

COMMON FIXED POINT THEOREMS FOR A FAMILY OF
MULTIVALUED F -CONTRACTIONS WITH AN
APPLICATION TO SOLVE A SYSTEM OF INTEGRAL
EQUATIONS

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ABSTRACT. Inspired by the work of Wardowski in [33] and Samet et al. in [26], in this article, we introduce some new contractive conditions for sequence of multi functions. We have constructed non-trivial examples to validate our results. We have applied our results to find a solution of a system of integral equations.

1. INTRODUCTION

The Banach contraction principle is a famous theorem in the field of fixed point theory and it is not wrong to say that it brought about a new era in metric fixed point theory. Since its inception, major and minor developments have been made regarding its generalization. In the recent past Wardowski ([33]) categorized some mappings into a new family and called it F or \mathfrak{F} family. Using the mappings from \mathfrak{F} family he introduced a new contraction condition namely the F -contractions, which effectively generalized the famous Banach contraction condition. Several researchers studying metric fixed point theory have comprehensively generalized the Banach contraction condition, see for example [2, 30, 25, 18, 13, 29, 22, 24, 28, 20, 1, 26, 6, 21, 7, 19, 8, 3–5, 15–17, 27, 12, 31, 14, 10, 11, 9, 23, 32, 33]. Semat *et al.* in [26] also succeeded in generalizing Banach contraction condition by introducing α - ψ -contraction. Many authors appreciated these two conditions which can be seen in [6, 21, 7, 19, 8, 3–5, 15, 16].

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Keeping in view both of these ideas, in this paper we introduce new contraction conditions for a sequence of multifunction and prove corresponding fixed point theorem. We also give a common fixed point theorem for sequence of bounded multifunctions by using the δ -distance. To conclude our findings we establish an existence theorem for a system of integral equations.

We gather some common results, notations and definitions, which are required for this paper. Let (X, d) be a metric space. We denote the set of all nonempty subsets of X by $N(X)$, the class of all nonempty closed subsets of X by $C(X)$ and the class of all nonempty bounded subsets of X by $B(X)$. For $b \in N(X)$, $d(a, B) = \inf\{d(a, b) : b \in N(X)\}$. For $A, B \in B(X)$, $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. Note that δ satisfies all conditions of a metric, except $A = B \Rightarrow \delta(A, B) = 0$. For $A, B \in C(X)$, the generalized Hausdorff metric on $C(X)$ is given as,

$$H(A, B) = \begin{cases} \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\} & \text{if the maximum exists} \\ \infty & \text{otherwise} \end{cases}$$

Wardowski [33] introduced the following definition.

DEFINITION 1.1. Let \mathfrak{F} be the class of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying:

- (F₁) F is increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.
- (F₂) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n \rightarrow \infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\mathfrak{d}_n) = -\infty$.
- (F₃) There exists $k \in (0, 1)$ such that $\lim_{\mathfrak{d} \rightarrow 0^+} \mathfrak{d}^k F(\mathfrak{d}) = 0$.

Following are some examples of such functions.

- (i) $F_a = \ln a$ for each $a \in (0, \infty)$.
- (ii) $F_b = b + \ln b$ for each $b \in (0, \infty)$.
- (iii) $F_c = -\frac{1}{\sqrt{c}}$ for each $c \in (0, \infty)$.

Wardowski ([33]) introduced F -contraction and proved corresponding fixed point theorem as,

DEFINITION 1.2 ([33]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is F -contraction if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Note that if T is F_a -contraction, then it is also Banach contraction. This it is not in the case for F_b -contraction.

THEOREM 1.3 ([33]). Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be F -contraction. Then T has a unique fixed point.

Sgroi and Vetro [29] introduced the following theorem.

