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COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE END RANDOM VARIABLES

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Abstract. The authors study complete convergence and complete moment convergence for arrays of rowwise extended negatively dependent (END) random variables and obtain some new results. The results extend and improve the corresponding theorems by Sung (2005), Hu and Taylor (1997), Hu et al. (1989), and Chow (1988).

1. Introduction

The concept of negatively orthant dependent (NOD) random variables was introduced by Ebrahimi and Ghosh ([4]).

Definition 1.1. *The random variables $X_1, \dots, X_k$ are said to be negatively upper orthant dependent (NUOD) if for all real $x_1, \dots, x_k$,*

$$P(X_i > x_i, i = 1, 2, \dots, k) \leq \prod_{i=1}^{k} P(X_i > x_i),$$

and negatively lower orthant dependent (NLOD) if

$$P(X_i \leq x_i, i = 1, 2, \dots, k) \leq \prod_{i=1}^{k} P(X_i \leq x_i).$$

Random variables $X_1, \dots, X_k$ are said to be NOD if they are both NUOD and NLOD.

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The concept of extended negatively dependent (END) random variables was introduced by Liu ([11]).

DEFINITION 1.2. *We call random variables* $\{X_i, i \geq 1\}$ *END if there exists a constant* $M > 0$ *such that both*

$$P(X_i \leq x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i)$$

and

$$P(X_i > x_i, i = 1, 2, \ldots, n) \leq M \prod_{i=1}^{n} P(X_i > x_i),$$

hold for each $n = 1, 2, \ldots$ *and all* $x_1, \ldots, x_n$.

Clearly the END structure is substantially more comprehensive than the NOD structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent. Joag-Dev and Proschan ([10]) also pointed out that negatively associated (NA) random variables must be NOD and NOD is not necessarily NA, thus NA random variables are END. Liu [11] also provided some interesting examples to illustrate that the extended negative dependence indeed allows a wide range of dependence structures. Since the article of Liu ([11]) appeared, Chen et al. ([2]), Wu and Guan ([14]) and Qiu et al. ([12]) studied the convergence properties for END random variables.

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In this case we write $U_n \to a$ completely. This notion was given by Hsu and Robbins ([5]).

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \text{ for some or all } \varepsilon > 0,$$

then the result was called the complete moment convergence by Chow ([3]).

In the following we let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, $\{k_n, n \geq 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$.

An array of rowwise random variables $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is said to be uniformly bounded by a random variable X (denoted by $\{X_{nk}\} \prec X$)

if there exists a constant $C > 0$ such that

$$\sup_{n,k} P(|X_{nk}| > x) \leq CP(|X| > x), \quad \text{for all } x > 0.$$

Clearly if $\{X_{nk}\} \prec X$, for $0 < p < \infty$ and any $1 \leq k \leq n, n \geq 1$, then $E|X_{nk}|^p \leq CE|X|^p$.

Hu et al. ([7]) stated the following complete convergence theorem for arrays of rowwise independent random variables.

THEOREM 1.3. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{c_n, n \geq 1\}$ be a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$. Suppose that for every $\varepsilon > 0$, some $\delta > 0$ and $\eta \geq 2$,*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty,$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \leq \delta) \right)^{\eta} < \infty$$

and

$$\sum_{k=1}^{k_n} EX_{nk} I(|X_{nk}| \leq \delta) \to 0 \quad \text{as } n \to \infty. \tag{1.1}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) < \infty \ \text{ for all } \varepsilon > 0. \tag{1.2}$$

The proof by Hu et al. given in [7] is mistakenly based on the fact that the assumptions of Theorem 1.3 imply

$$\sum_{k=1}^{k_n} X_{nk} \to 0 \ \text{ in probability} \tag{1.3}$$

as $n \to \infty$. Hu and Volodin ([9]) found that (1.3) does not necessarily follow from the assumptions of Theorem 1.3. Therefore, they replaced condition $\sum_{n=1}^{\infty} c_n = \infty$ by the condition $\liminf_{n\to\infty} c_n > 0$. In this case the assumptions of Theorem 1.3 imply (1.3).

Sung ([13]) proved Theorem 1.3 without the assumption $\liminf_{n\to\infty} c_n > 0$. Chen et al. ([1]) extended Theorem 1.3 for the case of arrays of rowwise negatively associated random variables.

Hu and Taylor ([8]) proved the following results.

THEOREM 1.4. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables and let $\{a_n, n \geq 1\}$ be a sequence of positive* *real numbers with $a_n \uparrow \infty$. Assume that $\Psi(t)$ is a positive even function that satisfies*

$$\frac{\Psi(|t|)}{|t|^p} \uparrow \quad \text{and} \quad \frac{\Psi(|t|)}{|t|^{p+1}} \downarrow \quad \text{as} \quad |t| \uparrow \tag{1.4}$$

for some integer $p \geq 2$. If

$$EX_{nk} = 0, 1 \leq k \leq n, n \geq 1, \tag{1.5}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi(X_{nk})}{\Psi(a_n)} < \infty \tag{1.6}$$

and

$$\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} E\Big(\frac{X_{nk}}{a_n}\Big)^2 \Big)^{2k} < \infty, \tag{1.7}$$

where k is a positive integer, then (1.5), (1.6), and (1.7) imply

$$\frac{1}{a_n} \sum_{k=1}^{n} X_{nk} \to 0 \quad a.s.. \tag{1.8}$$

THEOREM 1.5. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. If $\Psi(t)$ is a positive even function that satisfies (1.4) for $p = 1$, then (1.5) and (1.6) imply (1.8).*

In addition, Hu et al. ([6]) obtained the following complete convergence.

