

GENERALIZED SUSPENSION THEOREM IN EXTENSION THEORY

LEONARD R. RUBIN

University of Oklahoma, USA

ABSTRACT. A. Dranishnikov proved that for each CW-complex K and metrizable compactum X with $X\tau K$, it is true that $(X \times I)\tau(\Sigma K)$. Here, ΣK means the suspension of K in the CW-category, and by $X\tau K$ we mean that K is an absolute extensor for X . We are going to generalize this result so that X could be either a stratifiable space or a compact Hausdorff space. Since all metrizable spaces are stratifiable, then our result generalizes Dranishnikov's.

1. INTRODUCTION

In [3], Dranishnikov proved (see Theorem 4) that for each CW-complex K and metrizable compactum X with $X\tau K$, it is true that $(X \times I)\tau(\Sigma K)$, ΣK being the suspension of K in the CW-category. By $X\tau K$ we mean that K is an absolute extensor for X , or as we prefer to say, X is an absolute co-extensor for K . Although Theorem 4 of [3] was stated for an arbitrary metrizable space X , the proof that was given requires X to be compact. Here is our main theorem.

THEOREM 1.1. *Let X be either a stratifiable or compact Hausdorff space and K be a CW-complex such that $X\tau K$. Then $(X \times I)\tau(\Sigma K)$.*

As far as the class of stratifiable spaces is concerned, here are the defining terms.

2010 *Mathematics Subject Classification.* 54C55, 54C20.

Key words and phrases. Absolute co-extensor, absolute extensor, absolute neighborhood extensor, CW-complex, extension theory, paracompact, shrinking a cover, stratifiable space, stratification, suspension.

DEFINITION 1.2. A **stratification** on a space X is a sequence (s_k) , such that for each k , s_k is a function that assigns to every open set $U \subset X$ an open set $s_k(U) \subset X$ such that:

- (S1) $\text{cl}_X(s_k(U)) \subset U$,
- (S2) $\bigcup_{k=1}^{\infty} s_k(U) = U$,
- (S3) $U \subset V \Rightarrow s_k(U) \subset s_k(V)$.

A space is called **stratified** if it has a stratification and is called **stratifiable** if it is stratified and in addition is a T_1 -space.

This class of spaces was introduced in 1961 by J. G. Ceder [2] under the name M_3 -spaces, and was renamed by C. J. Borges [1] to stratifiable spaces. An exposition of generalized metrizable spaces, including stratifiable spaces, is given by G. Gruenhage [4] in the Handbook of Set-Theoretic Topology. What is important for us is that stratifiable spaces are hereditarily stratifiable, Hausdorff, paracompact, perfectly normal, and that any countable product of stratifiable spaces is stratifiable. They satisfy the subspace theorem in extension theory, namely, if K is a CW-complex, X is a stratifiable space, and $X \tau K$, then for each subspace $Y \subset X$, $Y \tau K$. This result appears as Theorem 18.12 of [6], but we will only need it in the weak form, i.e., when Y is a closed subspace of X . Moreover, every CW-complex is an absolute neighborhood extensor for the class of stratifiable spaces, and every metrizable space is stratifiable. Hence Theorem 1.1 has the following corollary.

COROLLARY 1.3. Let X be a metrizable space and K a CW-complex such that $X \tau K$. Then $(X \times I) \tau (\Sigma K)$. $\square$

2. OPEN SETS IN PRODUCTS

In this presentation, map will always mean continuous function. For each pair X, Y of spaces, let $C(X, Y)$ denote the set of maps of X to Y .

DEFINITION 2.1. Let X be a space and U an open subset of X . Denote $C(X, U, I) = \{f \in C(X, I) \mid (f(X \setminus U) \subset \{0\}) \wedge (f(U) \subset (0, 1])\}$. For each $f \in C(X, U, I)$ and $t \in I$, let $f_t^+ : X \rightarrow \mathbb{R}$ be the map given by $f_t^+(x) = f(x) + t$ and $f_t^- : X \rightarrow \mathbb{R}$ the map given by $f_t^-(x) = -f(x) + t$.

LEMMA 2.2. Let X be a space, U an open subset of X , $t \in I$, and $f \in C(X, U, I)$. Then

- (1) $\{f_t^+, f_t^-\} \subset C(X, \mathbb{R})$,
- (2) for each $x \in U$, $f_t^-(x) < f_t^+(x)$, and
- (3) for each $x \in X \setminus U$, $f_t^+(x) = t = f_t^-(x)$. $\square$

DEFINITION 2.3. Let X be a space, U an open subset of X , $t \in I$, and $f \in C(X, U, I)$. When we speak of the subspace of $X \times \mathbb{R}$ **between** f_t^- and f_t^+ we refer to $\{(x, y) \in U \times \mathbb{R} \mid f_t^-(x) < y < f_t^+(x)\}$. We shall denote this

subspace of $X \times \mathbb{R}$ $\text{betw}(f, U, t)$, noting that $\text{betw}(f, U, t) \subset U \times \mathbb{R}$. Let us use $\text{Betw}(X)$ to denote the collection of all such subsets $\text{betw}(f, U, t)$ of $X \times \mathbb{R}$.