THEOREM 1.4 ([29]). Let (X, d) be a complete metric space and let $T : X \rightarrow CB(X)$. Assume that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$(1.1) \quad 2\tau + F(H(Tx, Ty)) \leq F(a_1d(x, y) + a_2d(x, Tx) + a_3d(y, Ty) + a_4d(x, Ty) + Ld(y, Tx)),$$

for each $x, y \in X$ with $Tx \neq Ty$, where $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Then T has a fixed point.

2. MAIN RESULTS

We begin this section by introducing the following definitions.

DEFINITION 2.1. Let $\alpha : X \times X \rightarrow [0, \infty)$. A sequence of mappings $\{T_i : X \rightarrow N(X)\}_{i=1}^{\infty}$ is α -admissible sequence if for each $x \in X$ and $y \in T_i x$ for some $i \in \mathbb{N}$ such that $\alpha(x, y) \geq 1$, then we have $\alpha(y, z) \geq 1$ for each $z \in T_{i+1}y$. A sequence of mappings $\{T_i : X \rightarrow N(X)\}_{i=1}^{\infty}$ is α_* -admissible sequence if for each $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha_*(T_i x, T_j y) \geq 1$ for each $i, j \in \mathbb{N}$, where $\alpha_*(T_i x, T_j y) = \inf\{\alpha(u, v) : u \in T_i x \text{ and } v \in T_j y\}$.

The sequence of mappings is said to be strictly α -admissible and strictly α_* -admissible if we have strict inequality in the above definition.

REMARK 2.2. (i) Note that if a sequence of mappings $\{T_i : X \rightarrow N(X)\}_{i=1}^{\infty}$ is strictly α_* -admissible sequence, then it is strictly α -admissible sequence.

(ii) When $\{T_i\}_{i=1}^{\infty}$ is a constant sequence Definition 2.1 coincide with definition of α -admissible and α_* -admissible given in [21, Page 4] and [7, Page 1] respectively. Furthermore, if T is a singlevalued mapping then these definition 2.1 coincide with [26, Definition 2.2].

DEFINITION 2.3. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \rightarrow C(X)\}_{i=1}^{\infty}$ is an F_α -contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have

$$(2.1) \quad \tau + F(\alpha(x, y)H(T_i x, T_j y)) \leq F(N(x, y)),$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(T_i x, T_j y), N(x, y)\} > 0$, where

$$N(x, y) = a_1d(x, y) + a_2d(x, T_i x) + a_3d(y, T_j y) + a_4d(x, T_j y) + Ld(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

THEOREM 2.4. Let (X, d) be a complete metric space and let $\{T_i : X \rightarrow C(X)\}_{i=1}^{\infty}$ be an F_α -contraction of Hardy-Rogers-type satisfying the following conditions:

- (i) $\{T_i\}_{i=1}^{\infty}$ is strictly α -admissible sequence;
- (ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then the mappings in the sequence $\{T_i\}_{i=1}^\infty$ have a common fixed point.

PROOF. By hypothesis (ii), we assume without loss of generality that there exist $x_0 \in X$ and $x_1 \in T_1x_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in T_i x_1 \forall i \in \mathbb{N}$, then x_1 is a common fixed point. Let $x_1 \notin T_2x_1$, as $\alpha(x_0, x_1) > 1$ there exists $x_2 \in T_2x_1$ such that

$$(2.2) \quad d(x_1, x_2) \leq \alpha(x_0, x_1)H(T_1x_0, T_2x_1).$$

Since F is increasing, we have

$$(2.3) \quad F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)H(T_1x_0, T_2x_1)).$$

From (2.1) we have

$$(2.4) \quad \begin{aligned} \tau + F(d(x_1, x_2)) &\leq \tau + F(\alpha(x_0, x_1)H(T_1x_0, T_2x_1)) \\ &\leq F(a_1d(x_0, x_1) + a_2d(x_0, T_1x_0) + a_3d(x_1, T_2x_1) \\ &\quad + a_4d(x_0, T_2x_1) + Ld(x_1, T_1x_0)) \\ &\leq F(a_1d(x_0, x_1) + a_2d(x_0, x_1) + a_3d(x_1, x_2) \\ &\quad + a_4d(x_0, x_2) + L.0) \\ &\leq F(a_1d(x_0, x_1) + a_2d(x_0, x_1) + a_3d(x_1, x_2) \\ &\quad + a_4(d(x_0, x_1) + d(x_1, x_2))) \\ &= F((a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2)). \end{aligned}$$