THEOREM 1.6. *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise independent random variables with (1.5) and assume that $\{X_{nk}\} \prec X$. If $E|X|^{2p} < \infty$ for some $1 \leq p < 2$, then*

$$n^{-1/p} \sum_{k=1}^{n} X_{nk} \to 0 \quad \text{completely.} \tag{1.9}$$

Chow ([3]) obtained the following complete moment convergence.

THEOREM 1.7. *Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX_1 = 0$, $\alpha > 1/2$, $p \geq 1$ and $\alpha p > 1$. If $E\{|X_1|^p + |X_1| \log(1 + |X_1|)\} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} E\Big\{ \Big| \sum_{k=1}^{n} X_k \Big| - \varepsilon n^{\alpha} \Big\}_+ < \infty \ \text{for all}\ \varepsilon > 0. \tag{1.10}$$

In this work, we shall extend and improve Theorem 1.3 to END instead of independent or NA, and shall extend and improve Theorem 1.4-1.7 under some weaker conditions. It is worthy to point out that we study complete moment convergence for the arrays of END random variables under some similar conditions, which were not considered in Hu et al. ([7]), Sung ([13]) and Chen et al. ([1]).

In the paper, C will denote generic positive constants, whose value may vary from one application to another, $I(A)$ will indicate the indicator function of A.

2. Main results

We will present the main results of the paper and the proofs will be detailed in the next section.

Theorem 2.1. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise END random variables and let $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose that the following conditions hold:*

(i) *for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty; \tag{2.1}$$

(ii) *there exists $\eta \ge 1$ and $\delta > 0$ such that*

$$\sum_{n=1}^{\infty} c_n \biggl(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}| \le \delta) \biggr)^{\eta} < \infty. \tag{2.2}$$

Then

$$\sum_{n=1}^{\infty} c_n P\biggl(\Bigl| \sum_{k=1}^{k_n} \bigl(X_{nk} - EX_{nk} I(|X_{nk}| \le \delta) \bigr) \Bigr| > \varepsilon \biggr) < \infty \text{ for all } \varepsilon > 0. \tag{2.3}$$

Corollary 2.2. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise END random variables and let $\{c_n, n \ge 1\}$ be a sequence of positive constants. Then (2.1), (2.2) and (1.1) imply (1.2).*

Remark 2.3. Since independence implies END and we consider $\eta \ge 1$ instead of $\eta \ge 2$, Corollary 2.2 extends and improves Theorem 1.3. In addition, compared with the results of Qiu et al. ([12, Theorem 1]), Corollary 2.2 and Theorem 1 of Qiu et al. ([12]) do not completely overlap with each other, although the conditions of our result have some similarities to those of Qiu et al. in [12].

Let $c_n = 1$, $k_n = n$ for $n \ge 1$ and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Assuming that (1.5) holds and replacing X_{nk} by X_{nk}/a_n in formulation of Corollary 2.2, we can obtain the following corollary.

Corollary 2.4. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise END random variables with (1.5) and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Suppose that the following conditions hold:*

(i) *for every* $\varepsilon > 0$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > a_n\varepsilon) < \infty; \tag{2.4}$$

(ii) *there exists* $\eta \geq 1$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \leq a_n\delta)\right)^{\eta} < \infty; \tag{2.5}$$

(iii)

$$a_n^{-1}\sum_{k=1}^{n} EX_{nk} I(|X_{nk}| \leq a_n\delta) \to 0. \tag{2.6}$$

Then

$$\frac{1}{a_n}\sum_{k=1}^{n} X_{nk} \to 0 \quad \textit{completely.}$$

Remark 2.5. The following statements show that the conditions of Corollary 2.4 are weaker than those of Theorems 1.4 and 1.5.

Firstly, we state that (1.4)-(1.6) imply (2.4). Without loss of generality we may assume $0 < \varepsilon < 1$. If $p \geq 2$ or $p = 1$, by (1.4) and (1.6), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > a_n\varepsilon)\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n} E(I(|X_{nk}| > a_n\varepsilon)) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|}{a_n\varepsilon} I(|X_{nk}| > a_n\varepsilon)\\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^{p+1}}{(a_n\varepsilon)^{p+1}} I(a_n\varepsilon < |X_{nk}| \leq a_n)\\
&\quad + \varepsilon^{-1}\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^p}{a_n^p} I(|X_{nk}| > a_n)\\
&\leq (\varepsilon^{-(p+1)} + \varepsilon^{-1})\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi(X_{nk})}{\Psi(a_n)} < \infty.
\end{aligned}$$

Secondly, we take $\delta = 1$ and show that (1.4), (1.6) and (1.7) imply (2.5). By (1.4) and (1.6), we can get easily

$$\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \leq a_n)\right)^{\eta} \leq \left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi(X_{nk})}{\Psi(a_n)}\right)^{\eta} < \infty.$$

If $p \ge 2$, take $\eta = 2k$, where k is a positive integer. By (1.7), we can get

$$\sum_{n=1}^{\infty}\Big(a_n^{-2}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le a_n)\Big)^{\eta} \le \sum_{n=1}^{\infty}\Big(a_n^{-2}\sum_{k=1}^{n} EX_{nk}^2\Big)^{2k} < \infty.$$

Finally, we take $\delta = 1$ and show that (1.4)-(1.6) imply (2.6). By (1.4)-(1.6), we have

$$\begin{aligned} a_n^{-1}\Big|\sum_{k=1}^{n} EX_{nk} I(|X_{nk}| \le a_n)\Big| &= a_n^{-1}\Big|\sum_{k=1}^{n} EX_{nk} I(|X_{nk}| > a_n)\Big| \\ \le a_n^{-1}\sum_{k=1}^{n} E|X_{nk}| I(|X_{nk}| > a_n) &\le \sum_{k=1}^{n}\frac{E\Psi(X_{nk})}{\Psi(a_n)} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

To sum up, we know that Corollary 2.4 improve Theorems 1.4 and 1.5. Obviously, complete convergence implies almost sure convergence. Therefore, our conclusions are much stronger and conditions are much weaker.

