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PRESENTATIONS OF FEIGIN-STOYANOVSKY'S TYPE SUBSPACES OF STANDARD MODULES FOR AFFINE LIE ALGEBRAS OF TYPE $C_\ell^{(1)}$

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ABSTRACT. Feigin-Stoyanovsky's type subspace $W(\Lambda)$ of a standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ is a $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated by the highest-weight vector v_Λ, where $\tilde{\mathfrak{g}}_1$ is a certain commutative subalgebra of $\tilde{\mathfrak{g}}$. Based on the description of basis of $W(\Lambda)$ for $\tilde{\mathfrak{g}}$ of type $C_\ell^{(1)}$, we give a presentation of this subspace in terms of generators and relations

$$W(\Lambda) \simeq U(\tilde{\mathfrak{g}}_1^-)/J.$$

1. INTRODUCTION

B. Feigin and A. Stoyanovsky introduced principal subspaces of standard modules for affine Lie algebras of type $A_1^{(1)}$ and $A_2^{(1)}$ in [12] where they have recovered Rogers-Ramanujan type identities by considering graded dimensions of these subspaces. An important part of their investigation was the knowledge of presentations of these subspaces in terms of generators and relations. Another type of principal subspaces, called Feigin-Stoyanovsky's type subspaces, was introduced by M. Primc who constructed bases of these subspaces in different cases ([24], [25], [26], [18]). These kind of subspaces were further studied by many authors ([28], [15], [10], [11], [1], [13], [14], [2], [4], [5], [17], [29], etc.) and the knowledge of presentation presents an important question in this study ([6], [7], [8], [9], [27], [23]).

In our previous works we have described bases of Feigin-Stoyanovsky's type subspaces of standard modules for affine Lie algebras of type $C_\ell^{(1)}$ ([3])

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and obtained from them basis for the whole standard modules ([3]). In this note we use the description of bases of a Feigin-Stoyanovsky's type subspaces to give presentations of these subspaces in terms of generators and relations.

2. Feigin-Stoyanovsky's type subspaces

Let $\mathfrak{g}$ be a complex simple Lie algebra of type C_ℓ with a Cartan subalgebra $\mathfrak{h}$ and a root decomposition $\mathfrak{g} = \mathfrak{h} + \sum \mathfrak{g}_\alpha$. Let

$$R = \{\pm\epsilon_i \pm \epsilon_j \,|\, 1 \le i \le j \le \ell\}\backslash\{0\}$$

be the corresponding root system realized in $\mathbb{R}^\ell$ with the canonical basis $\epsilon_1, \dots, \epsilon_\ell$. Fix simple roots

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \dots, \quad \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell, \quad \alpha_\ell = 2\epsilon_\ell$$

and let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be the corresponding triangular decomposition. Let $\theta = 2\alpha_1 + \dots + 2\alpha_{\ell-1} + \alpha_\ell = 2\epsilon_1$ be the maximal root and

$$\omega_r = \epsilon_1 + \dots + \epsilon_r, \qquad r = 1, \dots, \ell$$

fundamental weights (cf. [16]). Fix root vectors $x_\alpha \in \mathfrak{g}_\alpha$. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form $\langle\, , \,\rangle$ normalized in such a way that $\langle \theta, \theta \rangle = 2$.

Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra of type $C_\ell^{(1)}$ associated to $\mathfrak{g}$,

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c + \mathbb{C}d,$$

with the canonical central element c and the degree element d (cf. [19]). Let

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+,$$

be a triangular decomposition of $\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{n}}_- = \mathfrak{n}_- + \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$, $\tilde{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}c + \mathbb{C}d$, $\tilde{\mathfrak{n}}_+ = \mathfrak{n}_+ + \mathfrak{g} \otimes t\mathbb{C}[t]$. Denote by $\Lambda_0, \dots, \Lambda_\ell$ fundamental weights of $\tilde{\mathfrak{g}}$.

For $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$ denote by $x(n) = x \otimes t^n$ and $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$, where z is a formal variable.

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module with the highest weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \dots + k_\ell\Lambda_\ell,$$

$k_i \in \mathbb{Z}_+$ for $i = 0, \dots, \ell$, and fix a highest weight vector v_Λ. Denote by $k = \Lambda(c)$ the level of $\tilde{\mathfrak{g}}$-module $L(\Lambda)$, $k = k_0 + k_1 + \dots + k_\ell$.

