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THE METRIC APPROXIMATION PROPERTY IN NON-ARCHIMEDEAN NORMED SPACES

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This paper is dedicated to the second author, who passed away on May 1, 2014.

Abstract. A normed space E over a rank 1 non-archimedean valued field K has the *metric approximation property* (MAP) if the identity on E can be approximated pointwise by finite rank operators of norm 1.

Characterizations and hereditary properties of the MAP are obtained. For Banach spaces E of countable type the following main result is derived: E has the MAP if and only if E is the orthogonal direct sum of finite-dimensional spaces (Theorem 4.9). Examples of the MAP are also given. Among them, Example 3.3 provides a solution to the following problem, posed by the first author in [8, 4.5]. Does every Banach space of countable type over K have the MAP?

1. Introduction

The study of Grothendieck's approximation in non-archimedean Banach spaces was initiated in [8]. In the present paper we derive new results leading to improvements of [8] (see e.g. Theorem 3.2). Also, we give (Example 3.3) a negative answer to the following problem, posed in [8, 4.5]. Does every Banach space of countable type over K have the MAP? As an application of Example 3.3 we additionally prove that the problem raised in [9, p. 95] has an affirmative answer, even for locally convex spaces of countable type.

In Section 5 we compare the results given in this paper with their classical versions, for Banach spaces over the real or complex field. This comparison, together with the one carried out in [8, Section 6], reveals sharp and interesting contrasts between the classical MAP and its non-archimedean counterpart.

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For explanation of terminology and symbols, see Section 2.

We recall the following fundamental notion from [8], where, for convenience (see Theorem 4.4 and Corollary 4.5), we include also non-complete spaces.

DEFINITION 1.1. Let $\lambda \in \mathbb{R}$, $\lambda \geq 1$. A normed space E over K has the *λ-bounded approximation property* (λ-BAP) if for each $\varepsilon > 0$ and each finite set $X \subset E$ there is a finite rank operator $T : E \to E$ with $\|T\| \leq \lambda$ and $\|T(x) - x\| \leq \varepsilon$ for all $x \in X$. E is said to have the *metric approximation property* (MAP) if it has the 1-BAP.

2. Preliminaries

By "classical theory" we mean functional analysis over $\mathbb{R}$ or $\mathbb{C}$.

Throughout $K := (K, |\,.\,|)$ is a non-archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\,.\,| : K \to [0, \infty)$.

For basics on valued fields, see [1, 10, 11, 13]. For background on non-archimedean functional analysis, see [9, 12, 13].

From now on in this paper E, F are non-archimedean normed spaces (over K).

For convenience we recall the following.

For a set $X \subset E$, $[X]$ denotes the linear hull of X. If $(D_i)_{i \in I}$ is a family of subspaces of E, then the linear hull of $\bigcup_i D_i$ is denoted by $\sum_i D_i$.

By $L(E, F)$ we mean the K-vector space of all continuous linear maps (or operators) $T : E \to F$ with the norm $T \mapsto \|T\| := \min\{M \geq 0 : \|T(x)\| \leq M\,\|x\| \text{ for all } x \in E\}$. If F is a Banach space then so is $L(E, F)$. If $T \in L(E, F)$ and D is a subspace of E, by $T|D$ we denote the restriction of T to D. We write $E' := L(E, K)$, $L(E) := L(E, E)$. By I_E we mean the identity $E \to E$. Also, $FR(E, F) := \{T \in L(E, F) : \dim T(E) < \infty\}$, is the space of the *finite rank operators* $E \to F$. We put $FR(E) := FR(E, E)$.

E is called *pseudoreflexive* ([13, p. 60]) if the canonical operator $j_E : E \to E''$ defined by $j_E(x)(f) := f(x)$ ($x \in E$, $f \in E'$) is isometric, i.e. if (for $E \neq \{0\}$) $\|x\| = \sup\{|f(x)|\,/\|f\| : f \in E', f \neq 0\}$ for all $x \in E$. If the valuation of K is dense, E is pseudoreflexive if and only if E is *norm-polar*, i.e. $\|x\| = \sup\{|f(x)| : f \in E', \|f\| \leq 1\}$ for all $x \in E$. If K is spherically complete every space E is pseudoreflexive ([13, 4.35]). But if K is not spherically complete the space ℓ^∞/c_0 is not pseudoreflexive; in fact, $(\ell^\infty/c_0)' = \{0\}$ ([13, 4.3]).