Taking $a_n = n^{1/p}$ for $1 \le p < 2$ in Corollary 2.4, we can obtain the following corollary.

COROLLARY 2.6. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise END random variables satisfying (1.5). Suppose that the following conditions hold:*

(i) *for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > n^{1/p}\varepsilon) < \infty;$$

(ii) *there exists $\eta > p/(2-p)$ and $\delta > 0$ such that*

$$\sum_{n=1}^{\infty}\Big(n^{-2/p}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le n^{1/p}\delta)\Big)^{\eta} < \infty;$$

(iii)

$$n^{-1/p}\sum_{k=1}^{n} EX_{nk} I(|X_{nk}| \le n^{1/p}\delta) \to 0,$$

where $1 \le p < 2$.

Then (1.9) holds.

REMARK 2.7. The following statements show that the conditions of Corollary 2.6 are weaker than those of Theorem 1.6.

Firstly, by $\{X_{nk}\} \prec X$ and $E|X|^{2p} < \infty$, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > n^{1/p}\varepsilon) \le C\sum_{n=1}^{\infty} nP(|X| > n^{1/p}\varepsilon) \le CE|X|^{2p} < \infty.$$

Secondly, since $E|X|^{2p} < \infty$ for $1 \le p < 2$, we know $E|X|^2 < \infty$. Hence, by $\eta > p/(2-p)$ and $\{X_{nk}\} \prec X$, we have

$$\sum_{n=1}^{\infty}\Big(n^{-2/p}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le n^{1/p}\delta)\Big)^{\eta} \le C\sum_{n=1}^{\infty} n^{(1-2/p)\eta}(E|X|^2)^{\eta} < \infty.$$

Finally, by (1.5), $\{X_{nk}\} \prec X$ and $E|X|^{2p} < \infty$, we have

$$n^{-1/p}\Big|\sum_{k=1}^{n} EX_{nk}I(|X_{nk}| \le n^{1/p}\delta)\Big| \le \; n^{-1/p}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > n^{1/p}\delta)$$
$$\le \delta^{1-2p}\sum_{k=1}^{n}\frac{E|X_{nk}|^{2p}}{n^2} I(|X_{nk}| > n^{1/p}\delta) \le C\delta^{1-2p}n^{-1}E|X|^{2p} \to 0$$

as $n \to \infty$.

To sum up, we know that Corollary 2.6 extends and improves Theorem 1.6.

The following theorem shows that, under some appropriate conditions, we can obtain complete moment convergence for the array of rowwise END random variables.

THEOREM 2.8. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise END random variables and let $\{c_n, n \ge 1\}$ be a sequence of positive constants. Suppose that (2.2) and the following conditions hold:*

(i) *for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \varepsilon) < \infty; \tag{2.7}$$

(ii) *there exists $\eta > 1$ and $\delta > 0$ such that*

$$\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta/16\eta) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Then

$$\sum_{n=1}^{\infty} c_n E\Big\{\Big|\sum_{k=1}^{k_n}\big(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| - \varepsilon\Big\}_+ < \infty \text{ for all } \varepsilon > 0. \tag{2.9}$$

COROLLARY 2.9. *Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise END random variables with (1.5). Then conditions (2.2), (2.7) and (2.8) imply*

$$\sum_{n=1}^{\infty} c_n E\Big\{\Big|\sum_{k=1}^{k_n} X_{nk}\Big| - \varepsilon\Big\}_+ < \infty \text{ for all } \varepsilon > 0.$$

PROOF. Note that, from (1.5) and (2.8), we can get

$$\begin{aligned}\Big|\sum_{k=1}^{k_n} EX_{nk}I(|X_{nk}|\le\delta)\Big| &= \Big|\sum_{k=1}^{k_n} EX_{nk}I(|X_{nk}|>\delta)\Big|\\ &\le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}|>\delta)\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Then for every given $\varepsilon>0$, while n is sufficiently large, $\big|\sum_{k=1}^{k_n} EX_{nk}I(|X_{nk}|\le\delta)\big|<\varepsilon$. Therefore, by (2.9), we have

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} c_nE\Big\{\Big|\sum_{k=1}^{k_n}\big(X_{nk}-EX_{nk}I(|X_{nk}|\le\delta)\big)\Big|-\varepsilon\Big\}_+\\ &\ge \sum_{n=1}^{\infty} c_nE\Big\{\Big|\sum_{k=1}^{k_n}X_{nk}\Big|-\Big|\sum_{k=1}^{k_n}EX_{nk}I(|X_{nk}|\le\delta)\Big|-\varepsilon\Big\}_+\\ &> \sum_{n=1}^{\infty} c_nE\Big\{\Big|\sum_{k=1}^{k_n}X_{nk}\Big|-2\varepsilon\Big\}_+.\end{aligned}$$

The proof is complete. □

Let $c_n=1$, $k_n=n$ for $n\ge1$ and let $\{a_n, n\ge1\}$ be a sequence of positive real numbers with $a_n\uparrow\infty$. Replacing X_{nk} by X_{nk}/a_n in formulation of Corollary 2.9, we can obtain the following corollary.