Fix the minuscule weight $\omega = \omega_\ell = \epsilon_1 + \dots + \epsilon_\ell \in \mathfrak{h}^*$; then $\langle \omega, \alpha \rangle \in \{-1, 0, 1\}$ for all $\alpha \in R$ and define *the set of colors*

$$\Gamma = \{\, \alpha \in R \mid \langle \omega, \alpha \rangle = 1 \} = \{\epsilon_i + \epsilon_j \,|\, 1 \le i \le j \le \ell\}.$$

Write

$$(ij) = \epsilon_i + \epsilon_j \in \Gamma \quad \text{and} \quad x_{ij} = x_{\epsilon_i + \epsilon_j}.$$

This gives a $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$; let $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\langle \omega, \alpha \rangle = 0} \mathfrak{g}_\alpha$, then

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1,$$

where

$$\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \tilde{\mathfrak{g}}_{\pm 1} = \sum_{\alpha \in \pm \Gamma} \mathfrak{g}_\alpha \otimes \mathbb{C}[t, t^{-1}].$$

The subalgebra $\tilde{\mathfrak{g}}_1$ is commutative, and $\mathfrak{g}_0$ acts on $\tilde{\mathfrak{g}}_1$ by adjoint action.

Feigin-Stoyanovsky's type subspace of $L(\Lambda)$ is a $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated by the highest-weight vector v_Λ,

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda = U(\tilde{\mathfrak{g}}_1^-) \cdot v_\Lambda \subset L(\Lambda),$$

where $\tilde{\mathfrak{g}}_1^- = \tilde{\mathfrak{g}}_1 \cap \tilde{\mathfrak{n}}_-$.

We use an exponential notation to describe monomials $\mathbf{m} \in U(\tilde{\mathfrak{g}}_1^-) = S(\tilde{\mathfrak{g}}_1^-)$:

$$\mathbf{m} = \cdots x_{i'j'}(-n)^{b_{i'j'}} \cdots x_{ij}(-1)^{a_{ij}}.$$

It will be clear from the context to which factors exponents a_{ij}'s, b_{ij}'s, c_{ij}'s correspond to.

A monomial $\mathbf{m}$ is said to satisfy *difference conditions* for $W(\Lambda)$, *DC* for short, if for any $n \in \mathbb{N}$ and $i_1 \le \cdots \le i_t \le j_t \le \ldots \le j_1 \le i_{t+1} \le \cdots \le i_s \le j_s \le \ldots \le j_{t+1}$, the exponents of $x_{ij}(-n)$'s and $x_{ij}(-n-1)$'s in $\mathbf{m}$, denoted by a_{ij}'s and b_{ij}', respectively, satisfy

$$b_{i_1 j_1} + \ldots + b_{i_t j_t} + a_{i_{t+1} j_{t+1}} + \ldots + a_{i_s j_s} \le k.$$

A monomial $\mathbf{m}$ satisfies *initial conditions* for $W(\Lambda)$, *IC* for short, if for every $i_1 \le \cdots \le i_t \le j_t \le \ldots \le j_1$,

$$a_{i_1 j_1} + \ldots + a_{i_t j_t} \le k_0 + k_1 + \ldots + k_{j_1 - 1}$$

where a_{ij}'s denote exponents of $x_{ij}(-1)$ in $\mathbf{m}$.

THEOREM 1 ([3]). *The set*

$$\{\mathbf{m} v_\Lambda \,|\, \mathbf{m} \text{ satisfies DC and IC for } W(\Lambda)\}$$

is a basis of $W(\Lambda)$.

3. PRESENTATION OF FEIGIN-STOYANOVSKY'S TYPE SUBSPACES

Difference conditions are consequences of the adjoint action of $\mathfrak{g}_0$ on the vertex-operator relation

$$x_\theta(z)^{k+1} = 0,$$

or, equivalently, on a family of relations

$$\sum_{\substack{n_1, \ldots, n_{k+1} \ge 1 \\ n_1 + \cdots + n_{k+1} = N}} x_{11}(-n_1) \cdots x_{11}(-n_{k+1}) = 0, \quad \text{for } N \ge k+1 \tag{3.1}$$

on $L(\Lambda)$ (cf. [3]; see also [21], [22], [20])).

