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COMMUTATIVE OPERATORS FOR DOUBLE YANGIAN $\mathbf{DY}(\mathfrak{sl}_n)$

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Abstract. We derive explicit formulae for certain commutative vertex operators associated with Iohara's realization of the level 1 $\mathrm{DY}(\mathfrak{sl}_n)$-modules. As an application, we construct combinatorial bases for the corresponding principal subspaces and recover the classical character formulae. In the end, we discuss the underlying nonlocal vertex algebra theory.

Introduction

The principal subspaces of the standard $\widehat{\mathfrak{sl}}_2$-modules were introduced by B. L. Feigin and A. V. Stoyanovsky in [9]. The authors found the correspondence between the principal subspaces' character formulae and the sum sides of some combinatorial identities, which was later extended to the other types of the affine Lie algebras. The corresponding combinatorial bases were constructed by G. Georgiev in type $A_n^{(1)}$, see [11], and by M. Butorac in types $(BC)_n^{(1)}$ and $G_2^{(1)}$; see [1, 2, 3]. The other closely related topics, such as presentations of principal subspaces, the underlying vertex algebraic structures or Feigin–Stoyanovsky's type subspaces, were also extensively studied; see, e.g., [4, 14, 20, 21, 22] and references therein.

In this paper, we consider the quantum double [6] of the Yangian $\mathrm{Y}(\mathfrak{sl}_n)$ over $\mathbb{C}[[h]]$ and, in particular, its Drinfeld realization; see [12, 15]. We find explicit formulae for certain commutative operators $\overline{E}_j(z) = \sum_r \overline{e}_j(r) z^{-r-1}$ in $\mathrm{Hom}(\mathcal{F}, \mathcal{F}((z)))$, associated with Iohara's realization [12] of level 1 modules for the double Yangian $\mathrm{DY}(\mathfrak{sl}_n)$, which is given on a certain $\mathbb{C}[[h]]$-module $\mathcal{F}$.

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These operators exhibit similar properties as the Ding–Feigin operators for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$, so, motivated by [5, 11], we define the principal subspaces $W_i \subset \mathcal{F}$ as the (h-adically completed) $\mathbb{C}[[h]]$-span of the monomials $\overline{e}_{i_m}(k_m)\dots\overline{e}_{i_1}(k_1)v_i$, where v_i is the highest weight vector. Finally, we construct monomial bases $\mathcal{B}_i$ for some dense $\mathbb{C}[[h]]$-submodule of W_i from which one can easily obtain the level 1 character formulae coinciding with those in [9, 11].

Even though our approach follows [11], some differences occur due to the underlying (nonlocal) vertex algebra theory, which we discuss in the last section. So far, the general construction of *quantum VOAs* by P. Etingof and D. Kazhdan [8] and of *h-adic nonlocal vertex algebras* by H.-S. Li [18, 19] gave rise to certain (quantum) vertex algebraic structures associated with the double Yangian. However, we employ the construction of *nonlocal h-vertex algebras* (which are, roughly speaking, the h-adic nonlocal vertex algebras without the h-adic completeness requirement) found in [16]. In contrast with the approach in [8, 18, 19], we consider the vertex operator products defined via $2h$-derivation $a(z) \mapsto (a(z+2h)-a(z))/2h$, which provides a simple interpretation of the Yangian integrability relations $\overline{E}_j(z)\overline{E}_j(z\pm h)=0$; cf. integrability relations in [17] and quantum integrability [5]. As with the construction of the basis $\mathcal{B}_i$, we show that these relations play an important role in the construction of basis for the nonlocal h-vertex algebra $V_1 \subset \mathrm{Hom}(\mathcal{F}, \mathcal{F}((z))[[h]])$ generated by the operator $\overline{E}_1(z)$. In the end, it is worth noting that, even though the operator $\overline{E}_1(z)$ is commutative, i.e. $[\overline{E}_1(z_1), \overline{E}_1(z_2)]=0$, the vertex operator products on V_1 are not commutative. Moreover, the product $a\cdot b := a_{-1}b$ for $a,b\in V_1$ defines a structure of a noncommutative associative algebra on V_1.

1. Iohara's realization

In this section, we follow [12] to introduce the double Yangian for $\mathfrak{sl}_n$ and its level 1 infinite-dimensional modules. Let h be a formal parameter and let M be an arbitrary $\mathbb{C}[[h]]$-module. Recall that the *h-adic topology* on M is the topology generated by the basis $v+h^mM$ with $v\in M$ and $m\in\mathbb{Z}_{\geqslant 0}$. Denote by $A=(a_{ij})_{i,j=1}^{n-1}$ the Cartan matrix of the simple Lie algebra $\mathfrak{sl}_n$. The *double Yangian* $\mathrm{DY}(\mathfrak{sl}_n)$ is defined as the h-adically completed associative algebra over $\mathbb{C}[[h]]$ generated by the central element c and the coefficients of the series

$$H_i^+(z) = 1 + h\sum_{k\geqslant 0} h_i(k)z^{-k-1}, \qquad H_i^-(z) = 1 - h\sum_{k<0} h_i(k)z^{-k-1},$$
$$E_i(z) = \sum_{k\in\mathbb{Z}} e_i(k)z^{-k-1}, \qquad F_i(z) = \sum_{k\in\mathbb{Z}} f_i(k)z^{-k-1},$$

where $i = 1, \dots, n-1$, subject to the defining relations

(1.1)
$$[H_i^{\pm}(z), H_j^{\pm}(w)] = 0,$$

(1.2)
$$(z_{\mp} - w_{\pm} + ha_{ij}/2)(z_{\pm} - w_{\mp} - ha_{ij}/2)H_i^{\pm}(z)H_j^{\mp}(w) \\ = (z_{\mp} - w_{\pm} - ha_{ij}/2)(z_{\pm} - w_{\mp} + ha_{ij}/2)H_j^{\mp}(w)H_i^{\pm}(z),$$

(1.3)
$$H_i^{\pm}(z)^{-1}E_j(w)H_i^{\pm}(z) = \frac{z_{\pm} - w - ha_{ij}/2}{z_{\pm} - w + ha_{ij}/2}E_j(w),$$

(1.4)
$$H_i^{\pm}(z)F_j(w)H_i^{\pm}(z)^{-1} = \frac{z_{\mp} - w - ha_{ij}/2}{z_{\mp} - w + ha_{ij}/2}F_j(w),$$

(1.5)
$$(z - w - ha_{ij}/2)E_i(z)E_j(w) = (z - w + ha_{ij}/2)E_j(w)E_i(z),$$

(1.6)
$$(z - w + ha_{ij}/2)F_i(z)F_j(w) = (z - w - ha_{ij}/2)F_j(w)F_i(z),$$

(1.7)
$$[E_i(z), F_j(w)] = \frac{\delta_{ij}}{h}\left(\delta(z_- - w_+)H_i^+(z_-) - \delta(z_+ - w_-)H_i^-(w_-)\right),$$