COROLLARY 2.10. *Let $\{X_{nk}, 1\le k\le n, n\ge1\}$ be an array of rowwise END random variables satisfying (1.5) and let $\{a_n, n\ge1\}$ be a sequence of positive real numbers with $a_n\uparrow\infty$. Suppose that (2.5) and the following conditions hold:*

(i) *for every $\varepsilon>0$*

$$\sum_{n=1}^{\infty} a_n^{-1}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}|>a_n\varepsilon)<\infty; \tag{2.10}$$

(ii) *there exists $\eta>1$ and $\delta>0$ such that*

$$a_n^{-1}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}|>a_n\delta/16\eta)\to 0 \text{ as } n\to\infty. \tag{2.11}$$

Then

$$\sum_{n=1}^{\infty} a_n^{-1}E\Big\{\Big|\sum_{k=1}^{n}X_{nk}\Big|-a_n\varepsilon\Big\}_+<\infty \text{ for all } \varepsilon>0.$$

REMARK 2.11. Wu and Zhu ([15]) discussed complete convergence and complete moment convergence for arrays of rowwise NOD random variables. The conditions in Wu and Zhu ([15]) are similar to those of Hu and Taylor

([8]). By some similar arguments in Remark 2.5, we can show that the conditions of Wu and Zhu ([15]) imply (2.4)-(2.6), (2.10) and (2.11). Here we omit the details. Since NOD implies END and the conditions in this paper are weaker than those of Wu and Zhu in [15], Corollary 2.4 and 2.10 improve Theorem 1.1 and 1.3 in [15] by Wu and Zhu, respectively.

Taking $k_n = n$ and $c_n = n^{\alpha p-2}$, and replacing X_{nk} by X_k/n^α for $1 \le k \le n$ in Corollary 2.9, we can obtain the following corollary.

Corollary 2.12. *Let $\{X_k, k \ge 1\}$ be a sequence of END random variables with $EX_k = 0$. Suppose that the following conditions hold:*

(i) *for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^\alpha \varepsilon) < \infty \tag{2.12}$$

(ii) *there exists $\eta > \max\{1, \frac{\alpha p-1}{2\alpha-1}\}$ and $\delta > 0$ such that*

$$n^{-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^\alpha \delta/16\eta) \to 0 \text{ as } n \to \infty$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \left(n^{-2\alpha} \sum_{k=1}^{n} EX_k^2 I(|X_k| \le n^\alpha \delta) \right)^\eta < \infty, \tag{2.13}$$

where $\alpha > 1/2$, $p \ge 1$ and $\alpha p > 1$.

Then conditions (2.12)-(2.13) imply (1.10).

Remark 2.13. The following statements show that the conditions of Corollary 2.12 are weaker than those of Theorem 1.7.

Firstly, we state the conditions of Theorem 1.7 imply (2.12). If $p > 1$, by $E|X_1|^p < \infty$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^\alpha \varepsilon) &= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E|X_1| I(|X_1| > n^\alpha \varepsilon) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} E|X_1| I(m^\alpha \varepsilon < |X_1| \le (m+1)^\alpha \varepsilon) \\
&\le \sum_{m=1}^{\infty} E|X_1| I(m^\alpha \varepsilon < |X_1| \le (m+1)^\alpha \varepsilon) \sum_{n=1}^{m} n^{\alpha p-1-\alpha} \\
&\le C \sum_{m=1}^{\infty} m^{\alpha p-\alpha} E|X_1| I(m^\alpha \varepsilon < |X_1| \le (m+1)^\alpha \varepsilon) \\
&\le C \sum_{m=1}^{\infty} E|X_1|^p I(m^\alpha \varepsilon < |X_1| \le (m+1)^\alpha \varepsilon) \le E|X_1|^p < \infty.
\end{aligned}$$

If $p = 1$, by $E\{|X_1| + |X_1| \log(1 + |X_1|)\} < \infty$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^{\alpha}\varepsilon) \\
&\quad \leq \sum_{m=1}^{\infty} E|X_1| I(m^{\alpha}\varepsilon < |X_1| \leq (m+1)^{\alpha}\varepsilon) \sum_{n=1}^{m} n^{-1} \\
&\quad \leq C \sum_{m=1}^{\infty} (1 + \log m) E|X_1| I(m^{\alpha}\varepsilon < |X_1| \leq (m+1)^{\alpha}\varepsilon) \\
&\quad \leq CE|X_1| + C \sum_{m=2}^{\infty} \log m E|X_1| I(m^{\alpha}\varepsilon < |X_1| \leq (m+1)^{\alpha}\varepsilon) \\
&\quad \leq CE|X_1| + C/\alpha \sum_{m=2}^{\infty} E\{|X_1| \log(|X_1|/\varepsilon)\} I(m^{\alpha}\varepsilon < |X_1| \leq (m+1)^{\alpha}\varepsilon) \\
&\quad \leq C(1 + 1/\alpha \log(1/\varepsilon)) E|X_1| \\
&\qquad + C/\alpha \sum_{m=2}^{\infty} E\{|X_1| \log |X_1|\} I(m^{\alpha}\varepsilon < |X_1| \leq (m+1)^{\alpha}\varepsilon) \\
&\quad \leq CE\{|X_1| + |X_1| \log(1 + |X_1|)\} < \infty.
\end{aligned}$$

