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ON THE CORRESPONDENCE BETWEEN SPECTRA OF THE OPERATOR PENCIL $A-\lambda B$ AND OF THE OPERATOR $B^{-1}A$

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Abstract. This paper is concerned with the reduction of the spectral problem for symmetric linear operator pencils to a spectral problem for the single operator. Also, a Rayleigh–Ritz–like bounds on eigenvalues of linear operator pencils are obtained.

1. Introduction and preliminaries

1.1. *Introduction.* In many areas of applied mathematics spectral problems for the operator pencil $L(\lambda) = A-\lambda B$ arise, where A and B are symmetric or selfadjoint operators acting in some Hilbert space $\mathcal{H}$. See, for instance, [15, 26, 16, 29] and references therein.

One way of dealing with this problem is reducing it to a classical spectral problem for a single operator. If B is bounded and boundedly invertible this can be done by introducing the operator $S = B^{-1}A$. It is easy to see that the spectral problems for L and S are equivalent. Additionally, one can think of the operator S as a symmetric or selfadjoint operator in the Krein space $(\mathcal{H}, (B\cdot,\cdot))$. Here $(\mathcal{H}, (B\cdot,\cdot))$ denotes the Banach space $(\mathcal{H}, \||B|^{1/2}\cdot\|)$, where $|B|^{1/2}$ is the positive square root of B, with the structure of the (possibly indefinite) inner product on this space given by $(B\cdot,\cdot)$.

If B is still bounded, but, say, zero is a point of continuous spectrum of the operator B, the space $(\mathcal{H}, (B\cdot,\cdot))$ is no longer complete, and S is no longer closed. If we drop the assumption that B is bounded, the problems are even more involved, and in the case when zero is an eigenvalue of B, S does not exists as an operator.

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There exists an extensive literature on the spectral theory of operator pencils with its applications in diverse mathematical and physical domains. Hence, we restrict ourselves to mentioning only two classical works [3] and [25].

The reduction of the spectral problem for operator pencils to the spectral problem for a single operator is a standard procedure used in numerous applications. One of the important steps in the reduction is to show that there is no loss of the relevant spectral information. Sometimes, it is sufficient to show that e.g. point spectra coincide, sometimes it is necessary to have a complete correspondence of various types of spectra. There is a plethora of such results in the literature, but usually for a specific setting and a specific type of spectra. Especially important in applications is the point spectrum, and it is treated in several papers, usually in different settings. See, for example [2, 29, 22, 9, 17]. A recent paper [30] contains some results about the correspondence of spectra in the case of bounded operators between Banach spaces. As far as we are aware, there are no papers which give a systematic treatment of the reduction problem.

This paper treats the case when the operators A and B are symmetric or selfadjoint operators in a Hilbert space $\mathcal{H}$ and the reduction operator is symmetric or selfadjoint in a Krein space $\mathcal{K}$, its topology generated by the operator B. So the aim is to investigate those reductions for which the symmetry of the problem is preserved. We treat various cases depending on the properties of the operator B.

We now give a brief outline of our main results.

In Sections 2 and 3 we investigate the case when B is bounded and $0 \in \sigma_c(B)$. In Section 2 we give a construction of a reduction operator in the case when A is a symmetric operator with finite and equal deficiency indices. The main result in Section 2 is Theorem 2.14 where we show that there exists a bijective correspondence between all selfadjoint extensions of the operator A and all selfadjoint extensions of the reduction operator S. Section 3 is concerned with the correspondence of the spectra. We show that there is a correspondence of spectra and point spectra, but that, in general, there does not exists correspondence for residual and continuous parts of spectra. We give necessary and sufficient conditions for the correspondence of residual, and hence also continuous, parts of spectra.

In Section 4 we investigate the case when B is unbounded and $0 \in \sigma_c(B)$. A construction of the reduction operator follows essentially the one given in [6], but the authors there do not investigate the questions we are concerned with, except the case of point spectrum for quasi–uniformly positive operators. The main result in this section is Theorem 4.1 in which we show that under the assumptions stated at the beginning of the section, the point spectrum of the pencil is contained in the point spectrum of the reduction operator and the spectrum of the reduction operator is contained in the spectrum of the

pencil. To obtain the correspondence of the spectra, additional assumptions are needed, and some are presented at the end of the section.

In Section 5, we apply the results from Sections 2 and 4 to obtain a Rayleigh–Ritz–like upper bound for the eigenvalues of the reduction operator given in terms of the eigenvalues of the matrix pair (A_V, B_V), where A_V, B_V are orthogonal projections of A, B on a finite dimensional subspace. To obtain a variational characterization for the reduction operator, we use the results from [5].

In Section 6 we investigate the case when 0 is an eigenvalue of B. By the use of the theory of linear relations we give constructions of linear relations which act as a substitute for reduced operators. We show that similar results as in Sections 3, 4 and 5 also hold in this case. Using a result from [28], we give some examples when a reduced operator can be constructed.

1.2. *Basic definitions and preliminaries.* Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and the corresponding norm $\|x\| = (x,x)^{1/2}$. By $D(T)$ we denote the domain, by $R(T)$ we denote the range of the operator T, and by $\mathrm{Ker}(T)$ we denote the null-space of T. We say that the operator T is symmetric if $T \subset T^*$ and T has a dense domain. The deficiency indices of a symmetric operator T are numbers $n_\pm = n_\pm(T)$ defined as $n_+ = \operatorname{codim} R(T-\mu)$, $n_- = \operatorname{codim} R(T-\overline{\mu})$, where $\mu \in \mathbb{C}\setminus\mathbb{R}$, $\Im\mu > 0$ is arbitrary. By $\dot{+}$ we denote the direct sum.

Let $L(\lambda) = A - \lambda B$ be an operator pencil in $\mathcal{H}$, where A and B are densely defined operators in $\mathcal{H}$, such that B is A–bounded, i.e. $D(B) \supset D(A)$ and $\|Bx\| \le C\|Ax\|$. Obviously, $D(L(\lambda)) = D(A)$.

Definition 1.1. *The complex number λ is in the resolvent set of the pencil L, (denoted by $\lambda \in \rho(L)$) if $0 \in \rho(L(\lambda))$. The spectrum of L is defined as $\sigma(L) = \mathbb{C}\setminus\rho(L)$, i.e. $\lambda \in \sigma(L)$ if $0 \in \sigma(L(\lambda))$. Analogously we define the point σ_p, residual σ_r, continuous σ_c, and approximative spectrum σ_{ap}.*

We also define the set of the *points of regular type* of the operator pencil L by

$$\Pi(L) = \{\lambda \in \mathbb{C} : \exists C > 0 \quad \text{such that} \quad \|L(\lambda)x\| \ge C\,\|x\| \quad \forall x \in D(L(\lambda))\},$$

i.e. the set of all $\lambda \in \mathbb{C}$ such that 0 is a point of regular type for the operator $L(\lambda)$.

We say that the vectors $x_0, x_1, \dots, x_k \in \mathcal{H}$, $x_0 \neq 0$, form a Jordan chain of length $k+1$ for L corresponding to the eigenvalue $\lambda_0 \in \mathbb{C}$ if

$$L(\lambda_0)x_i = Bx_{i-1}, \quad i = 0, 1, \dots, k,$$

where $x_{-1} = 0$.

In the next proposition we list some properties of linear operator pencils which are essentially known.

PROPOSITION 1.2. *Let $L(\lambda) = A - \lambda B$ be an operator pencil in the Hilbert space $\mathcal{H}$, where B is A–bounded.*

(i) *If L has a compact resolvent for some $\lambda \in \rho(L)$, i.e. if $L(\lambda)^{-1}$ is compact, then L has a compact resolvent in each point of the resolvent set.*
(ii) *If B is bounded, the set of points of regular type of L is an open set.*
(iii) *If B is bounded and A and B are symmetric operators, the deficiency indices of the symmetric operators A and $A - \lambda B$ coincide, for all $\lambda \in \mathbb{R}$.*

PROOF. The statement (i) easily follows from the generalized resolvent equation. The statement (ii) is evident. For the proof of (iii) see [18, Theorem 9.3]. □

The following lemma is known, but we give a proof for reader's convenience.

LEMMA 1.3. *Let $\mathcal{H}$ be a Hilbert space, D a dense subspace of $\mathcal{H}$, and N a finite dimensional subspace of $\mathcal{H}$. Then $D \cap N^\perp$ is dense in $N^\perp$.*

PROOF.[1] Let P be the orthogonal projection onto N. Since P is continuous and its range is N, the image $P(D)$ is a dense subspace of N. As N is finite dimensional, $P(D) = N$. Let D_0 be a subspace of D such that the restriction $P_0 \colon D_0 \to N$ of P onto D_0 is an isomorphism. Then $I - P_0^{-1}P$ is continuous on $\mathcal{H}$, its range is $N^\perp$ and $(I - P_0^{-1}P)(D) \subseteq D \cap N^\perp$. Since $(I - P_0^{-1}P)(D)$ is dense in $N^\perp$, so is $D \cap N^\perp$. □

Since we will mainly operate in Krein spaces, we introduce some basic notions from the Krein space theory. Let $\mathcal{K}$ be a vector space and let $[\cdot,\cdot]$ be an inner product on $\mathcal{K}$ (not necessarily definite). The pair $(\mathcal{K}, [\cdot,\cdot])$ is said to be a Krein space if there exists a direct sum decomposition $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ such that $(\mathcal{K}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and such that $[\mathcal{K}_+, \mathcal{K}_-] = \{0\}$. For such a decomposition the corresponding *fundamental symmetry* J is a linear operator defined by $J(x_+ + x_-) = x_+ - x_-$, where $x_+ \in \mathcal{K}_+$ and $x_- \in \mathcal{K}_-$. Let $(\cdot,\cdot) = [J\cdot,\cdot]$. Then the space $(\mathcal{K}, (\cdot,\cdot))$ is a Hilbert space. Its topology is independent of the choice of $\mathcal{K}_+$ and $\mathcal{K}_-$. The subspace F is said to be *ortho–complemented* if $F \dot{+} F^\perp = \mathcal{K}$, where $F^\perp = \{x : [x,y] = 0 \ \forall y \in F\}$.

The definitions of symmetric and selfadjoint operators in a Krein space are analogous to those in Hilbert space. A selfadjoint operator A is said to be *definitizable* if its resolvent set is nonempty and there exists a nonzero polynomial p such that $[p(A)x, x] \geq 0$ for all $x \in D(p(A))$. A definitizable operator has a spectral function with finitely many critical points in $\mathbb{R} \cup \{\infty\}$. A critical point z is regular if the spectral function is bounded in a neighborhood of z. A critical point is singular if it is not regular. We say that λ is an eigenvalue of positive (negative) type of an operator T if $[x,x] \geq 0$ (≤ 0)

[1] We thank the anonymous referee for this simplified version of the proof.

for all eigenvectors corresponding to λ. We say that an operator T is *quasi–uniformly positive* (qup) if there exists a subspace M of finite codimension in $D(T)$ such that

$$\inf\{[Tx, x] : x \in M,\ \|x\| = 1\} > 0.$$

For more details about Krein spaces see [23, 8, 1]. Basic properties of qup operators can be found in [11].