(1.8)
$$\sum_{\sigma\in\mathfrak{S}_m} [E_i(z_{\sigma(1)}), [E_i(z_{\sigma(2)}), \dots [E_i(z_{\sigma(m)}), E_j(w)] \dots]] = 0,\ i \neq j,\ m = 1 - a_{ij},$$

(1.9)
$$\sum_{\sigma\in\mathfrak{S}_m} [F_i(z_{\sigma(1)}), [F_i(z_{\sigma(2)}), \dots [F_i(z_{\sigma(m)}), F_j(w)] \dots]] = 0,\ i \neq j,\ m = 1 - a_{ij},$$

where $z_{\pm} = z \pm hc/4$ and $\delta(z - w) = \sum_{k\in\mathbb{Z}} z^{-k-1}w^k$ is the delta function.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_n$, let Q be the root lattice generated by the simple roots $\alpha_1, \dots, \alpha_{n-1} \in \mathfrak{h}^*$ and let $P \subset \mathfrak{h}^*$ be the weight lattice generated by the fundamental weights $\lambda_1, \dots, \lambda_{n-1}$. Denote by $(\cdot, \cdot)$ the standard bilinear form on $\mathfrak{h}^*$ defined by $(\alpha_i, \alpha_j) = a_{ij}$. Let $\mathfrak{s}$ be the Heisenberg algebra generated by the elements $a_{i,k}$, $i = 1, \dots, n-1$, $k \in \mathbb{Z}$, $k \neq 0$, and the central element c subject to the relations

$$[a_{i,k}, a_{j,l}] = k\delta_{i\,j}\delta_{k+l\,0}c. \tag{1.10}$$

For any $i = 0, \dots, n-1$ consider the h-adically completed tensor product

$$\mathcal{F}_i = \mathbb{C}[[h]][a_{j,k} : j = 1, \dots, n-1,\ k \in \mathbb{Z}_{<0}] \otimes \mathbb{C}[[h]][Q]e^{\lambda_i}, \tag{1.11}$$

where $\mathbb{C}[[h]][Q]$ denotes the group algebra of the root lattice Q over $\mathbb{C}[[h]]$ and $e^{\lambda_0} = 1$. Define the action of the elements c, $a_{j,k}$, ∂_{α_j}, e^{α_j}, $j = 1, \dots, n-1, k \in$

$\mathbb{Z}, k \neq 0$, on $\mathcal{F}_i$ by

$$c \cdot f \otimes e^\nu = f \otimes e^\nu, \qquad a_{j,k} \cdot f \otimes e^\nu = \begin{cases} a_{j,k} f \otimes e^\nu & \text{for } k < 0, \\ [a_{j,k}, f] \otimes e^\nu & \text{for } k > 0, \end{cases} \tag{1.12}$$

$$\partial_{\alpha_j} \cdot f \otimes e^\nu = (\alpha_j, \nu) f \otimes e^\nu \qquad \text{and} \qquad e^{\alpha_j} \cdot f \otimes e^\nu = f \otimes e^{\alpha_j + \nu} \tag{1.13}$$

for any $f \otimes e^\nu \in \mathcal{F}_i$. By action (1.12) $\mathcal{F}_i$ becomes a level 1 module for the Heisenberg algebra $\mathfrak{s}$. Set $c = 1$, so that $z_\pm = z \pm h/4$. We recall the realization of level 1 $\mathrm{DY}(\mathfrak{sl}_n)$-modules from [12, Theorem 4.7]:

THEOREM 1.1. *For any $i = 0, \dots, n-1$ the following assignments define a $\mathrm{DY}(\mathfrak{sl}_n)$-module structure on $\mathcal{F}_i$:*

$$\begin{aligned}
H_j^+(z) &= \exp\left(-\textstyle\sum_{k>0} \frac{a_{j,k}}{k} \left(\left(z + \frac{h}{2}\right)^{-k} - \left(z - \frac{h}{2}\right)^{-k}\right)\right) \left(\frac{z - \frac{h}{2}}{z + \frac{h}{2}}\right)^{-\partial_{\alpha_j}}, \\
H_j^-(z) &= \exp\left(-\textstyle\sum_{k>0} \frac{a_{j,-k}}{k} \left((z+h)^k - (z-h)^k\right)\right. \\
&\qquad \left. + \textstyle\sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} \left(\left(z + \frac{h}{2}\right)^k - \left(z - \frac{h}{2}\right)^k\right)\right), \\
E_j(z) &= \exp\left(\textstyle\sum_{k>0} \frac{a_{j,-k}}{k} \left(\left(z + \frac{h}{4}\right)^k + \left(z - \frac{3h}{4}\right)^k\right)\right. \\
&\qquad \left. - \textstyle\sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} \left(z - \frac{h}{4}\right)^k\right) \\
&\qquad \cdot \exp\left(-\textstyle\sum_{k>0} \frac{a_{j,k}}{k} \left(z + \frac{h}{4}\right)^{-k}\right) e^{\alpha_j} \left((-1)^{j-1} \left(z + \frac{h}{4}\right)\right)^{\partial_{\alpha_j}}, \\
F_j(z) &= \exp\left(-\textstyle\sum_{k>0} \frac{a_{j,-k}}{k} \left(\left(z + \frac{3h}{4}\right)^k + \left(z - \frac{h}{4}\right)^k\right)\right. \\
&\qquad \left. + \textstyle\sum_{k>0} \frac{a_{j+1,-k} + a_{j-1,-k}}{k} \left(z + \frac{h}{4}\right)^k\right) \\
&\qquad \cdot \exp\left(\textstyle\sum_{k>0} \frac{a_{j,k}}{k} \left(z - \frac{h}{4}\right)^{-k}\right) e^{-\alpha_j} \left((-1)^{j-1} \left(z - \frac{h}{4}\right)\right)^{-\partial_{\alpha_j}}.
\end{aligned}$$

The operators $H_j^\pm(z), E_j(z), F_j(z)$ clearly belong to

$$\mathcal{E}_h(\mathcal{F}_i) \coloneqq \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_i((z))[[h]]) \subset (\mathrm{End}\, \mathcal{F}_i)[[z^{\pm 1}]].$$

Notice that we required for tensor product (1.11) to be h-adically completed because otherwise, the coefficients of the given operators would not belong to $\mathrm{End}\, \mathcal{F}_i$. Clearly, the given operators can be also viewed as elements of $\mathcal{E}_h(\mathcal{F}) \coloneqq \mathrm{Hom}(\mathcal{F}, \mathcal{F}((z))[[h]])$, where $\mathcal{F}$ denotes the h-adically completed tensor product

$$\mathcal{F} = \mathbb{C}[[h]][a_{j,k} \,:\, j = 1, \dots, n-1,\ k \in \mathbb{Z}_{<0}] \otimes \mathbb{C}[[h]][P]$$

and $\mathbb{C}[[h]][P]$ is the group algebra of the weight lattice P over $\mathbb{C}[[h]]$.