Secondly, by $E|X_1|^p < \infty$ and $\alpha p > 1$, we have

$$\begin{aligned}
n^{-\alpha} &\sum_{k=1}^{n} E|X_k| I(|X_k| > n^{\alpha}\delta/16\eta) \\
&\leq (\delta/16\eta)^{1-p} n^{-\alpha p} \sum_{k=1}^{n} E|X_k|^p I(|X_k| > n^{\alpha}\delta/16\eta) \\
&\leq Cn^{1-\alpha p} E|X_1|^p \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Finally, we state the conditions of Theorem 1.7 imply (2.13). If $p \geq 2$, from $E|X_1|^p < \infty$, we know $EX_1^2 < \infty$. By $\eta > \max\{1, \frac{\alpha p-1}{2\alpha-1}\}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \left(n^{-2\alpha} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leq n^{\alpha}\delta) \right)^{\eta} \\
&\qquad \leq \sum_{n=1}^{\infty} n^{\alpha p-2-(2\alpha-1)\eta} (EX_1^2)^{\eta} < \infty.
\end{aligned}$$

If $1 \le p < 2$, by $\alpha p > 1$ and $\eta > 1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2}\Big(n^{-2\alpha}\sum_{k=1}^{n} EX_k^2 I(|X_k| \le n^{\alpha}\delta)\Big)^{\eta} &= \sum_{n=1}^{\infty} n^{\alpha p-2-(2\alpha-1)\eta}\big(EX_1^2 I(|X_1| \le n^{\alpha}\delta)\big)^{\eta} \\ &\le \delta^{(2-p)\eta}\sum_{n=1}^{\infty} n^{-1-(\alpha p-1)(\eta-1)}(E|X_1|^p)^{\eta} < \infty.\end{aligned}$$

To sum up, we know that Corollary 2.12 extends and improves Theorem 1.7.

3. Proofs

To prove main results in this paper, we need the following lemmas.

Lemma 3.1 ([11]). *If random variables $\{X_n, n \ge 1\}$ are END, then $\{g_n(X_n), n \ge 1\}$ are still END, where $\{g_n(\cdot), n \ge 1\}$ are either all monotone increasing or all monotone decreasing.*

Lemma 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of END random variables with mean zero and $0 < B_n = \sum_{k=1}^{n} EX_k^2 < \infty$. If $S_n = \sum_{k=1}^{n} X_k$, then there exists a constant $M > 0$ such that*

$$P(|S_n| \ge x) \le P(\max_{1\le k\le n} |X_k| \ge y) + 2M\exp\Big(\frac{x}{y} - \frac{x}{y}\log\big(1 + \frac{xy}{B_n}\big)\Big)$$

for $\forall x > 0, y > 0$.

Remark 3.3. Wu and Guan ([14]) established a similar conclusion, in which the term $P(\max_{1\le k\le n} |X_k| \ge y)$ was magnified as $\sum_{k=1}^{n} P(|X_k| \ge y)$. Here we omit the details of the proof.

We first state the proof of Theorem 2.1.

Proof. Let $\varepsilon > 0$ be given. Without loss of generality, we may assume $0 < \varepsilon < \delta$. For any $1 \le k \le k_n, n \ge 1$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > \varepsilon\Big) \\ &\quad\le \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > \varepsilon,\ \bigcup_{k=1}^{k_n}\{|X_{nk}| > \delta\}\Big) \\ &\quad+ \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > \varepsilon,\ \bigcap_{k=1}^{k_n}\{|X_{nk}| \le \delta\}\Big)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \\
&\quad + \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk}I(|X_{nk}| \le \delta) - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > \varepsilon\Big) \\
&=: I_1 + I_2.
\end{aligned}
$$

By (2.1), we can get $I_1 < \infty$. To prove (2.3), it suffices to show $I_2 < \infty$. Let

$$
\begin{aligned}
Y_{nk} &= -\delta I(X_{nk} < -\delta) + X_{nk} I(|X_{nk}| \le \delta) + \delta I(X_{nk} > \delta), \\
Z_{nk} &= -\delta I(X_{nk} < -\delta) + \delta I(X_{nk} > \delta).
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk} - EY_{nk} - Z_{nk} + EZ_{nk}\big)\Big| > \varepsilon\Big) \\
&\le \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk} - EY_{nk}\big)\Big| > \varepsilon/2\Big) \\
&\quad + \sum_{n=1}^{\infty} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(Z_{nk} - EZ_{nk}\big)\Big| > \varepsilon/2\Big) \\
&=: I_3 + I_4.
\end{aligned}
$$

By Markov inequality and (2.1), we have

$$
I_4 \le C \sum_{n=1}^{\infty} c_n E\Big|\sum_{k=1}^{k_n}\big(Z_{nk} - EZ_{nk}\big)\Big| \ \le\ C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) < \infty.
$$

For any $\varepsilon > 0$, let

$$
\mathbf{N}_1 = \Big\{ n : \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon/6\eta) \ge \varepsilon/(24\delta\eta) \Big\}, \quad \mathbf{N}_2 = \mathbf{N} - \mathbf{N}_1.
$$

We know

$$
\begin{aligned}
&\sum_{n \in \mathbf{N}_1} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk} - EY_{nk}\big)\Big| > \varepsilon/2\Big) \\
&\qquad \le \sum_{n \in \mathbf{N}_1} c_n \le 24\delta\eta/\varepsilon \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon/6\eta) < \infty.
\end{aligned}
$$

Then it suffices to show that $\sum_{n\in\mathbf{N}_2} c_n P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk} - EY_{nk}\big)\Big| > \varepsilon/2\Big) < \infty$. Let $B_n = \sum_{k=1}^{k_n} E(Y_{nk} - EY_{nk})^2$. Take $x = \varepsilon/2$, $y = \varepsilon/2\eta$ and $\eta \ge 1$. By