Since F is increasing, we get from above that

$$d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2).$$

That is,

$$(1 - a_3 - a_4)d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_1, x_2) < d(x_0, x_1).$$

From (2.4), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

If $x_2 \in T_i x_2 \forall i \in \mathbb{N}$ then x_2 is a common fixed point. Let $x_2 \notin T_3x_2$. Since $\{T_i\}_{i=1}^\infty$ is strictly α -admissible, we have $\alpha(x_1, x_2) > 1$. There exists $x_3 \in T_3x_2$ such that

$$(2.5) \quad d(x_2, x_3) \leq \alpha(x_1, x_2)H(T_2x_1, T_3x_2).$$

Since F is increasing, we have

$$(2.6) \quad F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2)H(T_2x_1, T_3x_2)).$$

From (2.1) we have

$$\begin{aligned}
 \tau + F(d(x_2, x_3)) &\leq \tau + F(\alpha(x_1, x_2)H(T_2x_1, T_3x_2)) \\
 &\leq F(a_1d(x_1, x_2) + a_2d(x_1, T_2x_1) + a_3d(x_2, T_3x_2) \\
 &\quad + a_4d(x_1, T_3x_2) + Ld(x_2, T_2x_1)) \\
 (2.7) \quad &\leq F(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3) \\
 &\quad + a_4d(x_1, x_3) + L.0) \\
 &\leq F(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3) \\
 &\quad + a_4(d(x_1, x_2) + d(x_2, x_3))) \\
 &= F((a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3)).
 \end{aligned}$$

Since F is increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4)d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (2.7) we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in T_n x_{n-1}, \quad x_{n-1} \neq x_n \quad \text{and} \quad \alpha(x_{n-1}, x_n) > 1 \quad \text{for each } n \in \mathbb{N}.$$

Furthermore,

$$(2.8) \quad F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \quad \text{for each } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in (2.8) we get $\lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = -\infty$. Thus by property (F_2) , we have $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0.$$

From (2.8) we have

$$(2.9) \quad d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \quad \text{for each } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in (2.9) we get,

$$(2.10) \quad \lim_{n \rightarrow \infty} n d_n^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for each $n \geq n_1$. Thus we have

$$(2.11) \quad d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1.$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n \rightarrow \infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By condition (iii) we have $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, T_i x^*) = 0 \forall i \in \mathbb{N}$. On contrary suppose that $d(x^*, T_{i_0} x^*) > 0$ for some $i_0 \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, T_{i_0} x^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$ and for above i_0 we have

$$\begin{aligned} (2.12) \quad d(x^*, T_{i_0} x^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*) H(T_{n+1} x_n, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) \\ &\quad + a_3 d(x^*, T_{i_0} x^*) + a_4 d(x_n, T_i x^*) + L d(x^*, x_{n+1}). \end{aligned}$$

Letting $n \rightarrow \infty$ in (2.12) we have

$$d(x^*, T_{i_0} x^*) \leq (a_3 + a_4) d(x^*, T_{i_0} x^*) < d(x^*, T_{i_0} x^*).$$

Which is a contradiction. Thus $d(x^*, T_i x^*) = 0 \forall i \in \mathbb{N}$. \square