REMARK 1.2. *An element* $a(z)=\sum_{r\in\mathbb{Z}} a_r z^{-r-1}$ *in* $(\operatorname{End}\mathcal{F})[[z^{\pm 1}]]$ *belongs to* $\mathcal{E}_h(\mathcal{F})$ *if and only if* $\lim_{r\to\infty} a_r v=0$ *for all* $v\in\mathcal{F}$*, where the limit is taken with respect to the* h*-adic topology on* $\mathcal{F}$*.*

In the latter part of the paper we will also need the operators

$$\Psi_j(z)=\exp\left(\sum_{k>0}\frac{a_{j,-k}}{k}\left(z+\frac{h}{4}\right)^k\right)\in\mathcal{E}_h(\mathcal{F}),\quad j=1,\dots,n-1, \tag{1.14}$$

which appear in the construction of the intertwiners in [12, Theorem 4.8]. Using (1.10) one can easily verify that operators (1.14) satisfy

$$E_i(z_1)\Psi_i(z_2)=\frac{z_1-z_2}{z_1+h/4}\Psi_i(z_2)E_i(z_1)\quad\text{and}\quad [E_i(z_1),\Psi_j(z_2)]=0 \text{ for } i\neq j. \tag{1.15}$$

2. Commutative operators

In this section, we introduce certain commutative operators associated with Iohara's realization and we study their properties. First, notice that the operator $E_j(z)\in\mathcal{E}_h(\mathcal{F})$ can be written as

$$E_j(z)=E_j^-(z)E_j^+(z)e^{\alpha_j}\left((-1)^{j-1}\left(z+\frac{h}{4}\right)\right)^{\partial_{\alpha_j}},$$

where

$$\begin{aligned}E_j^+(z)&=\exp\left(-\textstyle\sum_{k>0}\frac{a_{j,k}}{k}\left(z+\frac{h}{4}\right)^{-k}\right)\qquad\text{and}\\ E_j^-(z)&=\exp\Big(\textstyle\sum_{k>0}\frac{a_{j,-k}}{k}\left(\left(z+\frac{h}{4}\right)^k+\left(z-\frac{3h}{4}\right)^k\right)\\ &\qquad-\textstyle\sum_{k>0}\frac{a_{j+1,-k}+a_{j-1,-k}}{k}\left(z-\frac{h}{4}\right)^k\Big).\end{aligned}$$

For $j=1,\dots,n-1$ define the operator $k_j(z)$ on $\mathcal{F}$ by

$$\begin{aligned}k_j(z)=\exp\Big(&-\textstyle\sum_{k>0}\frac{a_{j,-k}}{k}\left(z^k-(z+h)^k\right)\\ &+\textstyle\sum_{k>0}\frac{a_{j+1,-k}+a_{j-1,-k}}{k}\left(z-\frac{h}{2}\right)^k\Big).\end{aligned} \tag{2.1}$$

Clearly, the operators

$$\overline{E}_j(z):=\sum_{k\in\mathbb{Z}}\overline{e}_j(k)z^{-k-1}:=k_j(z)E_j(z-h/4),\quad\text{where } j=1,\dots,n-1, \tag{2.2}$$

belong to $\mathcal{E}_h(\mathcal{F})$. Moreover, we have

$$\overline{E}_j(z)=\overline{E}_j^-(z)\overline{E}_j^+(z)e^{\alpha_j}\left((-1)^{j-1}z\right)^{\partial_{\alpha_j}}, \tag{2.3}$$

where

$$\begin{aligned}\overline{E}_j^+(z)&=\exp\left(-\textstyle\sum_{k>0}\frac{a_{j,k}}{k}z^{-k}\right)\quad\text{and}\\ \overline{E}_j^-(z)&=\exp\left(\textstyle\sum_{k>0}\frac{a_{j,-k}}{k}\left((z+h)^k+(z-h)^k\right)\right).\end{aligned}$$

THEOREM 2.1. *For any $i, j = 1, \dots, n-1$ we have*

$$[\overline{E}_i(z_1), \overline{E}_j(z_2)] = 0 \quad \text{if} \quad i - j \neq \pm 1, \tag{2.4}$$

$$z_1 \overline{E}_i(z_1) \overline{E}_j(z_2) = -z_2 \overline{E}_j(z_2) \overline{E}_i(z_1) \quad \text{if} \quad i - j = \pm 1. \tag{2.5}$$

Furthermore, we have

$$\overline{E}_j(z) \overline{E}_j(z \pm h) = 0 \quad \text{for all } j = 1, \dots, n-1. \tag{2.6}$$

Proof. Let us prove (2.4) for $i = j$. By using Theorem 1.1 and (1.10) one can easily verify that

$$(z_1 - z_2 + h) k_j(z_1) E_j(z_2 - h/4) = (z_1 - z_2) E_j(z_2 - h/4) k_j(z_1). \tag{2.7}$$

Observe that the elements $k_j(z_1)E_j(z_2 - h/4)$ and $E_j(z_2 - h/4)k_j(z_1)$ belong to $\mathrm{Hom}(\mathcal{F}, \mathcal{F}((z_2))[[z_1, h]])$, so we can multiply expression (2.7) by the inverse $(-z_2 + z_1)^{-1} \in \mathbb{C}[z_2^{-1}][[z_1]]$ of $z_1 - z_2$, thus getting

$$\frac{z_1 - z_2 + h}{-z_2 + z_1} k_j(z_1) E_j(z_2 - h/4) = E_j(z_2 - h/4) k_j(z_1). \tag{2.8}$$

Consider the inverses $(z_1 - z_2)^{-1}$ in $\mathbb{C}[z_1^{-1}][[z_2]]$ and $(-z_2 + z_1)^{-1}$ in $\mathbb{C}[z_2^{-1}][[z_1]]$ of the polynomial $z_1 - z_2$. Relation

$$E_j^+(z_1) E_j^-(z_2) = \frac{(z_1 - z_2)(z_1 - z_2 + h)}{(z_1 + h/4)^2} E_j^-(z_2) E_j^+(z_1), \tag{2.9}$$

which can be proved using (1.10), implies that the following equality of the well-defined elements of $\mathrm{Hom}(\mathcal{F}, \mathcal{F}((z_1, z_2))[[h]])$ holds:

$$\frac{z_1 - z_2 - h}{z_1 - z_2} E_j(w_1) E_j(w_2) = \frac{z_1 - z_2 + h}{-z_2 + z_1} E_j(w_2) E_j(w_1) \text{ for } w_k = z_k - h/4. \tag{2.10}$$

Since $[k_j(z_1), k_j(z_2)] = 0$, by multiplying (2.10) with the element $k_j(z_1)k_j(z_2)$, which belongs to $\mathrm{Hom}(\mathcal{F}, \mathcal{F}[[z_1, z_2]])$, we obtain

$$\begin{aligned} & k_j(z_1) \frac{z_1 - z_2 - h}{z_1 - z_2} k_j(z_2) E_j(w_1) E_j(w_2) \\ = & k_j(z_2) \frac{z_1 - z_2 + h}{-z_2 + z_1} k_j(z_1) E_j(w_2) E_j(w_1). \end{aligned} \tag{2.11}$$

Finally, due to (2.8), we can express (2.11) as

$$k_j(z_1) E_j(z_1 - h/4) k_j(z_2) E_j(z_2 - h/4) = k_j(z_2) E_j(z_2 - h/4) k_j(z_1) E_j(z_1 - h/4), \tag{2.12}$$

which is equivalent to

$$\overline{E}_j(z_1) \overline{E}_j(z_2) = \overline{E}_j(z_2) \overline{E}_j(z_1), \tag{2.13}$$

as required.