2. A reduction to the single operator problem for A symmetric and B bounded

In this section we assume:

1. A is a symmetric operator on a Hilbert space $\mathcal{H}$ with finite and equal deficiency indices $n_+(A) = n_-(A) =: n$,
2. B be is a bounded selfadjoint operator on $\mathcal{H}$ with $0 \in \sigma_c(B)$,
3. $\Pi(L) \cap \mathbb{R} \neq \emptyset$.

Let E be a spectral function of operator B. We define the operator J by $J = E(0, \infty) - E(-\infty, 0)$. Operator J is a bounded selfadjoint operator and $J^2 = I$ holds. We define $|B| := JB = BJ \geq 0$. Now, we define a space $\hat{\mathcal{K}}$ which consists of sequences (x_n), $x_n \in \mathcal{H}$ in the following way

$$\hat{\mathcal{K}} = \{(x_n) \colon |B|^{1/2} x_n \text{ converges in } \mathcal{H}\},$$

i.e. as a completion with respect to the norm generated by the operator $|B|^{1/2}$. We introduce equivalence relation $\sim$ on $\hat{\mathcal{K}}$:

$$(x_n) \sim (y_n) \text{ iff } |B|^{1/2}(x_n - y_n) \to 0 \text{ as } n \to \infty,$$

and define $\mathcal{K}$ as the corresponding quotient space $\mathcal{K} = \hat{\mathcal{K}}/\!\sim$. On $\mathcal{K}$ we introduce two inner products $(\cdot, \cdot)_{\mathcal{K}}$ and $[\cdot, \cdot]_{\mathcal{K}}$ as follows:

$$((x_n), (y_n))_{\mathcal{K}} = \lim_{n\to\infty} (|B|^{1/2} x_n, |B|^{1/2} y_n),$$
$$[(x_n), (y_n)]_{\mathcal{K}} = \lim_{n\to\infty} (J|B|^{1/2} x_n, |B|^{1/2} y_n).$$

It is easy to see that $(\mathcal{K}, (\cdot, \cdot)_{\mathcal{K}})$ is a Hilbert space, and that $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ is a Krein space. These spaces have the norm given by $\|(x_n)\|_{\mathcal{K}} = \lim_{n\to\infty} \big\||B|^{1/2} x_n\big\|$. If we identify $x \in \mathcal{H}$ with the sequence $(x, x, \ldots)$ in $\hat{\mathcal{K}}$, it is evident that $\mathcal{H}$ can be regarded as a subspace in $\mathcal{K}$, and in that case $\|x\|_{\mathcal{K}} = \big\||B|^{1/2} x\big\|$ holds. Also, $\mathcal{H}$ is a dense subspace in $\mathcal{K}$. Operators in $\mathcal{H}$ can also be regarded as operators in $\mathcal{K}$. Also, J can be extended to a bounded operator in $\mathcal{K}$, and this extension we also denote with J. It is easy to see that J is a fundamental symmetry in $\mathcal{K}$.

Proposition 2.1. *The spaces $\mathcal{H}$ and $\mathcal{K}$ are isomorphic.*

PROOF. We define

$$T : \mathcal{K} \to \mathcal{H}, \quad T(x_n) = \lim_{n\to\infty} |B|^{1/2} x_n. \tag{2.1}$$

Then it is easy to see that T is a linear surjection from $(\mathcal{K}, (\cdot,\cdot)_{\mathcal{K}})$ to $(\mathcal{H}, (\cdot,\cdot))$ which preserves the inner products, i.e. $(T(x_n), T(y_n)) = ((x_n), (y_n))_{\mathcal{K}}$. □

In applications, the space $\mathcal{K}$ can usually be represented as a function space as can be seen from the following example.

EXAMPLE 2.2. Let $\mathcal{H} = L_2(a,b)$, and let $w \in L_\infty(a,b)$ be a function which is zero on some non–empty set of measure zero. We define the operator B by $(Bf)(x) = w(x)f(x)$, $f \in L_2(a,b)$. The operator B is obviously bounded and $0 \in \sigma_c(B)$ holds.

The Krein space $\mathcal{K}$ can be identified with the space

$$\{f : \sqrt{|w(\cdot)|} f \in L_2(a,b)\}$$

(i.e. a weighted L_2 space with the weight w, see [31, Section 8.4]) with the inner products given by

$$[f,g]_{\mathcal{K}} = \int_a^b w(x) f(x) \overline{g(x)} \mathrm{d}x,$$
$$(f,g)_{\mathcal{K}} = \int_a^b |w(x)| f(x) \overline{g(x)} \mathrm{d}x.$$

The operator T is given by $Tf = \sqrt{|w(\cdot)|} f$, and the fundamental symmetry J is given by $Jf = (\operatorname{sgn} w) f$.

LEMMA 2.3. *Let D be a dense subset in $\mathcal{H}$. Then D is also dense in $\mathcal{K}$.*

PROOF. Let $\varepsilon > 0, x \in \mathcal{H}$ be arbitrary. Since $R(|B|^{1/2})$ is dense in $\mathcal{H}$, there exists $y \in \mathcal{H}$ such that $\left\| |B|^{1/2} y - x \right\| < \frac{\varepsilon}{2}$, and since D is dense in $\mathcal{H}$, there exists $z \in D$ such that $\|y - z\| < \frac{\varepsilon}{2} \left\| |B|^{1/2} \right\|^{-1}$. Now

$$\left\| |B|^{1/2} z - x \right\| \le \left\| |B|^{1/2} (z - y) \right\| + \left\| |B|^{1/2} y - x \right\| < \varepsilon.$$

This implies that $\{|B|^{1/2} x : x \in D\}$ is dense in $\mathcal{H}$. Now let $(x_n) \in \mathcal{K}$, $\varepsilon > 0$ be arbitrary. Set $\widetilde{x} = T(x_n) \in \mathcal{H}$. Then there is $y \in \mathcal{H}$ such that $\left\| \widetilde{x} - |B|^{1/2} y \right\| < \varepsilon$, which implies $\|(x_n) - y\|_{\mathcal{K}} < \varepsilon$. □

PROPOSITION 2.4. *The operator $B^{-1}A$ is symmetric in $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$.*

PROOF. Let $x, y \in D(B^{-1}A) = \{x \in D(A) : Ax \in R(B)\}$. Then

$$\begin{aligned} [B^{-1}Ax, y]_{\mathcal{K}} &= (J|B|^{1/2} B^{-1} Ax, |B|^{1/2} y) = (Ax, y) \\ &= (x, Ay) = (J|B|^{1/2} x, |B|^{1/2} B^{-1} Ay) = [x, B^{-1}Ay]_{\mathcal{K}}. \end{aligned}$$

Let $\lambda \in \Pi(L) \cap \mathbb{R}$ be arbitrary. Then $R(L(\lambda))$ is closed. From Proposition 1.2 and [18, Theorem 9.2] follows $\operatorname{codim} R(L(\lambda)) = n$. From Lemma 1.3 follows that $R(L(\lambda)) \cap R(B)$ is dense in $R(L(\lambda))$. Note that

$$\begin{aligned} R(L(\lambda)) \cap R(B) &= \{(A - \lambda B)x : x \in D(A), Ax \in R(B)\} \\ &= \{(A - \lambda B)x : x \in D(B^{-1}A)\}. \end{aligned}$$

Let $C > 0$ be such that $\|L(\lambda)x\| \geq C \|x\|$. Let $x \in D(A)$, $\varepsilon > 0$ be arbitrary. Set $\widetilde{x} = L(\lambda)x$. Then it exists $y \in D(B^{-1}A)$ such that $\|L(\lambda)y - \widetilde{x}\| < C\varepsilon$. Then

$$C\varepsilon > \|L(\lambda)y - \widetilde{x}\| = \|L(\lambda)(y - x)\| \geq C \|x - y\|,$$

hence $\|x - y\| < \varepsilon$. So $D(B^{-1}A)$ is dense in $D(A)$, which implies that $D(B^{-1}A)$ is dense in $\mathcal{H}$. Now Lemma 2.3 implies that $D(B^{-1}A)$ is dense in $\mathcal{K}$, hence $B^{-1}A$ is a symmetric operator in $\mathcal{K}$. □

We will need the following three lemmas.

LEMMA 2.5. *Let $(x_n) \in D(A^*)$ be a sequence such that*

$$A^* x_n \to 0 \tag{2.2}$$

and

$$|B|^{1/2} x_n \to 0. \tag{2.3}$$

Then $x_n \to 0$.

PROOF. From (2.3) it follows that $Bx_n \to 0$. Let $\lambda \in \Pi(L) \cap \mathbb{R}$ be arbitrary. Since $(A - \lambda B)^* = A^* - \lambda B$, we have

$$D(A^*) = D(A) \dotplus \operatorname{Ker}(A^* - \lambda B).$$

Let $x_n = x_n^0 + y_n$, $x_n^0 \in D(A)$, $y_n \in \operatorname{Ker}(A^* - \lambda B)$. Since $(A^* - \lambda B)x_n = (A - \lambda B)x_n^0$, and $(A - \lambda B)^{-1}$ is a bounded operator, it follows $x_n^0 \to 0$. Hence $Bx_n^0 \to 0$ and $By_n = B(x_n - x_n^0) \to 0$. Since the operator

$$B|_{\operatorname{Ker}(A^* - \lambda B)} : \operatorname{Ker}(A^* - \lambda B) \to B(\operatorname{Ker}(A^* - \lambda B))$$

acts between finite–dimensional spaces and is non–singular, we have $y_n \to 0$, hence $x_n \to 0$. □

LEMMA 2.6. *We have*

(i) *The set $D(|B|^{-1/2}A^*)$ is a core of A^*.*
(ii) *The set $D(|B|^{-1/2}A)$ is a core of A.*
(iii) *For each selfadjoint extension $\widetilde{A}$ of A, the set $D(|B|^{-1/2}\widetilde{A})$ is a core of $\widetilde{A}$.*

PROOF. Let $\lambda \in \Pi(L) \cap \mathbb{R}$ be arbitrary.

(i) As in the proof of Proposition 2.4, we have $\operatorname{codim} R(A-\lambda B) = n$. Also $(A-\lambda B)^* = A^* - \lambda B$ holds. This implies $D(A^*) = D(A) \dotplus \operatorname{Ker}(A^* - \lambda B)$.