Two subspaces D_1, D_2 of E are called *orthogonal* (notation $D_1 \perp D_2$) if $\|d_1 + d_2\| = \max(\|d_1\|, \|d_2\|)$ for all $d_1 \in D_1$, $d_2 \in D_2$. If, in addition, $D_1 + D_2 = E$ we say that D_1 and D_2 are each other's *orthocomplement.* For $x, y \in E$ we sometimes write $x \perp y$ in place of $Kx \perp Ky$ and say that x and y are *orthogonal.* By [13, 3.2] this holds if and only if $\|\mu\,x + y\| \geq \|\mu\,x\|$ (or $\geq \|y\|$) for all $\mu \in K$.

An operator $P \in L(E)$ is called a *projection* if $P^2 = P$, an *orthoprojection* if, in addition, $\mathrm{Ker}P \perp P(E)$ (which is equivalent to $\|P\| \leq 1$).

A system $(D_i)_{i\in I}$ of subspaces of E is called (*an*) *orthogonal* (*system*) if $D_i \perp \sum_{j\neq i} D_j$ for all $i \in I$. Analogously, a collection $(x_i)_{i\in I}$ of vectors in E is called *orthogonal* if $(Kx_i)_{i\in I}$ is orthogonal. If, in addition, $\|x_i\| = 1$ for all $i \in I$, it is called *orthonormal*. An orthogonal system $(x_i)_{i\in I} \subset E \setminus \{0\}$ is called an *orthogonal base* (*of* E) if each $x \in E$ has a (unique) expansion $x = \sum_i \lambda_i x_i$, where $\lambda_i \in K$ for all i. In the same spirit we have the notion of an *orthonormal base*. For example, in the Banach space c_0 of all null sequences in K (with the maximum norm), the unit vectors form an orthonormal base.

We also will need the following extension of the notion of orthogonality. Let $0 < t \leq 1$. A system $(x_i)_{i\in I}$ of vectors in E is called *t-orthogonal* if $\| \sum_{j\in J} \lambda_j x_j\| \geq t \max_{j\in J} \|\lambda_j x_j\|$ for all finite sets $J \subset I$ and $\lambda_j \in K$ $(j \in J)$. A t-orthogonal system $(x_i)_{i\in I} \subset E \setminus \{0\}$ is called a *t-orthogonal base* (*of* E) if each $x \in E$ has a (unique) expansion $x = \sum_i \lambda_i x_i$, where $\lambda_i \in K$ for all i. Notice that 1-orthogonal systems and bases are nothing but orthogonal systems and bases.

E is said to be *of countable type* if there is a countable set in E whose linear hull is dense. We quote the following result.

Theorem 2.1. ([9, 2.3.7, 2.3.25]) *A space of countable type has, for each $t \in (0, 1)$, a t-orthogonal base. It has an orthogonal base if K is spherically complete.*

Let $(E_i)_{i\in I}$ be a system of normed spaces. Its *orthogonal direct sum* $\bigoplus_i E_i$ is the space of all $(x_i)_{i\in I} \in \prod_i E_i$ for which $\lim_i \|x_i\| = 0$, normed by $(x_i)_{i\in I} \longmapsto \max_i \|x_i\|$. The subspace of all $(x_i)_{i\in I} \in \bigoplus_i E_i$ for which $\{i \in I : x_i \neq 0\}$ is finite, is called the *algebraic orthogonal direct sum* $\bigoplus_i^a E_i$. It is a dense subspace of $\bigoplus_i E_i$. If each E_i is a Banach space then so is $\bigoplus_i E_i$.

In classical Grothendieck's approximation theory the notion of the finite-dimensional decomposition property plays a role (see e.g. [2, 6.1]). In our theory we modify this concept as follows.

A Banach space E has the *finite-dimensional decomposition property* (FDDP) if it is the orthogonal direct sum of a system of finite-dimensional spaces. If K is spherically complete, every finite-dimensional space has an orthogonal base (Theorem 2.1), so E has the FDDP if and only if E has an orthogonal base. However, if K is not spherically complete there exist various kinds of finite-dimensional spaces without orthogonal base (see [6]); for these K the class of Banach spaces with the FDDP can be viewed as a natural proper generalization of the class of Banach spaces with an orthogonal base.

3. Examples and characterizations of the MAP

It was shown in [8] that a large amount of non-archimedean normed spaces have the λ-BAP ($\lambda > 1$) and the MAP. In fact, the following result holds.

Theorem 3.1 ([8, 3.3]).

(i) *Every norm-polar space E has the* λ-BAP *for all* $\lambda > 1$.