Root vectors of $\mathfrak{g}$ can be chosen so that the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is given by

$$[x_{-\alpha_i}, x_{ij}] = x_{i+1,j}, \quad [x_{-\alpha_j}, x_{ij}] = x_{i,j+1}, \quad [x_{-\alpha_i}, x_{ii}] = 2x_{i,i+1}$$

(cf. [16]). Then one easily sees that the adjoint action gives the following family of relations on $L(\Lambda)$:

$$\sum_{\substack{n_1+\cdots+n_{k+1}=N \\ \{i_1,\ldots,i_{k+1},j_1,\ldots,j_{k+1}\}=\{1^{m_1},\ldots,\ell^{m_\ell}\}}} C_{\mathbf{ij}} x_{i_1 j_1}(-n_1)\cdots x_{i_{k+1}j_{k+1}}(-n_{k+1}) = 0, \tag{3.2}$$

for some nonnegative integers $C_{\mathbf{ij}}$, where the sum runs over all such partitions $\mathbf{i}, \mathbf{j}$ of a multiset $\{1^{m_1},\ldots,\ell^{m_\ell}\}$, $m_1+\cdots+m_\ell = 2(k+1)$ (cf. [3]).

One obtains the difference conditions by finding minimal monomials of these relations, the so called *leading terms* of relations, whose multiples can be excluded from the spanning set. For this, a linear order on monomials is introduced. Define a linear order on the set of colors Γ: $(i'j') < (ij)$ if $i' > i$ or $i' = i$, $j' > j$. On the *set of variables* $\tilde{\Gamma}^- = \{x_\gamma(n) \,|\, \gamma \in \Gamma, n \in \mathbb{Z}_-\}$ define a linear order by $x_\alpha(n) < x_\beta(n')$ if $n < n'$ or $n = n'$, $\alpha < \beta$. For monomials, assume that factors descend from right to left, then use a lexicographic order (compare factors the greatest to the lowest one). Order $<$ is compatible with multiplication (see [24], [29]): if $\mathbf{m}_1 < \mathbf{m}_2$ then $\mathbf{m}\mathbf{m}_1 < \mathbf{m}\mathbf{m}_2$, for $\mathbf{m}, \mathbf{m}_1, \mathbf{m}_2 \in U(\tilde{\mathfrak{g}}_1^-)$.

For initial conditions consider decompositions

$$\Lambda = \Lambda^{(r)} + \Lambda_{(r)}, \qquad \Lambda^{(r)} = k_0\Lambda_0 + \cdots + k_{r-1}\Lambda_{r-1}, \quad \Lambda_{(r)} = k_r\Lambda_r + \cdots + k_\ell\Lambda_\ell,$$

for $1 \le r \le \ell$. By $v^{(r)}$ and $v_{(r)}$ denote highest weight vectors of the associated standard modules $L(\Lambda^{(r)})$ and $L(\Lambda_{(r)})$ of level $k^{(r)} = k_0 + \cdots + k_{r-1}$ and $k_{(r)} = k_r + \cdots + k_\ell$, respectively. Then $L(\Lambda)$ can be embedded in a tensor product $L(\Lambda) \subset L(\Lambda^{(r)}) \otimes L(\Lambda_{(r)})$. Since $x_{ij}(-1)v_{\Lambda_r} = 0$ if and only if $j \le r$ (cf. [3]), we have

$$\mathbf{m}(v^{(r)} \otimes v_{(r)}) = (\mathbf{m}v^{(r)}) \otimes v_{(r)}$$

for $\mathbf{m} = x_{i_1j_1}(-1)\cdots x_{i_tj_t}(-1)$ such that $j_s \le r$, $1 \le s \le t$. Hence, relations between such monomials corresponding to difference conditions for $W(\Lambda^{(r)})$ automatically become relations in $L(\Lambda)$ (cf. [3]). This gives the following family of relations on $L(\Lambda)$:

$$\sum_{\{i_1,\ldots,i_{k+1},j_1,\ldots,j_{k+1}\}=\{1^{m_1},\ldots,r^{m_r}\}} C_{\mathbf{ij}} x_{i_1j_1}(-1)x_{i_2j_2}(-1)\cdots x_{i_{k^{(r)}+1}j_{k^{(r)}+1}}(-1) = 0, \tag{3.3}$$

for some nonnegative integers $C_{\mathbf{ij}}$, where the sum runs over all such partitions $\mathbf{i}, \mathbf{j}$ of a multiset $\{1^{m_1},\ldots,r^{m_r}\}$, $m_1+\cdots+m_r = 2(k^{(r)}+1)$.