Let $x \in D(A^*)$ be arbitrary, and let $x = x_0 + y_0$, $x_0 \in D(A)$, $y_0 \in \operatorname{Ker}(A^* - \lambda B)$. Since $\operatorname{Ker}(A^* - \lambda B) \subset D(|B|^{-1/2}A^*)$, it is enough to find a sequence $(x_n^0) \in D(|B|^{-1/2}A)$ such that $x_n^0 \to x_0$ and $Ax_n^0 \to Ax_0$. Lemma 1.3 implies that $R(A-\lambda B) \cap R(|B|^{1/2})$ is dense in $R(A-\lambda B)$, so for $(A-\lambda B)x_0$ there exists a sequence $x_n^0 \in D(|B|^{-1/2}A)$ such that $(A-\lambda B)x_n^0 \to (A-\lambda B)x_0$. Since $(A-\lambda B)^{-1}$ is bounded it follows that $x_n^0 \to x_0$ which implies $Bx_n^0 \to Bx_0$, and consequently, $Ax_n^0 \to Ax_0$.

(ii) Since $R(A-\lambda B) \cap R(B)$ is dense in $R(A-\lambda B)$, for an arbitrary $x \in D(A)$ there exists a sequence $x_n \in D(B^{-1}A)$ such that $(A-\lambda B)x_n \to (A-\lambda B)x$, which implies $x_n \to x$ and $Ax_n \to Ax$.

(iii) Since $\widetilde{A}$ is a finite–dimensional extension of A, $\operatorname{codim} R(\widetilde{A}-\lambda B) \le n$, and $R(\widetilde{A}-\lambda B)$ is closed. Hence, again from Lemma 1.3, $R(\widetilde{A}-\lambda B) \cap R(|B|^{1/2})$ is dense in $R(\widetilde{A}-\lambda B)$. Now we can proceed as in the proof of the first statement of the Lemma. □

LEMMA 2.7. *The formulae* $(|B|^{-1/2}A)^* = A^*|B|^{-1/2}$ *and* $(B^{-1}A)^* = A^*B^{-1}$ *hold.*

PROOF. Since $|B|^{1/2}$ is bounded, [31, Satz 4.19] implies

$$(|B|^{1/2}(|B|^{-1/2}A))^* = (|B|^{-1/2}A)^*|B|^{1/2}. \tag{2.4}$$

From Lemma 2.5 follows

$$(|B|^{1/2}(|B|^{-1/2}A))^* = (A|_{D(|B|^{-1/2}A)})^* = (\overline{A|_{D(|B|^{-1/2}A)}})^* = A^*, \tag{2.5}$$

Now (2.4) and (2.5) imply $(|B|^{-1/2}A)^*|B|^{1/2} = A^*$. The second formula follows analogously. □

Let us define the operator S in $\mathcal{K}$ by

$$D(S) = \{x \in D(A) : Ax \in R(|B|^{1/2})\} = D(|B|^{-1/2}A)$$
$$Sx = T^{-1}J|B|^{-1/2}Ax,$$

where the operator T is defined by (2.1).

PROPOSITION 2.8. *The operator S is closed.*

PROOF. Let (x_n) be a sequence in $D(S)$, and let $x_n \to x$ in the norm $\|\cdot\|_{\mathcal{K}}$, which implies $x \sim (x_n)$. Set

$$\widetilde{x} = T(x_n), \tag{2.6}$$

and let $Sx_n \to y$ in the norm of $\mathcal{K}$. It follows

$$J|B|^{-1/2}Ax_n \to \widetilde{y} := Ty. \tag{2.7}$$

From (2.6) and (2.7) follows $Bx_n \to J|B|^{1/2}\widetilde{x}$ and $Ax_n \to J|B|^{1/2}\widetilde{y}$, respectively, which implies

$$(A-\lambda B)x_n \to J|B|^{1/2}(\widetilde{y}-\lambda\widetilde{x}), \tag{2.8}$$

for all $\lambda \in \mathbb{C}$. Especially, (2.8) holds for $\lambda_0 \in \Pi(L)$, which implies

$$x_n \to (A-\lambda_0 B)^{-1}J|B|^{1/2}(\widetilde{y}-\lambda_0\widetilde{x}) =: x_0.$$

Hence we found $x_0 \in \mathcal{H}$ such that $x_n \to x_0$ in the norm $\|\cdot\|$ and $Ax_n \to J|B|^{1/2}\widetilde{y}$, which implies $x_0 \in D(A)$ and $Ax_0 = J|B|^{1/2}\widetilde{y}$. Hence $Ax_0 \in R(|B|^{1/2})$, and $J|B|^{-1/2}Ax_0 = \widetilde{y}$, i.e. $Sx_0 = y$ and $x_0 \in D(S)$. □

PROPOSITION 2.9. *The adjoint of S in $\mathcal{K}$ is given by*

$$D(S^*) = \{x \in D(A^*) : A^*x \in R(|B|^{1/2})\} = D(|B|^{-1/2}A^*)$$
$$S^*x = (y_n), \quad \textit{where } T(y_n) = J|B|^{-1/2}A^*x.$$

PROOF. Let $(y_n), (z_n) \in \mathcal{K}$ be such that

$$[Sx, (y_n)]_{\mathcal{K}} = [x, (z_n)]_{\mathcal{K}}, \quad \forall x \in D(S).$$

This can be written as

$$\lim_n(|B|^{-1/2}Ax, |B|^{1/2}y_n) = \lim_n(J|B|^{1/2}x, |B|^{1/2}z_n) \quad \forall x \in D(S). \tag{2.9}$$

Let us denote $\widetilde{y} = T(y_n)$, $\widetilde{z} = T(z_n)$. Then (2.9) reads

$$(|B|^{-1/2}Ax, \widetilde{y}) = (x, J|B|^{1/2}\widetilde{z}) \quad \forall x \in D(|B|^{-1/2}A).$$

From Lemma 2.7 follows $\widetilde{y} \in D(A^*|B|^{-1/2})$ and

$$A^*|B|^{-1/2}\widetilde{y} = J|B|^{1/2}\widetilde{z}.$$

Now, since $\widetilde{y} \in R(|B|^{1/2})$ there exists $y \in \mathcal{H}$ such that $(y_n) \sim y$. Hence we can take $\widetilde{y} = |B|^{1/2}y$, which implies $A^*y = J|B|^{-1/2}\widetilde{z}$. Hence $J|B|^{-1/2}A^*y = \widetilde{z}$, which implies $y \in D(|B|^{-1/2}A^*)$ and $T(z_n) = J|B|^{-1/2}A^*y$.

On the other hand, take an arbitrary $y \in D(|B|^{-1/2}A^*)$. Then one can easily prove

$$(|B|^{-1/2}Ax, |B|^{1/2}y) = (|B|^{1/2}x, |B|^{-1/2}A^*y), \ \forall x \in D(S),$$

hence $y \in D(S^*)$ and $T(S^*y) = J|B|^{-1/2}A^*y$. □

REMARK 2.10. If $n = 0$, i.e. if A is selfadjoint, then S is also selfadjoint.

THEOREM 2.11. *The operator S is the closure of the operator $B^{-1}A$ in $\mathcal{K}$.*

PROOF. Let us denote with S' the closure of the operator $B^{-1}A$ in $\mathcal{K}$.

Let $x \in D(B^{-1}A)$, and set $(y_n) = Sx$, i.e. $\lim_{n\to\infty}|B|^{1/2}y_n = J|B|^{-1/2}Ax \in R(|B|^{1/2})$. It follows that there exists $y \in \mathcal{H}$ such that $y \sim (y_n)$, and $|B|^{1/2}y = J|B|^{-1/2}Ax$, hence $y = B^{-1}Ax$. Hence we have proved $B^{-1}A \subset S$, so $S' \subset S$.

Let $(y_n) \in D((B^{-1}A)^*)$ be arbitrary, where the adjoint is taken with respect to the geometry in $\mathcal{K}$, and set $(z_n) = (B^{-1}A)^* x$. Then

$$[B^{-1}Ax, (y_n)]_{\mathcal{K}} = [x, (z_n)]_{\mathcal{K}}$$

holds for all $x \in D(B^{-1}A)$. This implies

$$(|B|^{-1/2}Ax, \widetilde{y}) = (x, J|B|^{1/2}\widetilde{z}),\ \forall x \in D(B^{-1}A), \tag{2.10}$$

where $\widetilde{y} = T(y_n)$, $\widetilde{z} = T(z_n)$. The relation (2.10) can be written as

$$(B^{-1}Ax, J|B|^{1/2}\widetilde{y}) = (x, J|B|^{1/2}\widetilde{z}). \tag{2.11}$$

From Lemma 2.7 we know that $(B^{-1}A)^* = A^*B^{-1}$, hence (2.11) implies

$$A^*|B|^{-1/2}\widetilde{y} = J|B|^{1/2}\widetilde{z}.$$

This implies $\widetilde{y} \in R(|B|^{1/2})$, hence $(y_n) \sim y \in \mathcal{H}$ and $A^*y = J|B|^{1/2}\widetilde{z}$, which implies $y \in D(|B|^{-1/2}A^*)$. We have proved that $(B^{-1}A)^* \subset S^*$, hence $S \subset (B^{-1}A)^{**} = S'$. □

Let S be a closed symmetric operator in a Hilbert space with equal deficiency indices. It is well-known that the set $D(S)$ provided with the inner product $(x, y)_{D(S^*)} = (x, y) + (S^*x, S^*y)$ is a Hilbert space.

Definition 2.12 ([27,13,19]). *Let S be a symmetric operator in a Hilbert space $\mathcal{H}$. A collection $\{\Omega, \Gamma_1, \Gamma_2\}$, in which Ω is a Hilbert space, and $\Gamma_1, \Gamma_2 : D(C^*) \to \Omega$ are bounded operators, is called a space of boundary values (SBV) for S^*, if*

1. $(S^*x, y) - (x, S^*y) = (\Gamma_1 x, \Gamma_2 y)_\Omega - (\Gamma_2 x, \Gamma_1 y)_\Omega \quad \forall x, y \in D(S^*)$,
2. *the mapping $\Gamma : x \mapsto \{\Gamma_1 x, \Gamma_2 x\}$ from $D(S^*)$ to $\Omega \times \Omega$ is surjective.*

In the case of a symmetric operator S in a Krein space we have

Definition 2.13 ([20]). *Let S be a symmetric operator in a Krein space $(\mathcal{K}, [\cdot, \cdot])$. A collection $\{\Omega, \widehat{J}, \Gamma_1, \Gamma_2\}$, in which Ω is a Hilbert space, $\widehat{J}$ is an involution in Ω, and $\Gamma_1, \Gamma_2 : D(S^*) \to \Omega$ are bounded operators, is called a space of boundary values for S^*, if*

1. $[S^*x, y] - [x, S^*y] = (\Gamma_1 x, \widehat{J}\Gamma_2 y)_\Omega - (\Gamma_2 x, \widehat{J}\Gamma_1 y)_\Omega \quad \forall x, y \in D(S^*)$,
2. *the mapping $\Gamma : x \mapsto \{\Gamma_1 x, \Gamma_2 x\}$ from $D(S^*)$ to $\Omega \times \Omega$ is surjective.*

In the case of Hilbert space operators, it can be shown that a SBV exists for any symmetric operator with equal deficiency indices (see [13, 19]). In the case of Krein space operators, it can be shown that a SBV exists for an operator T if $\Pi(T) \neq \emptyset$ holds, and if the deficiency indices of T are equal (see [20]). For a fixed SBV for the operator T, there exists a bijective correspondence between the collection of the closed extensions $\widetilde{T}$ and the set of closed relations θ in Ω:

$$x \in D(\widetilde{T}) = D(\widetilde{T}_\theta) \Longleftrightarrow \{\Gamma_2 x, \Gamma_1 x\} \in \theta \subset \Omega \times \Omega. \tag{2.12}$$

An extension $\widetilde{T}$ is selfadjoint if and only if θ is a selfadjoint relation, or equivalently, if there exists a selfadjoint operator G on Ω given by the relation

$$(\cos G)\Gamma_2 x - (\sin G)\Gamma_1 x = 0 \Longleftrightarrow \{\Gamma_2 x, \Gamma_1 x\} \in \theta.$$

For more details on the theory of SBV see [19, 20, 13, 27].