EXAMPLE 2.5. Let $X = \mathbb{N}$ be endowed with the usual metric $d(x, y) = |x - y|$ for each $x, y \in X$. Define $\{T_i : X \rightarrow C(X)\}_{i=1}^{\infty}$ by

$$T_i x = \begin{cases} \{0, 1\} & \text{if } x = 0, 1, \\ \{2x - 2, 2x\} & \text{if } x > 1 \end{cases}$$

and $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in \{0, 1\}, \\ \frac{1}{4} & \text{if } x, y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F condition (2.1) reduces to

$$(2.13) \quad \frac{\alpha(x, y) H(T_i x, T_j y)}{N(x, y)} e^{\alpha(x, y) H(T_i x, T_j y) - N(x, y)} \leq e^{-\tau}$$

for each $x, y \in X$ with $\min\{\alpha(x, y)H(T_i x, T_j y), N(x, y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly,

$$\min\{\alpha(x, y)H(T_i x, T_j y), d(x, y)\} > 0$$

for each $x, y > 1$ with $x \neq y$. From (2.13) for each $x, y > 1$ with $x \neq y$ we have

$$\frac{1}{4}e^{-\frac{1}{2}|x-y|} < e^{-\frac{1}{2}}.$$

Thus $\{T_i\}_{i=1}^\infty$ is an α - F -contraction of Hardy-Rogers-type with $F(x) = x + \ln x$. For $x_0 = 1$ we have $x_1 = 0 \in T_1 x_0$ such that $\alpha(x_0, x_1) > 1$. Moreover, it is easy to see that $\{T_i\}_{i=1}^\infty$ is strictly α -admissible sequence and for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 2.4 $\{T_i\}_{i=1}^\infty$ has a common fixed point in X .

DEFINITION 2.6. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \rightarrow C(X)\}_{i=1}^\infty$ is an F_{α^*} -contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have

$$(2.14) \quad \tau + F(\alpha_*(T_i x, T_j y)H(T_i x, T_j y)) \leq F(N(x, y)),$$

for each $x, y \in X$, whenever

$$\min\{\alpha_*(T_i x, T_j y)H(T_i x, T_j y), N(x, y)\} > 0,$$

where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) + a_4 d(x, T_j y) + L d(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

THEOREM 2.7. Let (X, d) be a complete metric space and let $\{T_i : X \rightarrow C(X)\}_{i=1}^\infty$ be an α_* - F -contraction of Hardy-Rogers-type satisfying the following conditions:

- (i) $\{T_i\}_{i=1}^\infty$ is strictly α_* -admissible sequence;
- (ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) > 1$;
- (iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_i\}_{i=1}^\infty$ have a common fixed point.

PROOF. The proof of this theorem runs along the same lines as the proof of Theorem 2.9. \square

DEFINITION 2.8. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \rightarrow B(X)\}_{i=1}^\infty$ is an F_α -contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have

$$(2.15) \quad \tau + F(\alpha(x, y)\delta(T_i x, T_j y)) \leq F(N(x, y)),$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)\delta(T_i x, T_j y), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) \\ + a_4 d(x, T_j y) + L d(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Note that H is not a metric on the set of bounded subsets of X , as the following example shows.

Let $X = \mathbb{R}$, endowed with usual metric then $H(A, B) = 0$ but $A \neq B$ for $A = [0, 1)$ and $B = [0, 1]$. This implies that H is not a metric on Bounded subsets of \mathbb{R} . It would be interesting to see whether the conclusions of Theorem 2.4 hold for bounded subsets of X . We will show that the conclusions of Theorem 2.4 still hold for bounded subsets of X provided that the Hausdorff distance $H(A, B)$ in definition 2.3 is replaced with $\delta(A, B)$ and the strict inequality in (ii) of Theorem 2.4 is replaced by the soft inequality. More precisely we have the following result.

THEOREM 2.9. *Let (X, d) be a complete metric space and let $\{T_i : X \rightarrow B(X)\}_{i=1}^{\infty}$ be an F_α -contraction of Hardy-Rogers-type satisfying the following conditions:*

- (i) $\{T_i\}_{i=1}^{\infty}$ is α -admissible sequence;
- (ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) \geq 1$;
- (iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then the mappings in the sequence $\{T_i\}_{i=1}^{\infty}$ have a common fixed point.

