\mathbb{Q})$ and

$$\phi : \mathbb{Z}^r \to J(\mathbb{Q}), \quad (a_1, \ldots, a_r) = \sum_{k=1}^{r} a_k D_k.$$

LEMMA 2.3. *Let W be a finite subset of $J(\mathbb{Q})$, and let L be a sublattice of $\mathbb{Z}^r$. Suppose that $\jmath(C(\mathbb{Q})) \subset W + \phi(L)$. Let μ_1 be such that*

$$\mu_1 \leq h(D) - \hat{h}(D),$$

where $\hat{h}$ denotes the canonical height and h is an appropriately normalized logarithmic height on J. Let

$$\mu_2 = \max\left\{\sqrt{\hat{h}(w)} \ : \ w \in W\right\}.$$

Let M be the height-pairing matrix for the Mordell–Weil basis $D_1, \ldots, D_r$ and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min\left\{\sqrt{\lambda_j} \ : \ j = 1, \ldots, r\right\}.$$

Let $m(L)$ be the Euclidean norm of the shortest non-zero vector of L. Then, for any $P \in C(\mathbb{Q})$, either $\jmath(P) \in W$ or

$$h(\jmath(P)) \geq (\mu_3 m(L) - \mu_2)^2 + \mu_1.$$

3. PROOF OF THEOREM 1.1

To obtain an upper bound for the size of the S-integral points we use the following model

$$\mathcal{C}_2 : y^2 = F(x) := x^6 + 20x^4 + 12x^3 + 25x^2 + 24x + 16,$$

which is isomorphic to the curve $\mathcal{C}_1$ over $\mathbb{Z}[\frac{1}{7}]$, hence they have the same S-integral points. As an application of his theory of lower bounds for linear forms in logarithms, Baker [1] gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see e.g. [4], [11], [18] and [22]). In [10] an improved completely explicit upper bound for integral points were proved combining ideas from [11], [12], [15], [16], [17], [22] and in [13, 14] for S-integral points, the main results stated in Section 2. Let α be a root of F. We have that

$$x - \alpha = \kappa \xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. An appropriate finite set can be determined using Lemma 1. Using MAGMA [2] we get that $J(\mathbb{Q})$ is free of rank 2 with Mordell-Weil basis given by

$$D_1 =< x^2 - 2x + 8, 7x - 28 >,$$
$$D_2 =< x^2 + 1/2x + 2, 7/4x + 7 >$$

in Mumford representation, the torsion subgroup is trivial. The MAGMA procedures used to compute these data are based on Stoll's papers [19], [20], [21]. We obtain that

$$\mathcal{E} = \{1, \alpha^2 - 2\alpha + 8, 256\alpha^2 + 32\alpha + 32, 256\alpha^4 - 480\alpha^3 + 2016\alpha^2 + 192\alpha + 256\},$$

the discriminant of F is $-2^{24}7^8$ and the primes dividing the norms of the elements of $\mathcal{E}$ are $\{2, 7, 59, 8839\}$.

According to the Remark at page 42 in [13] we only need to compute bounds for some of these possible values. In our case only 4 values remain

$$\begin{aligned}
\kappa_1 &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839,\\
\kappa_2 &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot (\alpha^2 - 2\alpha + 8),\\
\kappa_3 &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot (256\alpha^2 + 32\alpha + 32),\\
\kappa_4 &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot\\
&\cdot(256\alpha^4 - 480\alpha^3 + 2016\alpha^2 + 192\alpha + 256).
\end{aligned}$$

For these values we have the following bounds

κ	κ_1	κ_2	κ_3	κ_4
Bound for the S-regulator	3.102×10^{123}	3.102×10^{123}	1.001×10^{292}	9.457×10^{292}
S-unit rank	64	64	113	113
bound for $h(x)$	1.741×10^{1792}	1.741×10^{1792}	3.449×10^{4165}	3.449×10^{4165}

It means that if (x, y) is an S-integral point on the curve $\mathcal{C}_2$ with $x = x_1/x_2, x_1, x_2 \in \mathbb{Z}, \gcd(x_1, x_2) = 1$, then Lemma 2 implies that

$$\max\{|x_1|, |x_2|\} \leq \exp(3.449 \times 10^{4165}),$$

here we used the MAGMA code `upperbounds.m` written by Gallegos-Ruiz to obtain bounds for the solutions. We note that the total running time of the calculations was 30.6 hours on an Intel Core i7-6700HQ 2.6GHz PC.

Let W be the image of the set of these known rational points in $J(\mathbb{Q})$, that is $W = \{0 \cdot D_1 + 0 \cdot D_2, -4 \cdot D_1 + 3 \cdot D_2, -5 \cdot D_1 + 0 \cdot D_2, -2 \cdot D_1 + 1 \cdot D_2, -1 \cdot D_1 - 1 \cdot D_2, -3 \cdot D_1 - 1 \cdot D_2, -4 \cdot D_1 + 1 \cdot D_2, -1 \cdot D_1 - 3 \cdot D_2\}$. Applying the Mordell-Weil sieve explained in [10] we obtain that $\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})$, where

$$B = 2^4 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 79 \cdot 83 \cdot 103 \cdot 107 \cdot 163 \cdot 167 \cdot 179 \cdot 181.$$

For this computation, we used information modulo good primes $p < 50000$ such that $\#J(\mathbb{F}_p)$ is 300-smooth. The total running time of this calculations was 34 minutes on an Intel Core i7-6700HQ 2.6GHz PC. We have that to 3 decimal places

$$\mu_1 = -7.873, \quad \mu_2 = 1.921, \quad \mu_3 = 0.283.$$

We apply Lemma 3 successively to primes of good reduction that satisfy the conditions of the lemma and Criteria (I)–(IV) [10, p. 878]. Using the first 50000 primes we obtain that a lower bound for the size of $\jmath(P)$ for P in the set of unknown rational points is

$$3.483 \times 10^{672}$$

and

$$\begin{aligned} B_1 &= 75631701145170013376999268729339294555 \\ &\quad 38174684977550374967399628867322197875 7 \\ &\quad 26365989785325666235115888371369266792 0 \\ &\quad 793326000000. \end{aligned}$$

We replace B by B_1 and start to sieve using primes that did not satisfied the criteria in the first application. After the second turn we have that the bound is

$$6.945 \times 10^{2510}$$

and the new value of B is of size 4.87×10^{567}. By applying the Mordell-Weil sieve using the first 50000 primes two more times we get that

$$h(\jmath(P)) \geq 2.157 \times 10^{9124}$$

for an unknown rational point P. Hence

$$h(x) \geq 1.079 \times 10^{9124}.$$

The total running time of this calculations was 21.8 hours on an Intel Core i7-6700HQ 2.6GHz PC. It contradicts the bound obtained earlier, hence the only S-integral points with $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$ on the hyperelliptic curve $\mathcal{C}_1$ are given by

$$(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32), (-7, \pm 196).$$

Acknowledgements.

The author would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper. The publication is supported by the EFOP-3.6.1-16-2016-00022 project. The project is co-financed by the European Union and the European Social Fund. Research supported in part by the OTKA grant K115479.

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