Let $\{\Omega, \Gamma_1, \Gamma_2\}$ be a fixed SBV for the operator A^*. Set $\widetilde{\Gamma}_i = \Gamma_i|_{D(|B|^{-1/2}A^*)}$, $i = 1, 2$. We will show that a collection $\{\Omega, I, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$ is a SBV for the operator S^*. Hence, we need to show that the statements 1. and 2. from Definition 2.13 hold. We start with

$$\begin{aligned}[S^*x, y]_{\mathcal{K}} - [x, S^*y]_{\mathcal{K}} &= (|B|^{-1/2}A^*x, |B|^{1/2}y) - (J|B|^{1/2}x, J|B|^{-1/2}A^*y) \\ &= (A^*x, y) - (x, A^*y) = (\Gamma_1 x, \Gamma_2 y)_\Omega - (\Gamma_2 x, \Gamma_1 y)_\Omega \\ &= (\widetilde{\Gamma}_1 x, \widetilde{\Gamma}_2 y)_\Omega - (\widetilde{\Gamma}_2 x, \widetilde{\Gamma}_1 y)_\Omega.\end{aligned}$$

To see that $\widetilde{\Gamma}_i$ is bounded for $i = 1, 2$, it is sufficient to prove that there exists $C > 0$ such that

$$\|x\|_{D(A^*)} \le C\|x\|_{D(S^*)} \quad \forall x \in D(S^*),$$

and since

$$\|A^*x\| \le \left\||B|^{1/2}|B|^{-1/2}A^*x\right\| \le \left\||B|^{1/2}\right\| \left\||B|^{-1/2}A^*x\right\|,$$

it is enough to see that there exists $C > 0$ such that

$$\|x\| \le C\left(\left\||B|^{1/2}x\right\| + \|A^*x\|\right) \quad \forall x \in D(|B|^{-1/2}A^*). \tag{2.13}$$

But this is implied by Lemma 2.5. Indeed, suppose that (2.13) does not hold. Then there is a sequence $(x_n) \in D(|B|^{-1/2}A^*)$, $\|x_n\| = 1$, such that

$$\|x_n\| \ge n\left(\left\||B|^{1/2}x_n\right\| + \|A^*x_n\|\right),$$

which implies $A^*x_n \to 0$ and $|B|^{1/2}x_n \to 0$, hence $x_n \to 0$, which is a contradiction with the statement of Lemma 2.5.

The density of $D(S^*)$ in $D(A^*)$ follows directly from Lemma 2.6. Our next aim is to show that $\widetilde{\Gamma} = \{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$ is surjective. Let $f, g \in \Omega$ be arbitrary. Then there exists $x \in D(A^*)$ such that $\Gamma_1 x = f$, $\Gamma_2 x = g$. Since $D(S^*)$ is dense in $D(A^*)$, there exists a sequence $(x_n) \in D(S^*)$ such that $x_n \to x$ in the norm of $D(A^*)$, and since Γ_i, $i = 1, 2$ are bounded, it follows that $\widetilde{\Gamma}_i x_n = \Gamma_i x_n \to \Gamma_i x$, $i = 1, 2$, so $R(\widetilde{\Gamma})$ is dense in $\Omega \times \Omega$. Now, since the space $\Omega \times \Omega$ is finite–dimensional, it follows $R(\widetilde{\Gamma}) = \Omega \times \Omega$.

Hence, we can parameterize the selfadjoint extensions of the operator S by the use of a SBV for the operator A^*. More precisely, let $\widetilde{A} = \widetilde{A}_\theta$ be an extension of the operator A generated by the closed relation θ on Ω. We define an extension $\widetilde{S}$ of S in the following way:

$$D(\widetilde{S}) = D(|B|^{-1/2}\widetilde{A}) \quad \text{and} \quad \widetilde{S}x = S^*x.$$

This definition implies

$$\begin{aligned} D(\widetilde{S}) &= \{x \in D(\widetilde{A}) : \widetilde{A}x = A^*x \in R(|B|^{1/2})\} \\ &= \left\{x \in D(A^*) : A^*x \in R(|B|^{1/2}) : \{\Gamma_2 x, \Gamma_1 x\} \in \theta\right\} \\ &= \left\{x \in D(S^*) : \{\widetilde{\Gamma}_2 x, \widetilde{\Gamma}_1 x\} \in D(\widetilde{S}_\theta)\right\}. \end{aligned}$$

On the other hand, if $\widetilde{S} = \widetilde{S}_\theta$ is a closed extension of the operator S generated by a closed relation θ on Ω, then it generates a closed extension $\widetilde{A}_\theta$ of the operator A. Thus, we have shown

THEOREM 2.14. *There exists a bijective correspondence between all closed extensions of the operator A and all closed extensions of the operator S. Specifically, there exists a bijective correspondence between all selfadjoint extensions of the operator A and all selfadjoint extensions of the operator S. Moreover, if $\widetilde{A}$ is a selfadjoint extension of A, then the operator $\widetilde{S}$ defined by $D(\widetilde{S}) = D(|B|^{-1/2}\widetilde{A})$, $\widetilde{S}x = (y_n)$, $T(y_n) = J|B|^{-1/2}\widetilde{A}$, is a selfadjoint extension of S.*

REMARK 2.15. In the case when a selfadjoint extension of A is given by the relation $(\cos G)\Gamma_2 x - (\sin G)\Gamma_1 x = 0$, where G is a selfadjoint operator in Ω, the relation $(\cos G)\widetilde{\Gamma}_2 x - (\sin G)\widetilde{\Gamma}_1 x = 0$ defines a selfadjoint extension of S.

EXAMPLE 2.16. Let $\mathcal{H} = L_2(0,1)$, $Af = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, $D(A) = C^2(0,1)$, $(Bf)(x) = xf(x)$. The operators A and B are obviously symmetric, B is bounded and $0 \in \sigma_c(B)$. One can easily see that $\Pi(L) \neq \emptyset$. A SBV for A^* can be chosen as follows:

$$\Omega = \mathbb{C}^2,\ \Gamma_1 = \{-f(0), f(1)\},\ \Gamma_2 = \{f'(0), f'(1)\}.$$

We have $D(A^*) = W_2^2(0,1)$, where $W_2^2(0,1)$ is the usual Sobolev space, and

$$D(|B|^{-1/2}A^*) = \left\{f \in W_2^2(01,) : \frac{1}{\sqrt{x}}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) \in L_2(0,1)\right\}.$$

The collection $\{\mathbb{C}^2, I, \Gamma_1|_{D(|B|^{-1/2}A^*)}, \Gamma_2|_{D(|B|^{-1/2}A^*)}\}$ is a SBV for S^*, hence all selfadjoint extensions of S are parameterized by the selfadjoint relations in $\mathbb{C}^2$ by the formula (2.12).

3. The correspondence of the spectra in the case of bounded B

In this section we have the same assumptions as in the previous section. Our aim in this section is to show the connection between the spectra of $\widetilde{L}$ and $\widetilde{S}$.

Let θ be a selfadjoint relation in $\Omega \times \Omega$ such that $\rho(\widetilde{L}) \neq \emptyset$ holds, where $\widetilde{L}(\lambda) = \widetilde{A}_\theta - \lambda B = \widetilde{A} - \lambda B$, and let $\widetilde{S} = \widetilde{S}_\theta$ be a selfadjoint extension generated by the relation θ.

PROPOSITION 3.1. *The operator $\widetilde{S}$ is not bounded.*

PROOF. Let us assume that $\widetilde{S}$ is bounded. This implies $D(\widetilde{S}) = \mathcal{K}$, hence $D(|B|^{-1/2}\widetilde{A}) = \mathcal{K}$, which implies $\mathcal{K} = \mathcal{H}$. This implies that for each sequence $(x_n) \in \mathcal{H}$ such that $|B|^{1/2}x_n \to y$, there exists $x \in \mathcal{H}$ such that $x \sim (x_n)$, i.e. $|B|^{1/2}x = y$. Hence $R(|B|^{1/2}) = \mathcal{H}$, a contradiction with the assumption $0 \in \sigma_c(B)$. □

For a symmetric operator S in a Hilbert or Krein space, by $\pi(S)$ we denote the number of negative squares of Hermitian sesquilinear form (Sx, y), $x, y \in D(s)$, where $(\cdot, \cdot)$ is the inner product on the Hilbert or Krein space. That is, $\pi(S)$ is the supremum of the dimensions of all subspaces $\mathcal{L}$ such that $(Sx, x) < 0$ for all $0 \neq x \in \mathcal{L}$.

THEOREM 3.2. *For each selfadjoint extension $\widetilde{A}$ of A, and $\widetilde{S}$ the corresponding extension of S, the formula $\pi(\widetilde{S}) = \pi(\widetilde{A})$ holds.*

PROOF. The inequality $\pi(\widetilde{S}) \leq \pi(\widetilde{A})$ is evident.

We will first prove the theorem in the case $\pi(\widetilde{S}) = 0$. In this case we have $[\widetilde{S}x, x]_{\mathcal{K}} \geq 0$ for all $x \in D(\widetilde{S})$, or, equivalently $(\widetilde{A}x, x) \geq 0$ for all $x \in D(|B|^{-1/2}\widetilde{A})$. Let us assume that $\pi(\widetilde{A}) > 0$. Then there exists some $x \in D(\widetilde{A})$ such that $(\widetilde{A}x, x) < 0$. Lemma 2.6 implies that there is a sequence $(x_n) \in D(|B|^{-1/2}\widetilde{A})$ such that $x_n \to x$ and $\widetilde{A}x_n \to \widetilde{A}x$, hence $(\widetilde{A}x_n, x_n) \to (\widetilde{A}x, x)$, so for n large enough we have $(\widetilde{A}x_n, x_n) < 0$; a contradiction.