PROOF. By hypothesis (ii), we assume without loss of generality that there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $\alpha(x_0, x_1) \geq 1$. If $x_1 \in T_i x_1 \forall i \in \mathbb{N}$, then x_1 is a common fixed point. Let $x_1 \notin T_2 x_1$. As $\alpha(x_0, x_1) \geq 1$, there exists $x_2 \in T_2 x_1$ such that

$$(2.16) \quad d(x_1, x_2) \leq \alpha(x_0, x_1)\delta(T_1 x_0, T_2 x_1).$$

Since F is increasing, we have

$$(2.17) \quad F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)\delta(T_1 x_0, T_2 x_1)).$$

From (2.15) we have

$$\begin{aligned}
 \tau + F(d(x_1, x_2)) &\leq \tau + F(\alpha(x_0, x_1)\delta(T_1x_0, T_2x_1)) \\
 &\leq F(a_1d(x_0, x_1) + a_2d(x_0, T_1x_0) + a_3d(x_1, T_2x_1) \\
 &\quad + a_4d(x_0, T_2x_1) + Ld(x_1, T_1x_0)) \\
 (2.18) \quad &\leq F(a_1d(x_0, x_1) + a_2d(x_0, x_1) + a_3d(x_1, x_2) \\
 &\quad + a_4d(x_0, x_2) + L.0) \\
 &\leq F(a_1d(x_0, x_1) + a_2d(x_0, x_1) + a_3d(x_1, x_2) \\
 &\quad + a_4(d(x_0, x_1) + d(x_1, x_2))) \\
 &= F((a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2)).
 \end{aligned}$$

Since F is increasing, we get from above that

$$d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2).$$

That is,

$$(1 - a_3 - a_4)d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_1, x_2) < d(x_0, x_1).$$

Now from (2.18), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

If $x_2 \in T_i x_2 \forall i \in \mathbb{N}$ then x_2 is a common fixed point. Let $x_2 \notin T_3 x_2$, since $\{T_i\}_{i=1}^\infty$ is α -admissible, we have $\alpha(x_1, x_2) \geq 1$. There exists $x_3 \in T_3 x_2$ such that

$$(2.19) \quad d(x_2, x_3) \leq \alpha(x_1, x_2)\delta(T_2x_1, T_3x_2).$$

Since F is increasing, we have

$$(2.20) \quad F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2)\delta(T_2x_1, T_3x_2)).$$

From (2.15) we have

$$\begin{aligned}
 \tau + F(d(x_2, x_3)) &\leq \tau + F(\alpha(x_1, x_2)\delta(T_2x_1, T_3x_2)) \\
 &\leq F(a_1d(x_1, x_2) + a_2d(x_1, T_2x_1) + a_3d(x_2, T_3x_2) \\
 &\quad + a_4d(x_1, T_3x_2) + Ld(x_2, T_2x_1)) \\
 (2.21) \quad &\leq F(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3) \\
 &\quad + a_4d(x_1, x_3) + L.0) \\
 &\leq F(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3) \\
 &\quad + a_4(d(x_1, x_2) + d(x_2, x_3))) \\
 &= F((a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3)).
 \end{aligned}$$

Since F is increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4)d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (2.21) we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in T_n x_{n-1}, \quad x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) \geq 1 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$(2.22) \quad F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in (2.22) we get $\lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2) , we have $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0.$$

From (2.22) we have

$$(2.23) \quad d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ in (2.23) we get

$$(2.24) \quad \lim_{n \rightarrow \infty} n d_n^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for each $n \geq n_1$. Thus we have

$$(2.25) \quad d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1.$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.25) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n \rightarrow \infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete so there exists