Equalities (2.4) for $i \neq j$ and (2.5) are clear. Let us prove (2.6). Recall that the left hand sides in (2.11) and (2.13) coincide. Hence, due to (2.9), we have

$$\begin{aligned}\overline{E}_j(z_1)\overline{E}_j(z_2) =&(z_1 - z_2 - h)(z_1 - z_2 + h)\\ &\cdot e^{2\alpha_j}(-1)^{2(j-1)\partial_{\alpha_j}}\,(z_1z_2)^{\partial_{\alpha_j}}\,k_j(z_1)k_j(z_2)\\ &\cdot E_j^-(z_1 - h/4)E_j^-(z_2 - h/4)E_j^+(z_2 - h/4)E_j^+(z_1 - h/4).\end{aligned} \tag{2.14}$$

Since the expression $k_j(z_1)k_j(z_2)E_j^-(w_1)E_j^-(w_2)E_j^+(w_2)E_j^+(w_1)$ belongs to $\mathrm{Hom}(\mathcal{F}, \mathcal{F}((z_1, z_2))[[h]])$, we obtain (2.6) by applying the substitution $z_1 = z$, $z_2 = z \pm h$ to (2.14). □

Remark 2.2. *Our construction of operators $\overline{E}_j(z)$ was motivated by the commutative operators for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$, which were introduced by J. Ding and B. Feigin in* [5]. *By Theorem* 2.1, *the operators* (2.2) *exhibit similar properties as the Ding–Feigin operators. However, the explicit formulae for the operators in* [5] *is, due to Frenkel–Jing realization* [10], *given in terms of elements of certain Heisenberg subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$, so the action of these operators can be directly extended to an arbitrary (restricted) $U_q(\widehat{\mathfrak{sl}}_n)$-module. On the other hand, the formulae for the operators $\overline{E}_j(z)$ are given in terms of elements of certain artificial Heisenberg algebra $\mathfrak{s}$ from the Iohara's realization, which does not need to be a subalgebra of the double Yangian* $\mathrm{DY}(\mathfrak{sl}_n)$. *Hence, it is not clear whether the operators $\overline{E}_j(z)$ can be naturally extended to an arbitrary* $\mathrm{DY}(\mathfrak{sl}_n)$*-module. However, the existence of the Hopf algebra structure on* $\mathrm{DY}(\mathfrak{sl}_2)$, *whose coproduct possesses a simple form in terms of the generating series $H_1^{\pm}(z), E_1(z), F_1(z)$, see* [7], *suggests that the extension of our operators to the tensor product of multiple copies of $\mathcal{F}_j$ might be plausible.*

Remark 2.3. *Equality* (2.6) *can be viewed as the Yangian version of the level* 1 *integrability relation $x_{\alpha_1}(z)^2 = 0$ for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$; see* [17]. *As with its classical analogue,* (2.6) *plays an important role in the construction of combinatorial bases for the principal subspaces, which is presented in the next section. In particular, see Lemma* 3.2.

3. Principal subspaces

In this section, we introduce the principal subspaces associated with $\mathrm{DY}(\mathfrak{sl}_n)$-modules $\mathcal{F}_i$ and construct the corresponding combinatorial bases. Our construction relies on the Georgiev's approach in [11], which we adapt to the Yangian setting.

For $i = 1, \dots, n-1$ introduce the elements

$$v_i = 1 \otimes e^{\lambda_i} \in \mathcal{F}_i \qquad \text{and} \qquad v_0 = 1 \otimes 1 \in \mathcal{F}_0.$$

Motivated by [5, 9, 11], we define the *principal subspace* W_i, $i = 0, \dots, n-1$, as the h-adic completion of the $\mathbb{C}[[h]]$-submodule of $\mathcal{F}_i$ which is spanned by the elements

(3.1)
$$\overline{e}_{i_m}(k_m) \dots \overline{e}_{i_1}(k_1) v_i, \quad \text{where } n-1 \geqslant i_1, \dots, i_m \geqslant 1,\ k_1, \dots, k_m \in \mathbb{Z},\ m \geqslant 0.$$

Due to (2.4) and (2.5), we can arrange the operators $\overline{e}_{i_r}(k_r)$ in (3.1), so that the indices $i_1, \dots, i_m$ are increasing from right to left. Hence, the $\mathbb{C}[[h]]$-span of the elements

(3.2)
$$\overline{e}_{i_m}(k_m) \dots \overline{e}_{i_1}(k_1) v_i, \text{ where } n-1 \geqslant i_m \geqslant \dots \geqslant i_1 \geqslant 1,\ k_1, \dots, k_m \in \mathbb{Z},\ m \geqslant 0$$

coincides with the $\mathbb{C}[[h]]$-span of (3.1). We will now further reduce spanning set (3.2).

Lemma 3.1. *For any $m \geqslant 0$ and $n-1 \geqslant i_m \geqslant \dots \geqslant i_1 \geqslant 1$ we have*

$$\overline{E}_{i_m}(z_m) \dots \overline{E}_{i_1}(z_1) v_i \ \in \ \prod_{s=1}^{m} z_s^{\delta_{i_s i} - r_{i_s - 1}} W_i[[z_m, \dots, z_1]], \tag{3.3}$$

where r_p denotes the number of indices $t = 1, \dots, m$ such that $i_t = p$ and, in particular, $r_0 = 0$.

Proof. The lemma is a consequence of (2.4) and the formulae

$$z^{\partial_{\alpha_j}} e^{\lambda_i} = e^{\lambda_i} z^{\delta_{ji} + \partial_{\alpha_j}}, \qquad [\overline{E}^+_{j+1}(z_2), \overline{E}^-_j(z_1)] = 0 \qquad \text{and}$$
$$e^{\alpha_{j+1}} \left((-1)^j z_2\right)^{\partial_{\alpha_{j+1}}} e^{\alpha_j} \left((-1)^{j-1} z_1\right)^{\partial_{\alpha_j}}$$
$$= (-1)^j e^{\alpha_j + \alpha_{j+1}} (-1)^{(j-1)\partial_{\alpha_j} + j\partial_{\alpha_{j+1}}} z_1^{\partial_{\alpha_j}} z_2^{\partial_{\alpha_{j+1}} - 1},$$

which can be verified by a direct calculation. □

Denote by $\mathcal{S}_i$ the set of all elements $\overline{e}_{i_m}(k_m) \dots \overline{e}_{i_1}(k_1) v_i$ satisfying

(3.4)
$$n-1 \geqslant i_1 \geqslant \dots \geqslant i_m \geqslant 1, \quad k_s \leqslant r_{i_s - 1} - \delta_{i_s i} - 1, \quad s = 1, \dots, m, \quad m \geqslant 0.$$

Notice that the element $\overline{e}_{i_m}(k_m) \dots \overline{e}_{i_1}(k_1) v_i$ is equal to the coefficient of $z_m^{-k_m - 1} \dots z_1^{-k_1 - 1}$ in expression (3.3). By Lemma 3.1 we see that the coefficient of $z_m^{-k_m - 1} \dots z_1^{-k_1 - 1}$ in (3.3) is zero if there exists an integer $s = 1, \dots, m$ such that $k_s > r_{i_s - 1} - \delta_{i_s i} - 1$. Hence, we conclude that W_i is equal to the h-adic completion of the $\mathbb{C}[[h]]$-span of $\mathcal{S}_i$.