Now we treat the case $\pi(\widetilde{S}) > 0$. Let integer n and elements $x_1, \ldots, x_n \in D(\widetilde{A})$ be arbitrarily chosen. Let $\mathcal{A}$ be the matrix $((\widetilde{A}x_n, x_n))_{i,j=1}^n$, and let k be the number of negative eigenvalues of $\mathcal{A}$. Then, by Lemma 2.6 (iii), for each $\varepsilon > 0$ there exist $x_1', \ldots, x_n' \in D(|B|^{-1/2}\widetilde{A})$ such that $\|x_i - x_i'\| < \varepsilon$ and $\|\widetilde{A}(x_i - x_i')\| < \varepsilon$. Since

$$|(\widetilde{A}x_i, x_j) - (\widetilde{A}x_i', x_j')| = |(\widetilde{A}(x_i - x_i'), x_j) + (\widetilde{A}x_i', x_j - x_j')| < \\ < \varepsilon(\|x_j\| + \|Ax_i\|) + \varepsilon^2,$$

for a sufficiently small $\varepsilon > 0$ the matrix $\mathcal{A}' = ((\widetilde{A}x_i', x_j'))_{i,j=1}^n$ will have at least k negative eigenvalues. Hence, the number of negative squares of the Hermitian form $(\widetilde{A}x, y)$, $x, y \in D(\widetilde{A})$ is smaller or equals the number of negative squares of the Hermitian form $[\widetilde{S}x, y]_{\mathcal{K}}$, $x, y \in D(\widetilde{S})$, which finishes the proof. □

THEOREM 3.3. *The spectra of $\widetilde{S}$ and $\widetilde{L}$ coincide. Moreover, their point spectra coincide, as well as their eigenspaces and Jordan chains.*

PROOF. First we will show $\sigma_p(\widetilde{L}) = \sigma_p(\widetilde{S})$. Let $\lambda \in \sigma_p(\widetilde{L})$ be arbitrary. Then there exists $x \in D(\widetilde{A})$, $x \neq 0$ such that

$$\widetilde{A}x = \lambda Bx. \tag{3.1}$$

It is evident that $x \in D(\widetilde{S})$. Multiplying (3.1) by $J|B|^{-1/2}$ we obtain $J|B|^{-1/2}\widetilde{A}x = \lambda|B|^{1/2}$, which implies $T(\widetilde{S}-\lambda)x = 0$, hence $(\widetilde{S}-\lambda)x = 0$.

On the other hand, for $\lambda \in \sigma_p(\widetilde{S})$ there exists $x \in D(\widetilde{S})$, $x \neq 0$, such that

$$J|B|^{-1/2}\widetilde{A}x = \lambda|B|^{1/2}x. \tag{3.2}$$

Multiplying (3.2) by $J|B|^{1/2}$ we obtain $\widetilde{A}x = \lambda Bx$, i.e. $\lambda \in \sigma_p(\widetilde{L})$. Hence, $\sigma_p(\widetilde{L}) = \sigma_p(\widetilde{S})$.

The relation $(\widetilde{S}-\lambda)x_1 = x_0$ implies

$$J|B|^{-1/2}\widetilde{A}x_1 - \lambda|B|^{1/2}x_1 = |B|^{1/2}x_0. \tag{3.3}$$

Multiplying (3.3) by $J|B|^{1/2}$ we obtain $(\widetilde{A}-\lambda B)x_1 = Bx_0$, hence a Jordan chain of $\widetilde{S}$ corresponding to an eigenvalue λ is also a Jordan chain of $\widetilde{L}$ for the same eigenvalue λ. The other direction can be seen analogously.

Let $\lambda \in \rho(\widetilde{S})$. This implies $T(R(\widetilde{S}-\lambda)) = \mathcal{H}$, i.e. $R(J|B|^{-1/2}\widetilde{A} - \lambda|B|^{1/2}) = \mathcal{H}$. This implies

$$R(\widetilde{A}-\lambda B) \supseteq R(|B|^{1/2}) \supseteq R(B). \tag{3.4}$$

Now, let $\mu \in \rho(\widetilde{L})$ be arbitrary. From (3.4) follows

$$(\widetilde{A}-\mu B)x = (\widetilde{A}-\lambda B)x + (\lambda-\mu)Bx \in R(\widetilde{A}-\lambda B),$$

hence $R(\widetilde{A}-\mu B) \subset R(\widetilde{A}-\lambda B)$, which implies $\lambda \in \rho(\widetilde{L})$.

On the other hand, let $\lambda \in \sigma(\widetilde{S}) \setminus \sigma_p(\widetilde{S})$. Then there exists $(y_n) \in \mathcal{K}$ such that

$$(y_n) \notin R(\widetilde{S}-\lambda). \tag{3.5}$$

We denote $\widetilde{y} = T(y_n)$. Then, the relation (3.5) can be written as

$$\widetilde{y} \notin R(J|B|^{-1/2}\widetilde{A} - \lambda|B|^{1/2})$$

which implies

$$J|B|^{-1/2}\widetilde{y} \notin R((\widetilde{A}-\lambda B)|_{D(|B|^{-1/2}\widetilde{A})}). \tag{3.6}$$

Let us assume that $J|B|^{1/2}\widetilde{y} \in R(\widetilde{A}-\lambda B)$. Then there exists $x \in D(\widetilde{A})$ such that $\widetilde{A}x - \lambda Bx = J|B|^{1/2}\widetilde{y}$, hence $\widetilde{A}x \in R(|B|^{1/2})$, a contradiction with (3.6). Hence, $J|B|^{1/2}\widetilde{y} \notin R(\widetilde{A}-\lambda B)$ and $\lambda \in \sigma(\widetilde{L})$. □

PROPOSITION 3.4. *We have*

(i) $\sigma_c(\widetilde{S}) \subset \sigma_c(\widetilde{L})$,

(ii) $\sigma_{ap}(\widetilde{S}) \subset \sigma_{ap}(\widetilde{L})$.

PROOF. (i) Let $\lambda \in \sigma_c(\widetilde{S})$. Then $R(\widetilde{S}-\lambda)$ is dense in $\mathcal{K}$. For each $(y_n) \in \mathcal{K}$, and for each $\varepsilon > 0$ there exists $x \in D(\widetilde{S})$ such that

$$\left\|(\widetilde{S}-\lambda)x - (y_n)\right\|_{\mathcal{K}} < \varepsilon.$$

We choose $y \in R(B)$ arbitrarily and set $\widetilde{y} = J|B|^{-1/2}y$, $y' = |B|^{-1/2}\widetilde{y}$. For $y' \in \mathcal{H} \subset \mathcal{K}$ there exists $x \in D(\widetilde{S})$ such that $\|(\widetilde{S} - \lambda)x - y'\|_{\mathcal{K}} < \varepsilon \||B|^{1/2}\|^{-1}$, or, equivalently,

$$\left\| J|B|^{-1/2}\widetilde{A}x - \lambda|B|^{1/2}x - \widetilde{y} \right\| < \varepsilon \left\| |B|^{1/2} \right\|^{-1}.$$

Now,

$$\left\| (\widetilde{A} - \lambda B)x - y \right\| = \left\| J|B|^{1/2}(J|B|^{-1/2}\widetilde{A}x - \lambda|B|^{1/2}x - \widetilde{y}) \right\| < \varepsilon.$$

Hence, $R(\widetilde{A} - \lambda B)$ is dense in $R(B)$, which implies that $R(\widetilde{A} - \lambda B)$ is dense in $\mathcal{H}$. Since $\lambda \notin \sigma_p(\widetilde{L})$, $\lambda \in \sigma_c(\widetilde{L})$ follows.

(ii) Let $\in \sigma_{ap}(\widetilde{S})$. Then there exists a sequence $(x_n) \in D(\widetilde{S})$, $\|x_n\|_{\mathcal{K}} = 1$ such that $\|(\widetilde{S} - \lambda)x_n\| \to 0$, or equivalently,

$$(J|B|^{-1/2}\widetilde{A} - \lambda|B|^{1/2})x_n \to 0 \tag{3.7}$$

and

$$\left\| |B|^{1/2}x_n \right\| = 1. \tag{3.8}$$

Multiplying relation (3.7) by $J|B|^{1/2}$ we obtain $(\widetilde{A} - \lambda B)x_n \to 0$. From (3.8) we see that $\|x_n\| \geq \left\| |B|^{1/2} \right\|^{-1}$. Set $\widehat{x}_n = x_n / \|x_n\|$. Then $\|\widehat{x}_n\| = 1$ and $(\widetilde{A} - \lambda B)\widehat{x}_n \to 0$, hence $\lambda \in \sigma_{ap}(\widetilde{L})$. □

COROLLARY 3.5. *The relation $\sigma_r(\widetilde{L}) \subset \sigma_r(\widetilde{S})$ holds.*

In general, Proposition 3.4 (and hence also Corollary 3.5) cannot be improved, as will be shown in Example 3.9.

First we give sufficient and necessary conditions for the complete correspondence of the spectra of $\widetilde{L}$ and $\widetilde{S}$.

LEMMA 3.6. *The subspace $R(\widetilde{A} - \lambda B)$ is dense in $\mathcal{H}$ if and only if $\mathcal{H} \cap R(\widetilde{S} - \lambda)^{\perp} = \{0\}$.*

PROOF. $\Longrightarrow$: First we show that the subspace $R(\widetilde{A} - \lambda B)$ is dense in $\mathcal{H}$ if and only if $R((\widetilde{A} - \lambda B)|_{D(|B|^{-1/2}\widetilde{A})})$ is dense in $\mathcal{H}$.

The case $\lambda = 0$ follows from Lemma 2.6, so we assume $\lambda \neq 0$. First we assume that $R(\widetilde{A} - \lambda B)$ is dense in $\mathcal{H}$. Let $x_0 \in \mathcal{H}$, $\varepsilon > 0$ be arbitrary. Then there exists $x \in D(\widetilde{A})$ such that $\|(\widetilde{A} - \lambda B)x - x_0\| < \varepsilon/2$. Lemma 2.6 implies that there exists $x' \in D(|B|^{-1/2}\widetilde{A})$ such that $\|x - x'\| < \frac{\varepsilon}{4|\lambda|\|B\|}$ and $\|\widetilde{A}(x - x')\| < \varepsilon/4$. Now,

$$\left\| (\widetilde{A} - \lambda B)x' - x_0 \right\| \leq \left\| (\widetilde{A} - \lambda B)(x' - x) + (\widetilde{A} - \lambda B)x - x_0 \right\| < \varepsilon.$$

The other direction is obvious.

We showed that $R((\widetilde{A}-\lambda B)|_{D(|B|^{-1/2}\widetilde{A})})$ is dense in $\mathcal{H}$, hence the relation

$$(x,(\widetilde{A}-\lambda B)y)=0,\ \forall y\in D(|B|^{-1/2}\widetilde{A})$$

implies $x=0$. Now, from

$$(x,(\widetilde{A}-\lambda B)y)=(|B|^{1/2}x,|B|^{-1/2}(\widetilde{A}-\lambda B)y)=[x,(\widetilde{S}-\lambda)y]_{\mathcal{K}}$$

our claim immediately follows.