$x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. By condition (iii) we have $\alpha(x_n, x^*) \geq 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, T_i x^*) = 0 \forall i \in \mathbb{N}$. On contrary suppose that $d(x^*, T_{i_0} x^*) > 0$ for some $i_0 \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, T_{i_0} x^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$ and for above i_0 , we have

$$(2.26) \quad \begin{aligned} d(x^*, T_{i_0} x^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*) \delta(T_{n+1} x_n, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) \\ &\quad + a_3 d(x^*, T_{i_0} x^*) + a_4 d(x_n, T_i x^*) + L d(x^*, x_{n+1}). \end{aligned}$$

Letting $n \rightarrow \infty$ in (2.26) we have

$$d(x^*, T_{i_0} x^*) \leq (a_3 + a_4) d(x^*, T_{i_0} x^*) < d(x^*, T_{i_0} x^*).$$

Which is a contradiction. Thus $d(x^*, T_i x^*) = 0$ for all $i \in \mathbb{N}$. \square

EXAMPLE 2.10. Let $X = \{0, 1, 2, 3, \dots\}$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y. \end{cases}$$

Define $\{T_i : X \rightarrow B(X)\}_{i=1}^\infty$ by

$$T_i x = \begin{cases} \{0\} & \text{if } x = 0, \\ \{0, 1, 2, 3, \dots, x\} & \text{if } x \neq 0 \end{cases}$$

and $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x = y = 0, \\ \frac{1}{2} & \text{if } x, y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln(x)$ for each $x \in (0, \infty)$. Under this F condition (2.15) reduces to

$$(2.27) \quad \frac{\alpha(x, y) \delta(T_i x, T_j y)}{N(x, y)} e^{\alpha(x, y) \delta(T_i x, T_j y) - N(x, y)} \leq e^{-\tau}$$

for each $x, y \in X$ with $\min\{\alpha(x, y) \delta(T_i x, T_j y), N(x, y)\} > 0$. Assume that $a_1 = 1, a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly

$$\min\{\alpha(x, y) \delta(T_i x, T_j y), d(x, y)\} > 0$$

for each $x, y > 1$ with $x \neq y$. From (2.15) for each $x, y > 1$ with $x \neq y$, we have

$$\frac{1}{2} e^{-\frac{1}{2}(x+y)} < e^{-\frac{1}{2}}.$$

Thus $\{T_i\}_{i=1}^\infty$ is an F_α -contraction of Hardy-Roger-type with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in T_1 x_0$ such that $\alpha(x_0, x_1) \geq 1$. Moreover, it is easy to see that $\{T_i\}_{i=1}^\infty$ is α -admissible sequence and for any sequence

$\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$. Therefore by Theorem 2.9 $\{T_i\}_{i=1}^\infty$ has a common fixed point in X .

DEFINITION 2.11. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \rightarrow B(X)\}_{i=1}^\infty$ is an F_{α^*} -contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have

$$(2.28) \quad \tau + F(\alpha_*(T_i x, T_j y)\delta(T_i x, T_j y)) \leq F(N(x, y)),$$

for each $x, y \in X$, whenever $\min\{\alpha_*(T_i x, T_j y)\delta(T_i x, T_j y), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) + a_4 d(x, T_j y) + L d(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

THEOREM 2.12. Let (X, d) be a complete metric space and let $\{T_i : X \rightarrow B(X)\}_{i=1}^\infty$ be an F_{α^*} -contraction of Hardy-Rogers-type satisfying the following conditions:

- (i) $\{T_i\}_{i=1}^\infty$ is α_* -admissible sequence;
- (ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) \geq 1$;
- (iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_i\}_{i=1}^\infty$ have a common fixed point.

PROOF. The proof of this theorem runs along the same lines as the proof of Theorem 2.9. \square

3. APPLICATION

In this section, as a consequence of our result we establish an existence theorem for a system of integral equations. Let $X = (C[a, b], \mathbb{R})$ be the space of all real valued continuous functions defined on $[a, b]$. Note that X is complete ([25]) with respect to the metric $d_\tau(x, y) = \sup_{t \in [a, b]} \{|x(t) - y(t)|e^{-|\tau t|}\}$.