Lemma 3.2, we have

$$\sum_{n\in\mathbf{N}_2} c_n P\Big(\Big|\sum_{k=1}^{k_n}(Y_{nk}-EY_{nk})\Big|>\varepsilon/2\Big)$$
$$\le \sum_{n\in\mathbf{N}_2} c_n P\big(\max_{1\le k\le k_n}|Y_{nk}-EY_{nk}|>\varepsilon/2\eta\big)+2C\sum_{n\in\mathbf{N}_2} c_n\Big(\frac{\mathrm{e}B_n}{B_n+\varepsilon^2/4\eta}\Big)^\eta$$
$$=: I_5+I_6.$$

For any $n\in\mathbf{N}_2$, by $\sum_{k=1}^{k_n}P(|X_{nk}|>\varepsilon/6\eta)<\varepsilon/(24\delta\eta)$ and $\varepsilon<\delta$, we can get

$$\max_{1\le k\le k_n}|EY_{nk}|\le \max_{1\le k\le k_n}E|Y_{nk}|$$
$$=\max_{1\le k\le k_n}\big\{E|X_{nk}|I(|X_{nk}|\le\varepsilon/6\eta)$$
$$+E|X_{nk}|I(\varepsilon/6\eta<|X_{nk}|\le\delta)+\delta P(|X_{nk}|>\delta)\big\}$$
$$\le \delta\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)+\delta\sum_{k=1}^{k_n}P(|X_{nk}|>\varepsilon/6\eta)+\varepsilon/6\eta \;\le\; \varepsilon/4\eta.$$

Therefore, for any $n\in\mathbf{N}_2$, we have

$$I_5\le\sum_{n\in\mathbf{N}_2}c_nP\big(\max_{1\le k\le k_n}|Y_{nk}|>\varepsilon/4\eta\big)\qquad(\text{ since }\ |Y_{nk}|\le|X_{nk}|\ )$$
$$\le\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}P\big(|X_{nk}|>\varepsilon/4\eta\big)<\infty.\qquad(\text{ by }\ (2.1)\ )$$

Note that for any $n\in\mathbf{N}_2$

$$\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)\le\sum_{k=1}^{k_n}P(|X_{nk}|>\varepsilon/6\eta)<\varepsilon/(24\delta\eta). \tag{3.1}$$

Note that $24\delta\eta/\varepsilon\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)<1$ if $n\in\mathbf{N}_2$. By C_r-inequality, (3.1), (2.1) and (2.2), we have

$$I_6\le C\sum_{n\in\mathbf{N}_2}c_n\big(B_n\big)^\eta\le C\sum_{n\in\mathbf{N}_2}c_n\Big(\sum_{k=1}^{k_n}EY_{nk}^2\Big)^\eta$$
$$\le C\sum_{n\in\mathbf{N}_2}c_n\Big(\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le\delta)\Big)^\eta+C\sum_{n\in\mathbf{N}_2}c_n\Big(\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)\Big)^\eta$$
$$\le C\sum_{n=1}^{\infty}c_n\Big(\sum_{k=1}^{k_n}EX_{nk}^2I(|X_{nk}|\le\delta)\Big)^\eta$$
$$+C\big(\varepsilon/(24\delta\eta)\big)^{\eta-1}\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)<\infty.$$

The proof is complete. ∎

Finally we state the proof of Theorem 2.8.

PROOF. Let $S_n = \sum_{k=1}^{k_n}\big(X_{nk} - EX_{nk}I(|X_{nk}| \le \delta)\big)$ and $\varepsilon > 0$ be given. Without loss of generality, we may assume $0 < \varepsilon < \delta$. We have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n E\big\{|S_n| - \varepsilon\big\}_+ &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\big(|S_n| - \varepsilon > t\big)\mathrm{d}t \\ &= \sum_{n=1}^{\infty} c_n \bigg\{ \int_0^{\delta} P\big(|S_n| > \varepsilon + t\big)\mathrm{d}t + \int_{\delta}^{\infty} P\big(|S_n| > \varepsilon + t\big)\mathrm{d}t \bigg\} \\ &\le \delta \sum_{n=1}^{\infty} c_n P\big(|S_n| > \varepsilon\big) + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\big(|S_n| > t\big)\mathrm{d}t \\ &=: I_7 + I_8.\end{aligned}$$

To prove (2.9), it suffices to show that $I_7 < \infty$ and $I_8 < \infty$. Noting that (2.7) implies (2.1), by Theorem 2.1, we have $I_7 < \infty$. Then we prove $I_8 < \infty$. Clearly

$$\begin{aligned}&P\big(|S_n| > t\big) \\ &= P\Big(|S_n| > t, \bigcup_{k=1}^{k_n}\{|X_{nk}| > t\}\Big) + P\Big(|S_n| > t, \bigcap_{k=1}^{k_n}\{|X_{nk}| \le t\}\Big) \\ &\le \sum_{k=1}^{k_n} P(|X_{nk}| > t) + P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk}I(|X_{nk}| \le t) - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > t\Big).\end{aligned}$$

Then we have

$$\begin{aligned}I_8 &\le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} P(|X_{nk}| > t)\mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk}I(|X_{nk}| \le t) - EX_{nk}I(|X_{nk}| \le \delta)\big)\Big| > t\Big)\mathrm{d}t \\ &=: I_9 + I_{10}.\end{aligned}$$