The other direction follows from the similar reasoning. □

LEMMA 3.7. *The subspace $R(\widetilde{S}-\lambda)$ is dense in $\mathcal{K}$ if and only if $R(|B|^{-1/2}(\widetilde{A}-\lambda B))$ is dense in $\mathcal{H}$.*

PROOF. $\Longrightarrow$: Let us assume that $R(|B|^{-1/2}(\widetilde{A}-\lambda B))$ is not dense in $\mathcal{H}$. Then there exists $\widetilde{x}\in\mathcal{H}$, $\widetilde{x}\neq 0$ such that

$$(\widetilde{x},|B|^{-1/2}(\widetilde{A}-\lambda B)y)=0\quad\forall y\in D(|B|^{-1/2}\widetilde{A}).$$

Set $(x_n)=T^{-1}J\widetilde{x}$. Then $((x_n),(\widetilde{S}-\lambda)y)_{\mathcal{K}}=0$ for all $y\in D(\widetilde{S})$, which implies the statement.

$\Longleftarrow$: Let us assume that $R(\widetilde{S}-\lambda)$ is not dense in $\mathcal{K}$. Then it exists $(y_n)\in\mathcal{K}$, $(y_n)\neq 0$, such that $((y_n),(\widetilde{S}-\lambda)x)_{\mathcal{K}}=0$ for all $x\in D(\widetilde{S})$. Set $y=T(y_n)$. Then

$$(\widetilde{y},(J|B|^{-1/2}\widetilde{A}-\lambda|B|^{1/2})x)=0\quad\forall x\in D(|B|^{-1/2}\widetilde{A}),$$

hence

$$(J\widetilde{y},|B|^{-1/2}(\widetilde{A}-\lambda B)x)=0\quad\forall x\in D(|B|^{-1/2}\widetilde{A}),$$

and the statement follows. □

Lemmata 3.6 and 3.7 imply the following result.

PROPOSITION 3.8. (i) *If $\lambda\in\sigma_r(\widetilde{S})$, then $\lambda\in\sigma_c(\widetilde{L})$ if and only if $\mathcal{H}\cap R(\widetilde{S}-\lambda)^{\perp}=\{0\}$.*

(ii) *If $\lambda\in\sigma_c(\widetilde{L})$, then $\lambda\in\sigma_r(\widetilde{S})$ if and only if $R(|B|^{-1/2}(\widetilde{A}-\lambda B))$ is not dense in $\mathcal{H}$.*

EXAMPLE 3.9. Let $\mathcal{H}=L_2(0,1)$. We define the operators A and B in $\mathcal{H}$ by:

$$(Af)(x)=\int_0^1 G(x,\xi)f(\xi)\mathrm{d}\xi,$$
$$(Bf)(x)=(2x-1)f(x),$$

where

$$G(x,\xi)=\begin{cases}x, & x\leq\xi,\\ \xi, & x\geq\xi.\end{cases}$$

Note that A is the inverse of the Dirichlet Laplacian. The operator B is bounded and $0 \in \sigma_c(B)$, $0 \in \sigma_c(A)$ holds. Note that this implies $0 \in \sigma_c(L)$. One can easily prove that $\Pi(L) \cap \mathbb{R} \neq \emptyset$ holds.

We will show that $R(|B|^{-1/2}A)$ is not dense in $\mathcal{H}$. Indeed, constant functions are not contained in $R(|B|^{-1/2}A)$. This follows from the fact that the function $x \mapsto \sqrt{|2x-1|}$ is not contained in $R(A)$. Now Proposition 3.8 (ii) implies $0 \in \sigma_r(S)$.

REMARK 3.10. If $\pi(B) < \infty$ or $\widetilde{A} \geq 0$, then $\sigma_r(\widetilde{L}) = \sigma_r(\widetilde{S}) = \emptyset$, and consequently, $\sigma_c(\widetilde{L}) = \sigma_c(\widetilde{S})$.

If $\widetilde{S}$ is definitizable, then $\sigma_r(\widetilde{S}) = \emptyset$, hence $\sigma_r(\widetilde{L}) = \emptyset$ and $\sigma_c(\widetilde{S}) = \sigma_c(\widetilde{L})$.

COROLLARY 3.11. *Let us suppose that $\pi(A) < \infty$. Then each selfadjoint extension $\widetilde{S}$ of S is definitizable.*

PROOF. Since deficiency indices of A are finite, $\pi(\widetilde{A}) < \infty$ holds for each selfadjoint extension $\widetilde{A}$, and Theorem 3.2 implies $\pi(\widetilde{S}) < \infty$. Now, let $\lambda_0 \in \Pi(L) \cap \mathbb{R}$ be arbitrary. It is well–known that there exists a selfadjoint extension $\widetilde{A}_0$ such that $0 \in \rho(\widetilde{A}_0 - \lambda_0 B)$ (see [21, 13]), which implies $\lambda_0 \in \rho(\widetilde{S}_0)$, where $\widetilde{S}_0$ is the corresponding operator. Also $\pi(\widetilde{S}_0) = \pi(\widetilde{A}_0) < \infty$ holds. Now, from [10, Proposition 1.1] follows that each selfadjoint extension of S has a nonvoid resolvent set, hence, see [23], each selfadjoint extension of S is definitizable. □

REMARK 3.12. If $\widetilde{A}$ is qup, the proof of Lemma 3.1 from [6] implies $0 \in \rho(\widetilde{L})$ for $\lambda \neq 0$, $|\lambda|$ sufficiently small. Hence $\lambda \in \rho(\widetilde{S})$ for such λ. From this fact and from $\pi(\widetilde{S}) = \pi(\widetilde{A}) < \infty$, it follows that $\widetilde{S}$ is also qup.

COROLLARY 3.13. *If $\widetilde{L}$ has a compact resolvent in one point λ (and then, from Proposition 1.2, in all points), and if*

$$\widetilde{A}x = \lambda Bx, x \neq 0 \Longrightarrow (Bx, x) \neq 0, \tag{3.9}$$

then $\widetilde{S}$ has a discrete spectrum.

PROOF. See [4]. □

REMARK 3.14. The relation (3.9) is equivalent with the fact that $\widetilde{S}$ does not have isotropic eigenvectors, i.e. if $[x, x] \neq 0$ for all eigenvectors x of $\widetilde{S}$.

4. B UNBOUNDED

In this section we will treat a more general case when B is possibly unbounded. In this section we assume:

1. A and B are selfadjoint and $D(A) \subset D(B)$,
2. 0 is not an eigenvalue of B, and
3. $\rho(L) \cap \mathbb{R} \neq \emptyset$.

Using a spectral shift (i.e. substituting A by $A-\lambda B$), if necessarily, we can assume $0\in\rho(A)$. The construction which we use here is essentially given in [6].

Similarly as in Section 2. we introduce a space $\mathcal{K}$ as the completion of $(D(|B|^{1/2}),[\cdot,\cdot])$, where $[\cdot,\cdot]=(J|B|^{1/2}\cdot,\cdot)$, by the use of Cauchy sequences. As in Section 2., we set $J=E(0,\infty)-E(-\infty,0)$, where E is the spectral function of B. By T we denote the operator $T:\mathcal{K}\to\mathcal{H}$, $T(x_n)=\lim_n|B|^{1/2}x_n$. It is easy to see that T is an isomorphism between spaces $\mathcal{H}$ and $\mathcal{K}$. As was shown in [6], $A^{-1}B$ extends to a bounded symmetric operator $\overline{R}$ in $\mathcal{K}$, and $\overline{R}$ is injective. Hence, the operator $S=\overline{R}^{-1}$ is selfadjoint and boundedly invertible in $\mathcal{K}$. Moreover, we can see that S is an extension of $B^{-1}A$ which is densely defined in $\mathcal{K}$. Indeed, since $\overline{R}$ is injective, it holds

$$(\overline{B^{-1}A})^{-1}=\overline{(B^{-1}A)^{-1}}=\overline{R},$$

so $\overline{B^{-1}A}=S$, where the closure is taken in $\mathcal{K}$. The domain of S is given by

$$D(S)=\{(x_n)\in\mathcal{K}:J|B|^{-1/2}Ax_n \text{ is convergent in } \mathcal{H}\}.$$

For $(x_n)\in\mathcal{K}$ we can always assume $x_n\in D(B)$. Then, if $(y_n)=(S-\lambda)(x_n)$ we have $T(y_n)=\lim_n J|B|^{-1/2}(A-\lambda B)x_n$.

If A is a qup, we can, instead of the assumption 3., make the assumption $\rho(L)\neq\emptyset$, since in [6, Lemma 3.1] it was proved that in this case there exists $\lambda_0\in\mathbb{R}$ such that $L(\lambda_0)$ is boundedly invertible. Also, in this case, S is a qup.

THEOREM 4.1. *We have*

$$\sigma_p(L)\subset\sigma_p(S) \tag{4.1}$$

and

$$\sigma(S)\subset\sigma(L). \tag{4.2}$$

PROOF. The relation (4.1) is obvious.

Let $\lambda\in\rho(L)$. Then it is easy to see that $|B|^{1/2}|A-\lambda B|^{-1/2}$ and $|B|^{1/2}|A-\overline{\lambda}B|^{-1/2}$ are everywhere defined bounded operators. From

$$(|A-\lambda B|^{-1/2}|B|^{1/2})^*=|B|^{1/2}|A-\overline{\lambda}B|^{-1/2}$$

follows

$$(|A-\lambda B|^{-1/2}|B|^{1/2})^{**}=(|B|^{1/2}|A-\overline{\lambda}B|^{-1/2})^*,$$

hence

$$|A-\lambda B|^{-1/2}|B|^{1/2}\subset(|B|^{1/2}|A-\overline{\lambda}B|^{-1/2})^*. \tag{4.3}$$

Since the right hand side in (4.3) is a bounded operator, it follows that also the left hand side in (4.3) is a bounded operator (but, in general, not everywhere defined!). Hence, the operator

$$|B|^{1/2}(A-\lambda B)^{-1}|B|^{1/2}=|B|^{1/2}|A-\lambda B|^{-1/2}\operatorname{sgn}(A-\lambda B)|A-\lambda B|^{-1/2}|B|^{1/2}$$

is bounded.

The subspace $D(|B|^{1/2}) \cap R(|B|^{1/2})$ is dense. Indeed, we have

$$D(|B|^{1/2}) \cap R(|B|^{1/2}) = \{x = |B|^{1/2}y : y \in D(B)\},$$

and the right hand side is dense since $D(B)$ is a core of $|B|^{1/2}$. Now, let $x \in \mathcal{H}$ be arbitrary. Then there exists a sequence $(x_n) \in D(|B|^{1/2}) \cap R(|B|^{1/2})$ such that $x_n \to Jx$. From the considerations given above it follows that $|B|^{1/2}(A - \lambda B)^{-1}|B|^{1/2}x_n$ converges to some $y \in \mathcal{H}$.