Alternatively, for $r \geq 2$ let $\mathfrak{g}_{(r)} \subset \mathfrak{g}_0$ be the subalgebra generated by elements $x_{\pm\alpha_t}$, $1 \leq t < r$. Start from a relation

$$x_{11}(-1)^{k^{(r)}+1}(v^{(r)} \otimes v_{(r)}) = (x_{11}(-1)^{k^{(r)}+1}v^{(r)}) \otimes v_{(r)} = 0. \tag{3.4}$$

Now the adjoint action of $\mathfrak{g}_{(r)}$ on the above relation gives relations (3.3). For $r = 1$, relations (3.3) come down to only one relation

$$x_{11}(-1)^{k_0+1}(v^{(1)} \otimes v_{(1)}) = 0. \tag{3.5}$$

Recall that Feigin-Stoyanovsky's type subspace $W(\Lambda)$ is

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1^-) \cdot v_\Lambda.$$

Since $\tilde{\mathfrak{g}}_1$ is commutative, universal enveloping algebra of $\tilde{\mathfrak{g}}_1^-$ is isomorphic to a polynomial algebra $\mathbb{C}[\tilde{\Gamma}^-]$. Hence, there is a surjection

$$f_\Lambda : \mathbb{C}[\tilde{\Gamma}^-] \to W(\Lambda), \qquad f : \mathbf{m} \to \mathbf{m} \cdot v_\Lambda.$$

We want to describe the kernel of this map, $\ker f_\Lambda \subset \mathbb{C}[\tilde{\Gamma}^-]$, so that

$$W(\Lambda) \simeq \mathbb{C}[\tilde{\Gamma}^-]/\ker f_\Lambda,$$

as vector spaces.

Theorem 2. *Let $J_\Lambda \subset \mathbb{C}[\tilde{\Gamma}^-]$ be the ideal generated by the following sets*

$$U(\mathfrak{g}_0) \cdot \left(\sum_{\substack{n_1,\ldots,n_{k+1} \geq 1 \\ n_1+\cdots+n_{k+1}=N}} x_{11}(-n_1) \cdots x_{11}(-n_{k+1}) \right), \qquad \textit{for } N \geq k+1,$$

$$U(\mathfrak{g}_{(r)}) \cdot x_{11}(-1)^{k^{(r)}+1}, \qquad \textit{for } r = 2, \ldots, \ell,$$

$$x_{11}(-1)^{k_0+1}.$$

Then $\ker f_\Lambda = J_\Lambda$.

Proof: From (3.1), (3.4) and (3.5), see also (3.2) and (3.3), it follows that the generators of J_Λ lie in the kernel of f_Λ. Hence f_Λ can be factorized to a quotient map

$$\bar{f}_\Lambda : \mathbb{C}[\tilde{\Gamma}^-]/J_\Lambda \to W(\Lambda).$$

This map is clearly a surjection, since f_Λ is a surjection.

We can imitate the proof for the spanning set for $W(\Lambda)$ (cf. Proposition 2 and 4 in [3]) to reduce the spanning set for $\mathbb{C}[\tilde{\Gamma}^-]/J_\Lambda$. Consider the generators of J_Λ and identify the minimal monomial inside each one; their multiples can be excluded from the spanning set. Like in [3], we get

$$\mathcal{B} = \{\mathbf{m} \,|\, \mathbf{m} \text{ satisfies DC and IC for } W(\Lambda)\}$$

as a spanning set of $\mathbb{C}[\tilde{\Gamma}^-]/J_\Lambda$.

To see that $\bar{f}_\Lambda$ is an injection, note that $\bar{f}_\Lambda$ maps $\mathcal{B}$ bijectively onto

$$\{\mathbf{m}v_\Lambda \,|\, \mathbf{m} \text{ satisfies DC and IC for } W(\Lambda)\} \subset W(\Lambda),$$

which is a basis of $W(\Lambda)$. This means that $\mathcal{B}$ is also linearly independent. Hence $\bar{f}_\Lambda$ maps a basis of $\mathbb{C}[\tilde{\Gamma}^-]/J_\Lambda$ onto a basis of $W(\Lambda)$ and therefore $\bar{f}_\Lambda$ is a bijection. ■

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