By (2.7), we have

$$I_9 \le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < \infty.$$

Then we prove $I_{10} < \infty$. Let

$$\begin{aligned}Y_{nk} &= -tI(X_{nk} < -t) + X_{nk}I(|X_{nk}| \le t) + tI(X_{nk} > t), \\ Z_{nk} &= -tI(X_{nk} < -t) + tI(X_{nk} > t),\end{aligned}$$

we have

$$
\begin{aligned}
&P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk}I(|X_{nk}|\le t)-EX_{nk}I(|X_{nk}|\le\delta)\big)\Big|>t\Big)\\
&=P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk}-EY_{nk}-Z_{nk}+EZ_{nk}+EX_{nk}I(\delta<|X_{nk}|\le t)\big)\Big|>t\Big)\\
&\le P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk}-EY_{nk}-Z_{nk}+EZ_{nk}\big)\Big|+\Big|\sum_{k=1}^{k_n}EX_{nk}I(\delta<|X_{nk}|\le t)\Big|>t\Big).
\end{aligned}
$$

From (2.8), we know

$$
\begin{aligned}
&\max_{t\ge\delta}t^{-1}\Big|\sum_{k=1}^{k_n}EX_{nk}I(\delta<|X_{nk}|\le t)\Big|\le\ \max_{t\ge\delta}t^{-1}\sum_{k=1}^{k_n}E|X_{nk}|I(\delta<|X_{nk}|\le t)\\
&\le\sum_{k=1}^{k_n}P(|X_{nk}|>\delta)\ \le\ \sum_{k=1}^{k_n}E|X_{nk}|I(|X_{nk}|>\delta)\to 0\quad\text{as}\ \ n\to\infty.
\end{aligned}
$$

Therefore, while n is sufficiently large,

$$
\Big|\sum_{k=1}^{k_n}EX_{nk}I(\delta<|X_{nk}|\le t)\Big|<t/2
$$

holds uniformly for $t\ge\delta$. Hence

$$
\begin{aligned}
&P\Big(\Big|\sum_{k=1}^{k_n}\big(X_{nk}I(|X_{nk}|\le t)-EX_{nk}I(|X_{nk}|\le\delta)\big)\Big|>t\Big)\\
&\qquad\le P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk}-EY_{nk}\big)-\sum_{k=1}^{k_n}\big(Z_{nk}-EZ_{nk}\big)\Big|>t/2\Big)\\
&\qquad\le P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk}-EY_{nk}\big)\Big|>t/4\Big)+P\Big(\Big|\sum_{k=1}^{k_n}\big(Z_{nk}-EZ_{nk}\big)\Big|>t/4\Big).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
I_{10}\le&\sum_{n=1}^{\infty}c_n\int_{\delta}^{\infty}P\Big(\Big|\sum_{k=1}^{k_n}\big(Z_{nk}-EZ_{nk}\big)\Big|>t/4\Big)\mathrm{d}t\\
&+\sum_{n=1}^{\infty}c_n\int_{\delta}^{\infty}P\Big(\Big|\sum_{k=1}^{k_n}\big(Y_{nk}-EY_{nk}\big)\Big|>t/4\Big)\mathrm{d}t\\
=:&\ I_{11}+I_{12}.
\end{aligned}
$$

For I_{11}, by Markov inequality and (2.7), we have

$$\begin{aligned}
I_{11} &\le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n}\int_{\delta}^{\infty} t^{-1}E|Z_{nk}|\mathrm{d}t \le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n}\int_{\delta}^{\infty} P(|X_{nk}|>t)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}|>\delta) < \infty.
\end{aligned}$$

Next we consider I_{12}. Let $B_n = \sum_{k=1}^{k_n} E(Y_{nk}-EY_{nk})^2$, $x = t/4$, $y = t/4\eta$ and $\eta > 1$. By Lemma 3.2, we have

$$\begin{aligned}
I_{12} &\le \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\big(\max_{1\le k\le k_n} |Y_{nk}-EY_{nk}| > t/4\eta\big)\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(\frac{B_n}{B_n + t^2/16\eta}\Big)^{\eta}\mathrm{d}t \\
&=: I_{13} + I_{14}.
\end{aligned}$$

From (2.8), we know that, while n is sufficiently large,

$$\sum_{k=1}^{k_n} P(|X_{nk}|>\delta/16\eta) \le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}|>\delta/16\eta) < 1/32\eta. \tag{3.2}$$

Hence, by (3.2), we have

$$\begin{aligned}
&\max_{t\ge\delta}\max_{1\le k\le k_n} t^{-1}|EY_{nk}| \le \max_{t\ge\delta}\max_{1\le k\le k_n} t^{-1}E|Y_{nk}| \\
&\quad\le \max_{t\ge\delta}\max_{1\le k\le k_n} \big\{t^{-1}E|X_{nk}|I(|X_{nk}|\le \delta/16\eta) \\
&\qquad + t^{-1}E|X_{nk}|I(\delta/16\eta < |X_{nk}| \le t) + P(|X_{nk}|>t)\big\} \\
&\quad\le \max_{t\ge\delta}\max_{1\le k\le k_n} \big\{t^{-1}\delta/16\eta + P(|X_{nk}|>\delta/16\eta) + P(|X_{nk}|>t)\big\} \\
&\quad\le 1/16\eta + \sum_{k=1}^{k_n} P(|X_{nk}|>\delta/16\eta) + \sum_{k=1}^{k_n} P(|X_{nk}|>\delta) \\
&\quad\le 1/16\eta + 2\sum_{k=1}^{k_n} P(|X_{nk}|>\delta/16\eta) \ < \ 1/8\eta.
\end{aligned}$$