Set $z_n = (A - \lambda B)^{-1}|B|^{1/2}x_n$. Then $|B|^{1/2}z_n \to y$, hence $(z_n) \in \mathcal{K}$ and $x_n = |B|^{-1/2}(A - \lambda B)z_n$. Now $|B|^{-1/2}(A - \lambda B)z_n \to Jx$, which implies $J|B|^{-1/2}(A - \lambda B)z_n \to x$, so $x \in T(R(S - \lambda))$. This implies $\lambda \in \rho(S)$. □

The other inclusion $\sigma(L) \subset \sigma(S)$ in general does not hold, since for large λ the operator $L(\lambda)$ need not be closed.

If we assume $0 \in \rho(B)$, then $\mathcal{K} = D(|B|^{1/2})$ and $\sigma_p(S) \subset \sigma_p(L)$, hence $\sigma_p(S) = \sigma_p(L)$. Also, it can be seen that $\sigma_{ap}(L) \subset \sigma_{ap}(S)$. Hence, if L has only approximative spectrum then $\sigma(L) = \sigma(S)$.

Another kind of construction is made in [24]. There it is assumed that $0 \in \rho(B)$ and $D(A) = D(|B|^{1/2})$. Then $L(\lambda)$ is a holomorphic function in Kato sense from $\mathcal{K}$ to $\mathcal{H}$. For the readers convenience, we give a definition.

Definition 4.2. *The function $T(\lambda) : \mathcal{X} \to \mathcal{Y}$ is holomorphic in Kato sence if there exist a bounded holomorphic functions $U(\lambda) : \mathcal{Z} \to \mathcal{X}$ and $V(\lambda) : \mathcal{Z} \to \mathcal{Y}$ such that $T(\lambda)U(\lambda) = V(\lambda)$ and $R(U(\lambda)) = D(T(\lambda))$ for all $\lambda \in \mathbb{C}$, where $\mathcal{Z}$ is some Banach space. An operator function $T(\lambda)$ is bounded holomorphic if it takes values in the set of bounded operators and if $T(\lambda)x$ is a holomorphic function in λ for all x.*

In our case, we can choose $U(\lambda) = B^{-1}$, $V(\lambda) = AB^{-1} - \lambda$, and $\mathcal{Z} = \mathcal{H}$.

We set $S = B^{-1}A$, and by the use of the similar methods as in the proofs of Theorem 3.3 and Proposition 3.4, it can be seen that $\sigma(L) = \sigma(S)$, $\sigma_{ap}(L) = \sigma_{ap}(S)$, $\sigma_c(L) = \sigma_c(S)$ and $\sigma_p(L) = \sigma_p(S)$, hence we have the complete equivalence of the spectra of L and S.

5. Variational characterization

In this section we apply the constructions of reduced operators from Sections 2 and 4 to obtain a Rayleigh–Ritz–like upper bound for the eigenvalues of the reduction operator given in terms of the eigenvalues of the matrix pair (A_V, B_V), where A_V, B_V are orthogonal projections of A, B on a finite dimensional subspace. Hence the bound is given in terms of the original data (A, B), which can be important in applications.

To obtain a variational characterization for the reduction operator, we need a result from [5].

Let Q be a qup operator such that all finite critical points are regular, and such that there are no critical points embedded in the positive continuous spectrum. For such a Q, in [5] a variational characterization of eigenvalues is obtained. More precisely, we define

$$\lambda_j^{\pm} = \sup\left\{\inf\left\{\frac{[Qx,x]}{[x,x]} : x \in M \cap \mathcal{C}^{\pm} \cap D(Q)\right\} : M \in \mathcal{M}_j\right\}, \tag{5.1}$$

where $\mathcal{C}^{\pm} = \{x : [x,x] \gtrless 0\}$, and $\mathcal{M}_j$ denotes the set of all subspaces of codimension $j-1$. Then, if by $d^{\pm}$ we denote the positive (negative) spectral shift of Q (for more details see [5]), a number $\lambda^{\pm}_{j+d^{\pm}}$ is either an eigenvalue of Q, or a point on the boundary of the essential spectrum of Q, for all $j \in \mathbb{N}$.

If the operator $\widetilde{S}$ from Section 2, or the operator S from Section 4 satisfy conditions stated above, we can variationally characterize the eigenvalues of $\widetilde{S}$ or S, respectively. If B is bounded, then we have the following variational characterization

$$\begin{aligned}\lambda_j^{\pm}(\widetilde{S}) &= \lambda_j^{\pm}(\widetilde{L}) \\ &= \sup\left\{\inf\left\{\frac{(\widetilde{A}x,x)}{(Bx,x)} : x \in M \cap \mathcal{C}^{\pm} \cap D(|B|^{-1/2}\widetilde{A})\right\} : M \in \mathcal{M}_j\right\}.\end{aligned} \tag{5.2}$$

In the case when B is unbounded, we have

$$\frac{[S(x_n),(x_n)]_{\mathcal{K}}}{[(x_n),(x_n)]_{\mathcal{K}}} = \lim_n \frac{(Ax_n,x_n)}{(Bx_n,x_n)},$$

where $(x_n) \in D(S)$. Since $D(|B|^{-1/2}A)$ is a core of the operator S, from [12, Theorem 4.5.3.] follows that in this case we have

$$\begin{aligned}\lambda_j^{\pm}(S) &= \lambda_j^{\pm}(L) \\ &= \sup\left\{\inf\left\{\frac{(Ax,x)}{(Bx,x)} : x \in M \cap \mathcal{C}^{\pm} \cap D(|B|^{-1/2}A)\right\} : M \in \mathcal{M}_j\right\}.\end{aligned} \tag{5.3}$$

Now, suppose that the operator $\widetilde{S}$ from Section 2 satisfies conditions stated above. Let $V \subset D(|B|^{-1/2}\widetilde{A})$ be a finite dimensional ortho–complemented subspace in $\mathcal{K}$. Let $\dim V = N$. Denote by P a orthogonal projector in $\mathcal{K}$ onto V. Set $\widetilde{A}_V = P\widetilde{A}P$, $B_V = PBP$. Let $\perp$ and $[\perp]$ denote the orthogonal complement in $\mathcal{H}$ and $\mathcal{K}$, respectively. The set of ortho–complemented subspaces of $\mathcal{K}$ we denote by $\mathcal{O}$. Let F be a subspace of V, with $\dim F = k$, and let $F = \operatorname{span}\{x_1,\dots,x_k\}$. Set $F' = \operatorname{span}\{B_V^{-1}x_1,\dots,B_V^{-1}x_k\}$. Then $\dim F' = k$ and

$$x[\perp]F' \Longleftrightarrow x \perp F, \quad \forall x \in V.$$

On the other hand, if $x[\perp]F$, then we define $F'' = \operatorname{span}\{B_V x_1,\dots,B_V x_k\}$ and then we have

$$x[\perp]F \Longleftrightarrow x \perp F'', \quad \forall x \in V.$$

For $m \leq N$, set

$$\mu_m = \sup_{\substack{F\subset V\\ \dim F=m-1}} \inf_{\substack{x\in V\\ x\perp F\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}_V x,x)}{(B_V x,x)}. \tag{5.4}$$

Then, from the considerations given above, we have

$$\begin{aligned}
\mu_m &= \sup_{\substack{F\subset V\\ \dim F=m-1}} \inf_{\substack{x\in V\\ x[\perp]F\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)}\\
&= \sup_{\substack{y_1,\dots,y_{m-1}\in V\\ \operatorname{span}\{y_1,\dots,y_{m-1}\}\in\mathcal{O}}} \inf_{\substack{x\in V\\ x\in\operatorname{span}\{y_1,\dots,y_{m-1}\}^{[\perp]}\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)}\\
&= \sup_{\substack{y_1,\dots,y_{m-1}\in \mathcal{K}\\ \operatorname{span}\{y_1,\dots,y_{m-1}\}\in\mathcal{O}}} \inf_{\substack{x\in V\\ x\in\operatorname{span}\{Py_1,\dots,Py_{m-1}\}^{[\perp]}\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)}\\
&= \sup_{\substack{y_1,\dots,y_{m-1}\in \mathcal{K}\\ \operatorname{span}\{y_1,\dots,y_{m-1}\}\in\mathcal{O}}} \inf_{\substack{x\in V\\ x\in\operatorname{span}\{y_1,\dots,y_{m-1}\}^{[\perp]}\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)} \geq\\
&\geq \sup_{\substack{y_1,\dots,y_{m-1}\in \mathcal{K}\\ \operatorname{span}\{y_1,\dots,y_{m-1}\}\in\mathcal{O}}} \inf_{\substack{x\in D(|B|^{-1/2}\widetilde{A})\\ x\in\operatorname{span}\{y_1,\dots,y_{m-1}\}^{[\perp]}\\ x\in\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)}\\
&= \sup_{M\in\mathcal{M}_j} \inf_{\substack{x\in D(|B|^{-1/2}\widetilde{A})\\ x\in M\cap\mathcal{C}^+}} \frac{(\widetilde{A}x,x)}{(Bx,x)}.
\end{aligned}$$

The analogous relation can be obtained for the eigenvalues of the negative type. From the finite dimensional principle given in [7], μ_m is an eigenvalue of the matrix pair $(\widetilde{A}_V, B_V)$. Hence we obtained a Rayleigh–Ritz–like upper bound for eigenvalues of $\widetilde{S}$:

$$\lambda_j^{\pm}(\widetilde{A}_V, B_V) \geq \lambda_j^{\pm}(\widetilde{S}).$$

The analogous Rayleigh–Ritz upper bound holds also for the operator S from Section 4, due to the formula (5.3).

6. Zero is an eigenvalue of B

If 0 is an eigenvalue of B, then in general a reduced operator cannot be constructed. But we can construct an reduced linear relation, a generalization of the notion of linear operator, and we can recover the results from previous

sections. Under some additional assumptions we can also construct a proper reduction operator.

In this section we assume:

1. A and B are selfadjoint operators,
2. B is A–bounded, $0 \in \sigma_p(B)$,
3. $\rho(A,B) \cap \mathbb{R} \neq \emptyset$.

The last relation implies that we can assume $0 \in \rho(A)$.

Then $B^{-1}A$ does not exist as a linear operator, but it can be introduced in terms of the linear relations, i.e. as a subspace of $\mathcal{H} \times \mathcal{H}$ (for the basic definitions see [28, 14]). We define

$$S_0 = B^{-1}A = \{\{x,y\} \in D(A) \times D(B) : Ax = By\}.$$

By S we denote the closure of S_0 as a subspace in $\mathcal{H} \times \mathcal{H}$.