Whenever X is a space and $f \in C(X, \mathbb{R})$, then by G_f we mean the graph of f , a closed subset of $X \times \mathbb{R}$. Recall that G_f is homeomorphic to X . In (2) of the next lemma, the two closed subsets of $X \times \mathbb{R}$ are the respective graphs of f_t^- and f_t^+ when these maps are restricted to $\text{cl}_X U$.

LEMMA 2.4. *Let X be a space and $W \in \text{Betw}(X)$. Then*

- (1) *W is an open subset of $X \times \mathbb{R}$, and*
- (2) *in case $W = \text{betw}(f, U, t)$, then $\text{bd}_{X \times \mathbb{R}} W$ equals the union of two closed subsets of $X \times \mathbb{R}$, each of which is homeomorphic to $\text{cl}_X U$. $\square$*

Our interest is in spaces of the form $X \times I$. Lemma 2.4 leads to the next fact.

LEMMA 2.5. *Let X be a space, $W \in \text{Betw}(X)$, and $M = W \cap (X \times I)$. Then*

- (1) *M is an open subset of $X \times I$, and*
- (2) *$\text{bd}_{X \times I} M$ equals the union of two closed subsets of $X \times I$, each of which is homeomorphic to a closed subspace of $\text{cl}_X U$. $\square$*

LEMMA 2.6. *Let Q, U , be open subsets of a normal space X with $Q \subset \text{cl}_X Q \subset U$, and J an interval in I . Then there exists an element $W \in \text{Betw} X$ such that $W \cap (X \times I) \subset U \times J$, and $Q \times J \subset W$. Moreover, we may choose $f \in C(X, I)$, an open subset U_0 of X with $U_0 \subset U$, and $t \in J$ such that $W = \text{betw}(f, U_0, t)$.*

PROOF. Let t be the midpoint of J , and $\epsilon > 0$ equal half the length of J . Choose $f \in C(X, [0, \epsilon]) \subset C(X, I) \subset C(X, \mathbb{R})$ so that $f(\text{cl}_X Q) \subset \{\epsilon\}$ and $f(X \setminus U) \subset \{0\}$. Let $U_0 = \{x \in U \mid f(x) > 0\}$. Thus, $U_0 \subset U$ is an open subset of X , and $\text{cl}_X Q \subset U_0$. One can see that $f \in C(X, U_0, I)$. Put $W = \text{betw}(f, U_0, t)$. We ask the reader, using Definition 2.3, to check that W meets the requirements stated above. $\square$

3. PROOF OF MAIN THEOREM

PROOF OF THEOREM 1.1. We denote the suspension of K as $\Sigma K = \{v^+, v^-\} * K$ where $\{v^+, v^-\}$ is a two-element CW-complex that is disjoint from K . Let A be a closed subset of $X \times I$ and $f : A \rightarrow \Sigma K$ a map. Designate $A^+ = f^{-1}(v^+)$ and $A^- = f^{-1}(v^-)$. Choose an open cover $\mathcal{E}$ of the regular space $X \times I$ such that if $E \in \mathcal{E}$ and $E \cap A^+ \neq \emptyset$, then $(\text{cl}_{X \times I} E) \cap A^- = \emptyset$.

Fix $x \in X$. There is a finite open cover $\mathcal{J}_x$ of I consisting of intervals, such that for each $J \in \mathcal{J}_x$, there exists an element $E_J \in \mathcal{E}$ with $\{x\} \times \text{cl}_I J \subset E_J$. Now choose an open neighborhood V_x of x in X having the property that:

- (*_1) $V_x \times J \subset E_J$ for each $J \in \mathcal{J}_x$.

Having done the preceding for each $x \in X$, choose a locally finite open cover $\{U_x | x \in X\}$ of X which is a shrinking of $\{V_x | x \in X\}$, i.e., for each $x \in X$, $\text{cl}_X U_x \subset V_x$. Let $\{Q_x | x \in X\}$ be an open cover of X that is a shrinking of $\{U_x | x \in X\}$. Hence,

($*_2$) for each $x \in X$, $\text{cl}_X Q_x \subset U_x \subset \text{cl}_X U_x \subset V_x$.

Making use of ($*_1$) and ($*_2$), we get:

($*_3$) for each $x \in X$ and $J \in \mathcal{J}_x$, $U_x \times J \subset E_J$.

Clearly, $\{U_x \times I | x \in X\}$ is a locally finite open cover of $X \times I$. Once again, fix $x \in X$; fix also $J \in \mathcal{J}_x$. Apply Lemma 2.6 with U_x , Q_x respectively in place of U and Q , to find $W_{x,J} \in \text{Betw}(X)$ such that,

($*_4$) there exist $f \in C(X, I)$, an open subset U_0 of X , $U_0 \subset U_x$, and $t \in J$ such that $W_{x,J} = \text{betw}(f, U_0, t)$,

($*_5$) $M_{x,J} = W_{x,J} \cap (X \times I) \subset U_x \times J$, and

($*_6$) $Q_x \times J \subset W_{x,J}$.