Finally, introduce the set $\mathcal{B}_i \subset \mathcal{S}_i$ by

$$\mathcal{B}_i = \{\overline{e}_{i_m}(k_m) \dots \overline{e}_{i_1}(k_1) v_i \in \mathcal{S}_i \ : \\ i_r = i_{r+1} \text{ implies } k_{r+1} \leqslant k_r - 2 \text{ for all } 1 \leqslant r \leqslant m-1\}.$$

Lemma 3.2. *For any $i = 0, \dots, n-1$ the principal subspace W_i coincides with the h-adically completed $\mathbb{C}[[h]]$-span of $\mathcal{B}_i$.*

Proof. Observe that, due to (2.6), we have

$$\overline{E}_j(z+h)\overline{E}_j(z)v = 0 \qquad \text{and} \qquad \frac{\overline{E}_j(z+h)-\overline{E}_j(z-h)}{2h}\overline{E}_j(z)v = 0 \tag{3.5}$$

for any $v \in W_i$. We can view the expressions in (3.5) as Taylor series in the variable h. By considering their constant terms with respect to h we conclude

$$\overline{E}_j(z)\overline{E}_j(z)v = 0 \mod hW_i \quad \text{and} \quad \left(\frac{d}{dz}\overline{E}_j(z)\right)\overline{E}_j(z)v = 0 \mod hW_i. \tag{3.6}$$

Fix an arbitrary positive integer N and the monomial

$$E_1 \coloneqq \overline{e}_{i_m}(k_m)\dots\overline{e}_{i_{p+2}}(k_{p+2})\overline{e}_{i_{p+1}}(k_{p+1})\overline{e}_{i_p}(k_p)\overline{e}_{i_{p-1}}(k_{p-1})\dots\overline{e}_{i_1}(k_1)v_i \in \mathcal{S}_i. \tag{3.7}$$

We will prove that there exist an element K_1 in the h-adically completed $\mathbb{C}[[h]]$-span of $\mathcal{B}_i$ such that $E_1 = K_1 \mod h^N W_i$. Since N was arbitrary and $\mathcal{F}_i$ is h-adically complete, this implies the statement of the lemma.

Due to (2.4), we can assume without loss of generality that for all indices r the equality $i_r = i_{r+1}$ implies $k_{r+1} \leqslant k_r$. Let p be the minimal integer such that $i_p = i_{p+1}$ and $k_{p+1} > k_p - 2$. If such integer p does not exist, the monomial E_1 already belongs to $\mathcal{B}_i$, so the proof is over.

By considering the coefficients of the variable z in (3.6) and arguing as in [11], we can find the $\mathbb{C}$-linear combination L_1', which consists of some monomials of the form

$$\overline{e}_{j_m}(l_m)\dots\overline{e}_{j_{p+2}}(l_{p+2})\overline{e}_{j_{p+1}}(l_{p+1})\overline{e}_{j_p}(l_p)\overline{e}_{j_{p-1}}(l_{p-1})\dots\overline{e}_{j_1}(l_1)v_i \in \mathcal{S}_i \tag{3.8}$$

satisfying that for all $r = 1,\dots,p$ equality $j_r = j_{r+1}$ implies $l_{r+1} \leqslant l_r - 2$, such that

$$E_1 = L_1' \mod hW_i.$$

By applying the aforementioned procedure for an appropriate number of times, so that in each step the procedure is applied on all monomials (3.8) obtained in the previous step, we obtain the $\mathbb{C}$-linear combination L_1 of some monomials in $\mathcal{B}_i$ such that

$$E_1 = L_1 \mod hW_i.$$

Note that $E_2 \coloneqq (E_1 - L_1)/h$ is a well-defined element of W_i. Hence, we can find a $\mathbb{C}[[h]]$-linear combination of some monomials of the form (3.8), which coincides with E_2 modulo hW_i. Next, by arguing as above, we conclude that every such monomial coincides modulo hW_i with some $\mathbb{C}$-linear combination of some elements of $\mathcal{B}_i$. Therefore, there exists a $\mathbb{C}[[h]]$-linear combination L_2 of some elements of $\mathcal{B}_i$ satisfying $E_2 = L_2 \mod hW_i$. This implies

$$E_1 = L_1 + hL_2 \mod h^2W_i.$$

Clearly, we can now proceed inductively and find $\mathbb{C}[[h]]$-linear combinations $L_3, \ldots, L_N$ of some elements of $\mathcal{B}_i$ such that

$$E_1 = L_1 + hL_2 + \ldots + h^{N-1}L_N \mod h^N W_i,$$

thus proving the lemma. □

The following theorem is the main result in this section.

THEOREM 3.3. *For any $i = 0, \ldots, n-1$ the set $\mathcal{B}_i$ forms a basis for a dense $\mathbb{C}[[h]]$-submodule of W_i.*

Proof. Due to Lemma 3.2, it is sufficient to prove that the set $\mathcal{B}_i$ is linearly independent over $\mathbb{C}[[h]]$. Since the proof of linear independence closely follows [11], we will only provide a brief outline, so that we cover the differences. In particular, the proof in [11] relies on certain intertwining operators. Therefore, our first goal is to find the explicit formulae for the operators which exhibit similar properties, with respect to $\overline{E}_j(z)$, as those in the classical setting. These operators will be denoted by $\mathcal{Y}_j(z)$.

For any series $a = a(z)$ denote by $\widehat{a}$ its classical limit $(a)|_{h=0}$. In order to simplify our notation, we will denote the classical limits $\widehat{\overline{E}}_j(z)$ of the operators $\overline{E}_j(z)$ and the classical limits $\widehat{\overline{e}}_j(r)$ of their coefficients $\overline{e}_j(r)$ by $\widehat{E}_j(z)$ and $\widehat{e}_j(r)$ respectively. This notation should not cause any confusion because the classical limits of the operators $E_j(z)$ are not considered anywhere in this paper. For example, we see from (2.3) that

(3.9)
$$\widehat{E}_j(z) = \exp\left(\sum_{k>0} \tfrac{2a_{j,-k}}{k} z^k\right) \exp\left(-\sum_{k>0} \tfrac{a_{j,k}}{k} z^{-k}\right) e^{\alpha_j} \left((-1)^{j-1} z\right)^{\partial_{\alpha_j}}.$$

By applying (3.9) on v_i we get

$$\widehat{E}_j(z) v_i \;=\; \widehat{e}_j(-1-\delta_{ij}) v_i z^{\delta_{ij}} \;+\; \text{higher powers of } z,$$

where

$$\widehat{e}_j(-1-\delta_{ij}) v_i = (-1)^{(j-1)\delta_{ij}} (1 \otimes e^{\alpha_j + \lambda_i}). \tag{3.10}$$