Consider the system of integral equations of the form

$$(3.1) \quad x(t) = f(t) + \int_a^b K_i(t, s, x(s)) ds,$$

for $t, s \in [a, b]$ and $i \in \{1, 2, 3, \dots, N\}$ with $N \in \mathbb{N}$. Where $K_i : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ are continuous functions.

THEOREM 3.1. Let $X = (C[a, b], \mathbb{R})$ and let $\{T_i : X \rightarrow X\}_{i=1}^N$ be the operators defined as

$$(3.2) \quad T_i x(t) = f(t) + \int_a^b K_i(t, s, x(s)) ds,$$

for $t, s \in [a, b]$. Where $K_i : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ are continuous functions. Assume that there exist $\gamma : X \rightarrow (0, \infty)$, $\alpha : X \times X \rightarrow (0, \infty)$ and the following conditions hold:

(i) for each $i, j \in \{1, 2, 3, \dots, N\}$ there exists $\tau > 0$ such that

$$|K_i(t, s, x) - K_j(t, s, y)| \leq \frac{e^{-\tau}}{\gamma(x+y)} |x - y|$$

for each $t, s \in [a, b]$ and $x, y \in X$. Moreover,

$$\left| \int_a^b \frac{e^{|\tau s|}}{\gamma(x+y)} ds \right| \leq \frac{e^{|\tau t|}}{\alpha(x, y)}$$

for each $t \in [a, b]$;

(ii) for $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(T_i x, T_j y) \geq 1$ for each $i, j \in \{1, 2, 3, \dots, N\}$;

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, T_i x_0) \geq 1$ for some $i \in \{1, 2, 3, \dots, N\}$;

(iv) for any sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then the system of integral equations (3.1) has a solution in X .

PROOF. First we show that $\{T_i\}$ is an F_α -contraction of Hardy-Rogers-type. For each $i, j \in \{1, 2, 3, \dots, N\}$, we have

$$\begin{aligned} |T_i x(t) - T_j y(t)| &\leq \int_a^b |K_i(t, s, x(s)) - K_j(t, s, y(s))| ds \\ &\leq \int_a^b \frac{e^{-\tau}}{\gamma(x(s) + y(s))} |x(s) - y(s)| ds \\ &= \int_a^b \frac{e^{-\tau} e^{|\tau s|}}{\gamma(x(s) + y(s))} |x(s) - y(s)| e^{-|\tau s|} ds \\ &\leq e^{-\tau} d_\tau(x, y) \int_a^b \frac{e^{|\tau s|}}{\gamma(x(s) + y(s))} ds \leq \frac{e^{|\tau t|}}{\alpha(x, y)} e^{-\tau} d_\tau(x, y). \end{aligned}$$

Thus we have

$$\alpha(x, y) |T_i x(t) - T_j y(t)| e^{-|\tau t|} \leq e^{-\tau} d_\tau(x, y).$$

Equivalently,

$$\alpha(x, y) d_\tau(T_i x, T_j y) \leq e^{-\tau} d_\tau(x, y).$$

Clearly natural logarithm belongs to \mathfrak{F} . Applying it on above inequality we get

$$\ln(\alpha(x, y) d_\tau(T_i x, T_j y)) \leq \ln(e^{-\tau} d_\tau(x, y)),$$

after some simplification we get

$$\tau + \ln(\alpha(x, y) d_\tau(T_i x, T_j y)) \leq \ln(d_\tau(x, y)).$$

Thus $\{T_i\}_{i=1}^N$ is an F_α -contraction of Hardy-Rogers-type with $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $F(x) = \ln x$. Therefore by 2.9 it follows that the system of operators (3.2) have a common fixed point, that is, the system of integral equations (3.1) has a solution in X . \square

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