Indeed, since $Q_x \times J \subset X \times I$, then:

($*_7$) $Q_x \times J \subset M_{x,J}$.

Let $\mathcal{P}_x = \{M_{x,J} | J \in \mathcal{J}_x\}$. Then $\mathcal{P}_x$ is a finite collection of open subsets of $X \times I$ that covers $Q_x \times I$, and $\bigcup \mathcal{P}_x \subset U_x \times I$. Denote $\mathcal{P} = \bigcup \{\mathcal{P}_x | x \in X\}$. Then $\mathcal{P}$ is a locally finite open cover of $X \times I$. If we apply ($*_5$), ($*_2$), and ($*_1$), we find that for each $P \in \mathcal{P}$, there is an element $E \in \mathcal{E}$ with $P \subset E$. Therefore, if $P \cap A^+ \neq \emptyset$, then $(\text{cl}_{X \times I} P) \cap A^- = \emptyset$.

Let $\mathcal{P}^+ = \{P \in \mathcal{P} | P \cap A^+ \neq \emptyset\}$. Define $W = \bigcup \mathcal{P}^+$. For each $P \in \mathcal{P}^+$, $(\text{cl}_{X \times I} P) \cap A^- = \emptyset$. Since $\mathcal{P}^+$ is a locally finite collection of subsets of $X \times I$, then $A^+ \subset W \subset \text{cl}_{X \times I} W \subset X \times I \setminus A^-$. Again using the fact that $\mathcal{P}^+$ is a locally finite collection of subsets of $X \times I$, one sees that $\text{bd}_{X \times I} W$ is a closed subset of $\bigcup \{\text{bd}_{X \times I} P | P \in \mathcal{P}^+\}$. The latter is a locally finite collection of closed subsets of $X \times I$ each having the form $M_{x,J}$ as in ($*_5$), so an application of Lemma 2.5(2) yields that $(\text{bd}_{X \times I} W)\tau K$.

Of course, $\text{bd}_{X \times I} W$ is a normal space, so $(\text{bd}_{X \times I} W)\tau \mathbb{R}$. The space $\Sigma K \setminus \{v^+, v^-\}$ is homeomorphic to $K \times \mathbb{R}$, so $(\text{bd}_{X \times I} W)\tau(K \times \mathbb{R})$. Therefore there exists a map $\varphi : \text{bd}_{X \times I} W \rightarrow \Sigma K \setminus \{v^+, v^-\} \subset \Sigma K$ that extends the map $f|(A \cap \text{bd}_{X \times I} W)$. Let $\psi = \varphi \cup f : (\text{bd}_{X \times I} W) \cup A \rightarrow \Sigma K$. Certainly, ψ is a map. Moreover, $\psi((A \cap \text{cl}_{X \times I} W) \cup \text{bd}_{X \times I} W) \subset \Sigma K \setminus v^-$. However, $\Sigma K \setminus v^-$ is an open subset of ΣK , so it is an absolute neighborhood extensor for both the class of compact Hausdorff spaces and the class of stratifiable spaces. It is also a contractible space. So Theorem 4.7.1 (page 43) of [5] shows that $(\text{cl}_{X \times I} W)\tau(\Sigma K \setminus v^-)$.

Consider the restriction $\psi^+ = \psi|((A \cap \text{cl}_{X \times I} W) \cup \text{bd}_{X \times I} W)$ with image in $\Sigma K \setminus v^-$. There is a map $\xi^+ : \text{cl}_{X \times I} W \rightarrow \Sigma K \setminus v^-$ that extends ψ^+ . Similarly, we can find a map $\xi^- : (X \times I) \setminus W \rightarrow \Sigma K \setminus v^+$ that extends

$\psi^- = \psi|((A \setminus W) \cup \text{bd}_{X \times I} W)$ whose image is in $\Sigma K \setminus v^+$. The map $\xi^+ \cup \xi^- : X \times I \rightarrow \Sigma K$ is an extension of f . This ends our proof of Theorem 1.1. $\square$

REFERENCES

1. C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.
2. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.
3. A. N. Dranishnikov, *Extension of maps into CW-complexes*, Math. USSR-Sb. **74** (1993), 47–56.; Mat. Sb. **182** (1991), 1300–1310 (Russian).
4. G. Gruenhage, *Generalized metric spaces*, in Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, 423–501.
5. S. T. Hu, Theory of retracts, Wayne State University Press, Detroit, 1965.
6. I. Ivanšić and L. Rubin, Dimension, extension, and shape, in preparation.

L. R. Rubin
 Department of Mathematics
 University of Oklahoma
 Norman, Oklahoma 73019
 USA
E-mail: lrubin@ou.edu
Received: 27.8.2016.