We will also need the following simple formula:

$$\widehat{e}_j(r) e^{\alpha_k} = (-1)^{(j-1)(\delta_{j\,k-1} + \delta_{j\,k+1})} e^{\alpha_k} \widehat{e}_j(r + 2\delta_{jk} - \delta_{j\,k-1} - \delta_{j\,k+1}). \tag{3.11}$$

Consider the operators $\mathcal{Y}_j(z) \in \mathcal{E}_h(\mathcal{F})$ defined by

$$\mathcal{Y}_j(z) = \mathcal{Y}_j^-(z) \mathcal{Y}_j^+(z) e^{\lambda_j} \left((-1)^j z\right)^{\partial_{\lambda_j}}, \qquad j = 1, \ldots, n-1,$$

where

$$\mathcal{Y}_j^+(z) = \exp\left(-\sum_{k>0} \tfrac{a_{j,k}}{2k} z^{-k}\right) \quad \text{and} \quad \mathcal{Y}_j^-(z) = \exp\left(\sum_{k>0} \tfrac{a_{j,-k}}{k} z^k\right).$$

Notice that $\widehat{\mathcal{Y}}_j(z) = \mathcal{Y}_j(z)$. Using (1.10) one can easily prove that

$$[\widehat{E}_j(z_1), \mathcal{Y}_k(z_2)] = 0 \quad \text{for all } j, k = 1, \ldots, n-1. \tag{3.12}$$

In particular, the classical limits $\widehat{e}_j(r)$ of coefficients of $\overline{E}_j(z)$ commute with the coefficients of the operator $\mathcal{Y}_k(z)$. Moreover, for any $i = 0, \dots, n-1$ and $j, k = 1, \dots, n-1$ we have

$$z^{\partial_{\lambda_k}} v_i = z^{(\lambda_k, \lambda_i)} v_i \qquad \text{and} \qquad \widehat{e}_j(r) e^{\lambda_k} = e^{\lambda_k} \widehat{e}_j(r + \delta_{jk}). \tag{3.13}$$

We are now ready to prove the linear independence of the set $\mathcal{B}_i$. Clearly, it is sufficient to prove that the set

$$\widehat{\mathcal{B}}_i = \Big\{ \widehat{b} \,:\, b \in \mathcal{B}_i \Big\}$$

is linearly independent. Our first goal is to show that the set $\widehat{\mathcal{B}}_i$ does not contain zero. We proceed as in the proof of [11, Theorem 4.2]. For an arbitrary element

$$b = \overline{e}_{j_m}(l_m) \dots \overline{e}_{j_2}(l_2) \overline{e}_{j_1}(l_1) v_i \in \mathcal{B}_i$$

set

$$\widehat{b} = \widehat{e}_{j_m}(l_m) \dots \widehat{e}_{j_2}(l_2) \widehat{e}_{j_1}(l_1) v_i \in \widehat{\mathcal{B}}_i.$$

By applying $\operatorname{Res}_z z^{-(\lambda_{j_1}, \lambda_i) - 1} \mathcal{Y}_{j_1}(z)$ on $\widehat{b}$ and using (3.12) we get

$$\operatorname*{Res}_z z^{-(\lambda_{j_1}, \lambda_i) - 1} \mathcal{Y}_{j_1}(z) \widehat{b} = \operatorname*{Res}_z z^{-(\lambda_{j_1}, \lambda_i) - 1} \widehat{e}_{j_m}(l_m) \dots \widehat{e}_{j_2}(l_2) \widehat{e}_{j_1}(l_1) \mathcal{Y}_{j_1}(z) v_i.$$

By the first equality in (3.13) this equals to

$$\widehat{e}_{j_m}(l_m) \dots \widehat{e}_{j_2}(l_2) \widehat{e}_{j_1}(l_1) e^{\lambda_{j_1}} v_i.$$

Next, we employ the second equality in (3.13) to move the invertible element $e^{\lambda_{j_1}}$ to the left and then, we multiply the expression by $e^{-\lambda_{j_1}}$ thus canceling $e^{\lambda_{j_1}}$:

$$\widehat{e}_{j_m}(l_m + \delta_{j_1 j_m}) \dots \widehat{e}_{j_2}(l_2 + \delta_{j_1 j_2}) \widehat{e}_{j_1}(l_1 + 1) v_i. \tag{3.14}$$

In (3.14), all indices l_s such that $j_s = j_1$ are increased by 1, while the indices l_p for $p > s$ remain unchanged.

We can repeat this procedure of increasing indices until the rightmost element in (3.14) becomes $\widehat{e}_{j_1}(-1 - \delta_{i j_1})$. Then, we use (3.10) to express the resulting monomial as

$$\begin{aligned} &\widehat{e}_{j_m}(l_m - (1 + \delta_{i j_1} + l_1) \delta_{j_1 j_m}) \dots \\ &\quad \dots \widehat{e}_{j_2}(l_2 - (1 + \delta_{i j_1} + l_1) \delta_{j_1 j_2}) \widehat{e}_{j_1}(-1 - \delta_{i j_1}) v_i \\ =&(-1)^{(j_1 - 1)\delta_{i j_1}} \widehat{e}_{j_m}(l_m - (1 + \delta_{i j_1} + l_1) \delta_{j_1 j_m}) \dots \\ &\quad \dots \widehat{e}_{j_2}(l_2 - (1 + \delta_{i j_1} + l_1) \delta_{j_1 j_2}) e^{\alpha_{j_1}} v_i. \end{aligned} \tag{3.15}$$

Finally, we employ (3.11) to move the invertible element $e^{\alpha_{j_1}}$ to the left and then, we multiply the expression by $\pm e^{-\alpha_{j_1}}$. As a result, we obtain another element of $\mathcal{B}_i$, a monomial consisting of $m-1$ operators $\widehat{e}_j(r)$ acting on v_i.

We can now continue to reduce the length of the resulting monomial via the above described procedure, until we obtain the nonzero element v_i.

This proves that $\widehat{b}$ is nonzero, so the original monomial b is nonzero as well. Therefore, the sets $\widehat{\mathcal{B}}_i$ and $\mathcal{B}_i$ do not contain zero.

A similar technique can be used to prove that the set $\widehat{\mathcal{B}}_i$ is linearly independent, which implies the statement of the theorem; see [11, Theorem 4.2] for details. □

We can now easily recover the character formulae from [9, 11] as follows. Define the degree of the monomial

$$b = \overline{e}_{j_m}(l_m)\dots\overline{e}_{j_1}(l_1)v_i \in \mathcal{B}_i \qquad \text{by} \qquad \deg b = -l_m - \dots - l_1.$$

Denote by $r_p = r_p(b)$ the number of indices $t = 1, \dots, m$ such that $i_t = p$.