Therefore, while n is sufficiently large, we know that $\max_{1\le k\le k_n}|EY_{nk}| < t/8\eta$ holds uniformly for $t\ge\delta$. Hence, by (2.7), we have

$$\begin{aligned}
I_{13} &\le \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\big(\max_{1\le k\le k_n}|Y_{nk}| > t/8\eta\big)\mathrm{d}t \quad (\text{ since } |Y_{nk}|\le|X_{nk}|\,)\\
&\le \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} P\big(\max_{1\le k\le k_n}|X_{nk}| > t/8\eta\big)\mathrm{d}t\\
&\le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n}\int_{\delta}^{\infty} P\big(|X_{nk}| > t/8\eta\big)\mathrm{d}t\\
&\le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I\big(|X_{nk}| > \delta/8\eta\big) < \infty.
\end{aligned}$$

Finally, we prove $I_{14}<\infty$. By C_r-inequality, we have

$$\begin{aligned}
I_{14} &\le C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \big(t^{-2}B_n\big)^{\eta}\mathrm{d}t \le C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(t^{-2}\sum_{k=1}^{k_n} EY_{nk}^2\Big)^{\eta}\mathrm{d}t\\
&= C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(t^{-2}\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}|\le\delta)\\
&\quad + t^{-2}\sum_{k=1}^{k_n} EX_{nk}^2 I(\delta<|X_{nk}|\le t) + \sum_{k=1}^{k_n} P(|X_{nk}|>t)\Big)^{\eta}\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(t^{-2}\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}|\le\delta)\Big)^{\eta}\mathrm{d}t\\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(t^{-1}\sum_{k=1}^{k_n} E|X_{nk}|I(\delta<|X_{nk}|\le t)\Big)^{\eta}\mathrm{d}t\\
&\quad + C\sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \Big(\sum_{k=1}^{k_n} P(|X_{nk}|>t)\Big)^{\eta}\mathrm{d}t \;=:\; I'_{14}+I''_{14}+I'''_{14}.
\end{aligned}$$

By $\eta>1$ and (2.2), we have

$$\begin{aligned}
I'_{14} &= C\sum_{n=1}^{\infty} c_n \Big(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}|\le\delta)\Big)^{\eta}\int_{\delta}^{\infty} t^{-2\eta}\mathrm{d}t\\
&\le C\sum_{n=1}^{\infty} c_n \Big(\sum_{k=1}^{k_n} EX_{nk}^2 I(|X_{nk}|\le\delta)\Big)^{\eta} < \infty.
\end{aligned}$$

From (2.8), while n is sufficiently large, we can get $\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < 1$. Hence, by $\eta > 1$ and (2.7), we have

$$I''_{14} \le C \sum_{n=1}^{\infty} c_n \Big(\sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) \Big)^{\eta} \int_{\delta}^{\infty} t^{-\eta} \mathrm{d}t$$
$$\le C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < \infty.$$

From (2.8), while n is sufficiently large, we know

$$\sum_{k=1}^{k_n} P(|X_{nk}| > t) \le \sum_{k=1}^{k_n} P(|X_{nk}| > \delta) \le \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < 1$$

holds uniformly for $t \ge \delta$. Hence, by (2.7), we have

$$I'''_{14} \le C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > t) \mathrm{d}t$$
$$\le C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E|X_{nk}|I(|X_{nk}| > \delta) < \infty.$$

The proof is complete. □

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References

[1] P. Y. Chen, T. C. Hu, X. Liu and A. Volodin, *On complete convergence for arrays of rowwise negatively associated random variables,* Theory Probab. Appl. **52** (2008), 323–328.

[2] Y. Q. Chen, A. Y. Chen and K. W. Ng, *The strong law of large numbers for extended negatively dependent random variables,* J. Appl. Probab. **47** (2010), 908–922.

[3] Y. S. Chow, *On the rate of moment convergence of sample sums and extremes,* Bull. Inst. Math. Acad. Sinica **16** (1988), 177–201.

[4] N. Ebrahimi and M. Ghosh, *Multivariate negative dependence,* Comm. Statist. ATheory Methods **10** (1981), 307–337.

[5] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers,* Proc. Nat. Acad. Sci. U. S. A. **33** (1947), 25–31.
[6] T. C. Hu, F. Móricz and R. L. Taylor, *Strong laws of large numbers for arrays of rowwise independent random variables,* Acta Math. Hungar. **54** (1989), 153162.
[7] T. C. Hu, D. Szynal and A. Volodin, *A note on complete convergence for arrays,* Statist. Probab. Lett. **38** (1998), 27–31.
[8] T. C. Hu and R. L. Taylor, *On the strong law for arrays and for the bootstrap mean and variance,* Int. J. Math. Math. Sci. **20** (1997), 375–382.
[9] T. C. Hu and A. Volodin, *Addendum to "A note on complete convergence for arrays",* Statist. Probab. Lett. **47** (2000), 209–211.
[10] K. Joag-Dev and F. Proschan, *Negative association of random variables with applications,* Ann. Statist. **11** (1983), 286–295.
[11] L. Liu, *Precise large deviations for dependent random variables with heavy tails,* Statist. Probab. Lett. **79** (2009), 1290–1298.
[12] D. H. Qiu, P. Y. Chen, R. A. Giuliano and A. Volodin, *On the complete convergence for arrays of rowwise extended negatively dependent random variables,* J. Korean Math. Soc. **50** (2013), 379–392.
[13] S. H. Sung, A. Volodin and T. C. Hu, *More on complete convergence for arrays,* Statist. Probab. Lett. **71** (2005), 303–311.
[14] Y. F. Wu and M. Guan, *Convergence properties of the partial sums for sequences of END random variables,* J. Korean Math. Soc. **49** (2012), 1097–1110.
[15] Y. F. Wu and D. J. Zhu, *Convergence properties of partial sums for arrays of rowwise negatively orthant dependent random variables,* J. Korean Statist. Soc. **39** (2010), 189–197.

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