Let $\mathcal{K}$ denote the factor space $D(B)/_{\operatorname{Ker} B}$, and let $\widetilde{\mathcal{K}}$ denote the completion of $(\mathcal{K}, (B\cdot,\cdot))$. Evidently, $\widetilde{\mathcal{K}}$ is a Krein space, and it consists of all Cauchy sequences in $D(|B|^{1/2})$, where we identify such two sequences (x_n) and (y_n) if $|B|^{1/2}(x_n - y_n) \to 0$. This equivalence relation we denote by the $\sim$ symbol. Let $[x]$ denote a class in $\widetilde{\mathcal{K}}$ represented by x.

Let $R(\lambda, S)$ be a resolvent of S. Then the family of linear operators $R(\lambda, S)$ in $\mathcal{H}$ induces a family $\widetilde{R}(\lambda, S)$ of linear operators in the space $\mathcal{K}$, since $R(\lambda, S) \operatorname{Ker} B \subset \operatorname{Ker} B$ (see [24]). We define the linear relation $\widetilde{S}$ in $\widetilde{\mathcal{K}}$ by $\widetilde{S} = \widetilde{R}(0,S)^{-1}$.

We say that λ is a *discrete eigenvalue* of the operator T if λ is an isolated eigenvalue of finite multiplicity of the operator T.

First we treat the case when B is a bounded operator and zero is a discrete eigenvalue. Then it is easy to see that we can substitute Cauchy sequences with elements of $\mathcal{H}$, and that $x \sim y$ if and only if $x - y \in \operatorname{Ker} B$. The inner product on $\widetilde{K}$ is given by $[\cdot,\cdot] = (B\cdot,\cdot)$. It is easy to see that

$$\widetilde{S} = \{\{[x],[y]\} : x - A^{-1}By \in \operatorname{Ker} B\},$$

and that $\widetilde{S}$ is a selfadjoint relation.

THEOREM 6.1. *The spectra of $\widetilde{S}$ and L coincide. Moreover, their point spectra coincide.*

PROOF. The relation $\sigma_p(L) \subset \sigma_p(\widetilde{S})$ is obvious. On the other hand, let $\lambda \in \sigma_p(\widetilde{S})$. Then there exists $x \in \mathcal{H}$ such that $x - \lambda A^{-1}Bx \in \operatorname{Ker} B$, hence $Bx - \lambda BA^{-1}Bx = 0$. There exists $y \neq 0$ such that $Bx = Ay$, hence

$$0 = (I - \lambda BA^{-1})Bx = (I - \lambda BA^{-1})Ay = Ay - \lambda By,$$

which implies $\lambda \in \sigma_p(L)$. It is easy to see that

$$(S - \lambda)^{-1} = (A - \lambda B)^{-1}B \quad \forall \lambda \in \mathbb{C} \setminus \sigma_p(L) = \mathbb{C} \setminus \sigma_p(\widetilde{S}).$$

From this relation it is easy to obtain $\rho(L) = \rho(\widetilde{S})$. □

If B is bounded, but zero is not a discrete eigenvalue of B, it can be seen that

$$\widetilde{S} = \{\{[x], [(y_n)]\} : x - A^{-1}J|B|^{1/2} \lim_n |B|^{1/2} y_n \in \operatorname{Ker} B\},$$

where J is defined as in Section 2, is a selfadjoint relation. Using similar techniques as in Section 2, it can be shown that $\sigma(\widetilde{S}) = \sigma(L)$ and $\sigma_p(\widetilde{S}) = \sigma_p(L)$.

This kind of procedure can be implemented also in the case when B is not bounded, using the techniques from Section 4. For instance, if all assumptions given in the beginning of the Section 4 hold, except from $0 \notin \sigma_p(B)$, we can define $\widetilde{S}$ by

$$\widetilde{S} = \{\{[(x_n)], [(y_n)]\} : (x_n) \sim (A^{-1}By_n)\}.$$

Reasoning analogously as in the proof of Theorem 4.1, it can be shown that $\sigma_p(L) \subset \sigma_p(\widetilde{S})$ and $\sigma(\widetilde{S}) \subset \sigma(L)$.

When is it possible to reduce the relation $\widetilde{S}$ to an operator, without the loss of the information about spectra? To answer this question we introduce the notion of the *multi–valued part* $T_\infty = \{\{0, g\} \in T\}$ of the relation T (see [28]). We also set $T(0) = \{g \in \mathcal{K} : \{0, g\} \in T\}$. It is easy to see that T_∞ is ortho–complemented if and only if $T(0)$ is ortho–complemented. We have

THEOREM 6.2 ([28]). *Let T be a closed symmetric relation in a Krein space $\mathcal{K}$ such that T_∞ is ortho–complemented. Then $T_s = T \cap (T_\infty)^\perp$ is an operator with $D(T_s) = D(T)$ and $R(T_s) \subset T(0)^\perp$. We also have*

(i) $\sigma_p(T) = \sigma_p(T_s)$,
(ii) $\sigma(T) = \sigma(T_s)$, *and*
(iii) *T is a selfadjoint relation if and only if T_s is a selfadjoint operator in $T(0)^\perp$.*

Hence, we can reduce $\widetilde{S}$ to an operator if the set

$$\widetilde{S}(0) = \{[(y_n)] : \lim_n |B|^{1/2} A^{-1} B y_n \in \operatorname{Ker} B\}$$

is ortho–complemented.

If, for instance, $R(B) \cap A \operatorname{Ker} B = \{0\}$, or if $\operatorname{Ker} B$ is A–invariant, $\widetilde{S}(0)$ is ortho–complemented.

If $d_0 = \dim \operatorname{Ker} B < \infty$ and if $\widetilde{S}(0)$ is ortho–complemented, we can also obtain Rayleigh–Ritz–like upper bound for the eigenvalues of $\widetilde{S}(0)$ in terms of the non-zero eigenvalues of the matrix pair (A_V, B_V), where $V \subset D(A)$, in the case when the dimension of V is greater then the sum of d_0 and the spectral shift of $\widetilde{S}_s = \widetilde{S} \cap (\widetilde{S}(0))^\perp$. We omit the details.

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References

[1] T. Ya. Azizov and I. S. Iokhvidov, Linear operators in spaces with an indefinite metric, John Wiley & Sons Ltd., Chichester, 1989.

[2] T. Ya. Azizov, *Operator theory in Krein spaces and operator pencils*, In H. Helson, B. Sz.-Nagy, F.-H. Vasilescu, and Gr. Arsene, editors, Linear Operators in Function Spaces, volume 43 of Operator Theory: Advances and Applications, Birkhäuser Basel, 1990, 111–121.

[3] H. Baumgärtel, Analytic perturbation theory for matrices and operators, Birkhäuser Basel, 1985.

[4] P. Binding, D. Eschwé and H. Langer, *Variational principles for real eigenvalues of self-adjoint operator pencils*, Integral Equations Operator Theory **38** (2000), 190–206.

[5] P. Binding and B. Najman, *A variational principle in Kreĭn space* Trans. Amer. Math. Soc. **342** (1994), 489–499.

[6] P. Binding and B. Najman, *The minimal index of a self-adjoint pencil*, Glas. Mat. Ser. III **35(55)** (2000) 25–44.

[7] P. Binding, B. Najman and Q. Ye, *A variational principle for eigenvalues of pencils of Hermitian matrices*, Integral Equations Operator Theory **35** (1999) 398–422.

[8] J. Bognár, Indefinite inner product spaces, Springer-Verlag, New York, 1974.

[9] P. A. Cojuhari, *Estimates of the discrete spectrum of a linear operator pencil*, J. Math. Anal. Appl. **326** (2007), 1394–1409.

[10] B. Ćurgus and H. Langer, *A Kreĭn space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989), 31–61.

[11] B. Ćurgus and B. Najman, *Quasi-uniformly positive operators in Kreĭn space*, in Operator theory and boundary eigenvalue problems (Vienna, 1993), Birkhäuser, Basel, 1995, 90–99.

[12] E. B. Davies, Spectral theory and differential operators, Cambridge University Press, Cambridge, 1995.

[13] V. A. Derkach and M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.

[14] A. Dijksma and H. S. V. de Snoo, *Symmetric and selfadjoint relations in Kreĭn spaces. I*, in Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985), Birkhäuser, Basel, 1987, 145–166.

[15] A. Dijksma and H. Langer, *Operator theory and ordinary differential operators*, in Lectures on operator theory and its applications (Waterloo, ON, 1994), Amer. Math. Soc., Providence, 1996, 73–139.

[16] M. V. Falaleev, *The Cauchy problem for a degenerate heat equation in Banach spaces* Differ. Uravn. **44** (2008), 1120–1130.

[17] Y. L. Gao, Z. Wang and H. Y. Wu, *Spectrum of a self-adjoint operator pencil and its applications*, Chinese Ann. Math. Ser. A **33** (2012), 101–112.

[18] I. C. Gohberg and M. G. Kreĭn, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Amer. Math. Soc. Transl. (2) **13** (1960), 185–264.

[19] V. I. Gorbachuk and M. L. Gorbachuk, Boundary value problems for operator differential equations, Kluwer Academic Publishers Group, Dordrecht, 1991.

[20] A. N. Kočubeĭ, *Extensions of J-symmetric operators*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. **31** (1979), 74–80, 167.

[21] M.G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I*, Mat. Sb. **20(62)** (1947), 431–495.

[22] P. Lancaster, A. Shkalikov and Q. Ye, *Strongly definitizable linear pencils in Hilbert space*, Integral Equations and Operator Theory **17** (1993), 338–360.

[23] H. Langer, *Spectral functions of definitizable operators in Kreĭn spaces*, in Functional analysis (Dubrovnik, 1981), Springer, Berlin, 1982, 1–46.
[24] H. Langer, R. Mennicken and C. Tretter, *A self-adjoint linear pencil $Q-\lambda P$ of ordinary differential operators*, Methods Funct. Anal. Topology **2** (1996), 38–54.
[25] A. S. Marcus, Introduction to the spectral theory of polynomial operator pencils, translation of Mathematical monograph, vol. 71. Amer. Math. Soc., Providence, 1988.
[26] A. B. Mingarelli, *Indefinite Sturm–Liouville problems*, in Ordinary and partial differential equations (Dundee, 1982), Springer, Berlin, 1982, 519–528.
[27] K. Schmüdgen, Unbounded self-adjoint operators on Hilbert space, Springer, Dordrecht, 2012.
[28] P. Sorjonen, *On linear relations in an indefinite inner product space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 169–192.
[29] C. Tretter, *Linear operator pencils $A - \lambda B$ with discrete spectrum*, Integral Equations and Operator Theory **37** (2000), 357–373.
[30] O. Verdier, *Reductions of operator pencils*, Math. Comp. **83** (2014), 189–214.
[31] J. Weidmann, Lineare Operatoren in Hilberträumen. B. G. Teubner, Stuttgart, 1976.

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