Example 1. *For* $b = e_3(-9)e_2(-6)e_2(-4)e_1(-1)v_2 \in \mathcal{B}_2$ *we have*

$$\deg(b) = 9 + 6 + 4 + 1 = 20, \quad r_3(b) = r_1(b) = 1 \quad \textit{and} \quad r_2(b) = 2.$$

For $\widetilde{W}_i := \operatorname{span}_{\mathbb{C}[[h]]} \mathcal{B}_i$ define

$$\operatorname{ch}_q \widetilde{W}_i = \sum_{d \geqslant 0} \dim W_i^d \cdot q^d, \quad \text{where} \quad W_i^d = \operatorname*{span}_{\mathbb{C}[[h]]} \{b \in \mathcal{B}_i \,:\, \deg b = d\}. \tag{3.16}$$

Note that the direct sum of all W_i^d is only a dense $\mathbb{C}[[h]]$-submodule of W_i, so writing $\operatorname{ch}_q W_i$ instead of $\operatorname{ch}_q \widetilde{W}_i$ in (3.16) might be misleading.

Corollary 3.4. *For any* $i = 0, \dots, n-1$ *we have*

$$\operatorname{ch}_q \widetilde{W}_i = \sum_{r_1 \geqslant 0} \frac{q^{r_1^2 + r_1\delta_{i1}}}{(q)_{r_1}} \sum_{r_2 \geqslant 0} \frac{q^{r_2^2 + r_2(\delta_{i2} - r_1)}}{(q)_{r_2}} \dots \sum_{r_{n-1} \geqslant 0} \frac{q^{r_{n-1}^2 + r_{n-1}(\delta_{i\,n-1} - r_{n-2})}}{(q)_{r_{n-1}}}.$$

4. Nonlocal h-vertex algebras

In this section, we follow the approach in [16] to construct the nonlocal h-vertex algebra generated by the operator $\overline{E}_1(z) \in \mathcal{E}_h(\mathcal{F})$. Since we restrict our considerations to $\mathrm{DY}(\mathfrak{sl}_2)$, we will omit index 1 and write $\overline{E}(z)$ and $\overline{e}(r)$ instead of $\overline{E}_1(z)$ and $\overline{e}_1(r)$.

Recall that the $\mathbb{C}[[h]]$-module V is said to be *separable* if $\cap_{m \geqslant 1} h^m V = 0$ and that V is said to be *torsion-free* if $hv \neq 0$ for all nonzero $v \in V$. Moreover, V is said to be *topologically free* if it is separable, torsion-free and complete with respect to the h-adic topology. For example, the $\mathbb{C}[[h]]$-modules $\mathcal{F}_i$ and $\mathcal{F}$, constructed in Section 1, are topologically free. Any topologically free $\mathbb{C}[[h]]$-module V can be written as $V = V_0[[h]]$ for some complex vector space V_0; see [13] for details.

Definition 4.1. *A* nonlocal h-vertex algebra *is a triple* $(V, Y, \mathbf{1})$, *where* V *is a separated and torsion-free* $\mathbb{C}[[h]]$*-module with a* $\mathbb{C}[[h]]$*-module map*

$$Y(\cdot, z_0) \colon V \to \mathcal{E}_h(V) \tag{4.1}$$

$$a \mapsto Y(a, z_0) = \sum_{r\in\mathbb{Z}} a_r z_0^{-r-1}$$

and a distinguished element $\mathbf{1}$ *in* V *such that the following conditions hold:*

$$Y(\mathbf{1}, z_0)a = a \quad \textit{for all } a \in V, \tag{4.2}$$

$$Y(a, z_0)\,\mathbf{1} \in V[[z_0]] \quad \textit{and} \quad \lim_{z_0\to 0} Y(a, z_0)\,\mathbf{1} = a \quad \textit{for all } a \in V, \tag{4.3}$$

$$Y(a, z_0 + z_2)Y(b, z_2)c = Y(Y(a, z_0)b, z_2)c \quad \textit{for all } a, b, c \in V. \tag{4.4}$$

Axiom (4.4) is equivalent to

$$(a_r b)_s\, c = \sum_{l\geqslant 0} \binom{l-r-1}{l} a_{r-l}\,(b_{s+l}c) \quad \text{for all } a, b, c \in V,\ r, s \in \mathbb{Z}. \tag{4.5}$$

In Definition 4.1, we implicitly assume that the infinite sum in (4.5) contains finitely many nonzero summands, so that the right hand side in (4.4) is well-defined.

Remark 4.2. *The above definition, which was given in* [16]*, is a minor modification of the notion of* h-adic nonlocal vertex algebra*, originally introduced by Li in* [19]*. In contrast with* h*-adic nonlocal vertex algebras, Definition* 4.1 *does not require for the underlying* $\mathbb{C}[[h]]$*-module* V *to be* h*-adically complete. This will slightly simplify the construction in this section, resulting in "smaller" objects which posses nice combinatorial interpretation. Also, Definition* 4.1 *requires for a much stronger form of the associativity axiom to hold, which is useful in this particular setting. However, in general theory, the weaker form as in* [19] *should be considered.*

For any $a(z)$ in $\mathcal{E}_h(\mathcal{F})$ define its $2h$-derivation by

$$a^{(1)}(z) = \frac{d_{2h}}{d_{2h}z} a(z) = \frac{a(z+2h) - a(z)}{2h}. \tag{4.6}$$

Observe that $a^{(1)}(z)$ and $a(z+h)$ belong to $\mathcal{E}_h(\mathcal{F})$ for any $a(z) \in \mathcal{E}_h(\mathcal{F})$ because $\mathcal{F}$ is h-adically complete. Even though we will construct some nonlocal h-vertex algebras contained in $\mathcal{E}_h(\mathcal{F})[t^{1/2}]$, which are not h-adically complete, the completeness of the underlying $\mathbb{C}[[h]]$-module $\mathcal{F}$ will be important for our construction.

Definition 4.3. *Let*

$$(a(z, t^{1/2}), b(z, t^{1/2})) = (\textstyle\sum_{i=m_1}^{m_2} a_i(z)t^{i/2}, \sum_{j=k_1}^{k_2} b_j(z)t^{j/2})$$

be a pair in $\mathcal{E}_h(\mathcal{F})[t^{\pm 1/2}]$ *satisfying*

$$[a_i(z_1), b_j(z_2)] = 0 \quad \textit{for all } i = m_1, \dots, m_2,\ j = k_1, \dots, k_2.$$

For any integer r define the element $a(z,t^{1/2})_{-r-1}b(z,t^{1/2})$ in $\mathcal{E}_h(\mathcal{F})[t^{\pm 1/2}]$ by

$$a(z,t^{1/2})_{-r-1}b(z,t^{1/2}) = \sum_{i=m_1}^{m_2}\sum_{j=k_1}^{k_2}\left(a_i(z)t^{i/2}\right)_{-r-1}\left(b_j(z)t^{j/2}\right),$$

where for $r<0$ we set $\left(a_i(z)t^{i/2}\right)_{-r-1}\left(b_j(z)t^{j/2}\right)=0$ while for $r\geqslant 0$ we define

$$(a_i(z)t^{i/2})_{-r-1}(b_j(z)t^{j/2}) = \frac{1}{r!}a_i^{(r)}(z)b_j(z+(i/2+r)\cdot 2h)t^{i/2+j/2+r}.$$

The products of vertex operators, given in Definition 4.3, were already studied in [16]. However, instead of using (4.6), they were introduced via h-derivation $a(z)\mapsto (a(z+h)-a(z))/h$ and integral powers of the variable t. These minor differences do not affect the construction of nonlocal h-vertex algebras provided therein, so, by [16, Theorem 1.11], we have

THEOREM 4.4. *There exists a unique, smallest nonlocal h-vertex algebra $(V_t,Y_t,\mathbf{1})$ which satisfies the folowing:*

- *V_t is a $\mathbb{C}[[h]]$-submodule of $\mathcal{E}_h(\mathcal{F})[t^{\pm 1/2}]$;*
- *The operator $\overline{E}(z,t^{1/2}):=\overline{E}(z)t^{-1/2}\in\mathcal{E}_h(\mathcal{F})[t^{\pm 1/2}]$ belongs to V_t;*
- *The vertex operator map is defined by*

$$Y_t(a,z)b := \sum_{r\in\mathbb{Z}} a_{-r-1}b\,z^r, \quad \text{where } a,b\in V_t,$$

  and the vertex operator products are given by Definition 4.3;
- *The element $\mathbf{1}$ denotes the identity $V_t\to V_t$.*

Moreover, we have

$$V_t = \operatorname*{span}_{\mathbb{C}[[h]]}\left\{\overline{E}(z,t^{1/2})_{r_m}\dots\overline{E}(z,t^{1/2})_{r_1}\mathbf{1}\,:\,r_1,\dots,r_m\in\mathbb{Z},\,m\geqslant 0\right\}. \tag{4.7}$$

Our goal is to obtain a nonlocal h-vertex algebra structure on the evaluation of V_t at $t^{1/2}=1$. In order to achieve this, we first need to derive a basis for V_t. Observe that (2.6) is equivalent to

$$\overline{E}(z,t^{1/2})_{-1}\overline{E}(z,t^{1/2}) = \overline{E}(z,t^{1/2})_{-2}\overline{E}(z,t^{1/2}) = 0, \tag{4.8}$$

see also (3.5). We can use associativity (4.5), together with (4.8), to reduce spanning set (4.7). The following lemma can be proved as [16, Lemma 2.6].

LEMMA 4.5. *Nonlocal h-vertex algebra V_t is spanned by the set*

$$\mathcal{B}_{V_t} = \Big\{\overline{E}(z,t^{1/2})_{r_m}\dots\overline{E}(z,t^{1/2})_{r_1}\mathbf{1}\,:\,$$
$$r_1,\dots,r_m\in\mathbb{Z},\,r_1\leqslant -1,\,r_2,\dots,r_m\leqslant -3,\,m\geqslant 0\Big\}.$$

Consider the evaluation $V_1 \subseteq \mathcal{E}_h(\mathcal{F})$ of V_t at $t = 1$,

$$V_1 = (V_t)|_{t^{1/2}=1} = \left\{ a(z,1) \,:\, a(z,t^{1/2}) \in V_t \right\}.$$

THEOREM 4.6. *The sets $\mathcal{B}_{V_t}$ and $\mathcal{B}_{V_1} := (\mathcal{B}_{V_t})|_{t^{1/2}=1}$ are linearly independent over $\mathbb{C}[[h]]$. Hence, they form bases for V_t and V_1 respectively. Moreover, the evaluation* $\mathrm{ev}\colon V_t \to V_1$

$$\mathrm{ev} \,:\, a(z,t^{1/2}) \mapsto a(z,1)$$

is injective. Consequently, the $\mathbb{C}[[h]]$-module V_1 possesses the structure of a nonlocal h-vertex algebra $(V_1, Y_1, \mathbf{1})$ such that the vertex operator map Y_1 satisfies

$$Y_1(a,z_0)b = \sum_{r\in\mathbb{Z}} \mathrm{ev}\left(\mathrm{ev}^{-1}(a)_r \,\mathrm{ev}^{-1}(b)\right) z_0^{-r-1} \quad \textit{for any } a,b \in V_1.$$

Proof. Observe that the first equation in (1.15) and relation $[k_1(z_1), \Psi_1(z_2)] = 0$ imply

$$\overline{E}(z_1)\Psi_1(z_2 - h/4) = \left(1 - \frac{z_2}{z_1}\right)\Psi_1(z_2 - h/4)\overline{E}(z_1). \tag{4.9}$$

Next, for any $a(z) \in \mathcal{E}_h(\mathcal{F})$ and a positive integer r we have

$$a^{(r)}(z) = \frac{1}{2^r h^r} \sum_{l=0}^{r} \binom{r}{l} (-1)^l a(z + 2(r-l)h), \tag{4.10}$$

which can be easily verified by induction over r. The theorem can be now proved by using (4.9), (4.10) and by repeating the arguments in the proof of [16, Theorem 2.11] (almost) verbatim. □

REMARK 4.7. *The appropriately completed nonlocal h-vertex algebra V_1 would provide an example of an h-adic nonlocal vertex algebra, as defined in* [19]. *However, in contrast with* [19], *the structures studied in this paper are constructed using different vertex operator products; the products in Li's general theory of h-adic nonlocal vertex algebras utilize the derivation*

$$a(z) \mapsto \frac{d}{dz} a(z), \tag{4.11}$$

while the example constructed in Theorem 4.6 *relies on $2h$-derivation* (4.6).

REMARK 4.8. *Nonlocal h-vertex algebra V_1 is not commutative. Indeed, for*

$$a^{(n)} := \overline{E}(z,1)_{-n}\,\mathbf{1} \qquad \textit{we have, for example,} \qquad a^{(1)}_{-1} a^{(3)}_{-1}\,\mathbf{1} \neq a^{(3)}_{-1} a^{(1)}_{-1}\,\mathbf{1}\,.$$

Consequently, the underlying associative algebra structure, defined by $a \cdot b := a_{-1}b$ for $a,b \in V_1$, is not commutative; recall (4.5). *However, since the classical limit of $2h$-derivation* (4.6) *is derivation* (4.11), *the classical limit of V_1 is a commutative vertex algebra.*

In the end, it is worth noting that

$$V_1 = \bigoplus_{d\geqslant 0} V_1^d, \quad \text{where} \quad V_1^d = \operatorname*{span}_{\mathbb{C}[[h]]} \{b \in \mathcal{B}_{V_1} \,:\, \deg b = d\}$$

and $\deg b$ for $b = \overline{E}(z,1)_{r_m} \dots \overline{E}(z,1)_{r_1}\, \mathbf{1} \in \mathcal{B}_{V_1}$ is defined by

$$\deg b = m - \sum_{j=1}^{m} (m-j+1)(r_j+1).$$

Finally, as with the bases found in [16], we have

$$\operatorname{ch}_q V_1 \;:=\; \sum_{d\geqslant 0} \dim V_1^d \cdot q^d \;=\; \sum_{r\geqslant 0} \frac{q^{r^2}}{(q)_r}.$$

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