(ii) *Suppose either K is spherically complete or E has an orthogonal base. Then E has the* MAP.

For examples of Banach spaces (e.g. valued field extensions; spaces of continuous (analytic, differentiable) functions) with an orthogonal base and hence with the MAP (Theorem 3.1.(ii)) see [9, Section 2.5].

We now extend Theorem 3.1.(i) by proving that pseudoreflexivity is equivalent to having the λ-BAP for all $\lambda > 1$.

Theorem 3.2. *E is pseudoreflexive if and only if E has* λ-BAP *for all* $\lambda > 1$.

Proof. The "only if" follows directly from Theorem 3.1. To prove the "if", let $x \in E$, $x \neq 0$, let $0 < t < 1$; we construct an $f \in E' \setminus \{0\}$ with $|f(x)| \geq t\,\|f\|\,\|x\|$. By assumption there is a $T \in FR(E)$ with $\|T\| \leq t^{-1/2}$ and $\|T(x) - x\| < \|x\|$. Then $\|T(x)\| = \|x\|$. Now $T(E)$, being finite-dimensional, is pseudoreflexive ([13, 3.16(iv)]), so there is a $g \in (T(E))'$, $g \neq 0$ such that $|g(T(x))| \geq t^{1/2}\,\|g\|\,\|T(x)\|$. Then $f := g \circ T$ is in $E' \setminus \{0\}$ and $\|f\| \leq \|g\|\,\|T\| \leq t^{-1/2}\,\|g\|$. Thus, $|f(x)| = |g(T(x))| \geq t\,\|f\|\,\|T(x)\| = t\,\|f\|\,\|x\|$, and we are done. □

In the real and complex theory, the λ-BAP for all $\lambda > 1$ implies the MAP ([8, 6.III]), but not in our theory. In fact, it is shown in [8, 4.1] that, for non-spherically complete K, the Banach space ℓ^∞ of all bounded sequences in K (with the supremum norm), has the λ-BAP for all $\lambda > 1$ but does not have the MAP.

Now ℓ^∞ is not of countable type ([9, 2.5.15]) and it was asked in [8, 4.5], whether spaces of countable type automatically had the MAP. The next example gives a negative answer to this question.

Example 3.3. There exists a reflexive Banach space E of countable type that does not have the MAP.

Proof. Let K be not-spherically complete, let $K^\vee$ be its spherical completion. Then $K^\vee$ is in particular a K-Banach space.

We first prove that no pair of non-zero vectors in $K^\vee$ is an orthogonal system. In fact, let $x, y \in K^\vee \setminus \{0\}$. To show that Kx is not orthogonal to Ky we may assume that $|x| = |y| = 1$ (as $|K^\vee| = |K|$). Now the residue class fields of $K^\vee$ and K are isomorphic, so $\left|x\,y^{-1} - \mu\right| < 1$ for some $\mu \in K$. It follows that $|x - \mu\,y| < 1$, i.e. Kx is not orthogonal to Ky.

Next we show that no infinite-dimensional subspace of $K^\vee$ has the MAP. In fact, suppose there is an infinite-dimensional subspace G of $K^\vee$ with the MAP; we derive a contradiction. Let $x \in G \setminus \{0\}$. There is a $T \in FR(G)$ with $\|T\| \leq 1$ and $\|T(x) - x\| < \|x\|$. Then $\|T(x)\| = \|x\|$. For each $z \in \operatorname{Ker} T \setminus \{0\}$

we have $\|x - z\| \geq \|T(x) - T(z)\| = \|T(x)\| = \|x\|$, so $Kx \perp \operatorname{Ker} T$, a contradiction with the assertion proved above.

Finally, $K^\vee/K$ is spherically complete ([13, 4.2]), so $(K^\vee/K)' = \{0\}$ ([13, 4.3]). In particular, $K^\vee$ is not of countable type as a K-normed space ([13, 3.16]), so certainly admits closed infinite-dimensional subspaces E of countable type (which are reflexive, [13, 4.18]), finishing the construction. □

APPLICATION. In [9, p. 95] the following problem was posed: *Does there exist an absolutely convex edged set C in some locally convex space G over K such that its closure $\overline{C}$ is not edged?*

We shall use Example 3.3 to provide an affirmative answer when K is not spherically complete and G is even of countable type.

Let ρ be the topology of pointwise convergence on $L(E)$, i.e. the Hausdorff locally convex topology on $L(E)$ defined by the family of seminorms $\{p_x : x \in E\}$, where $p_x(T) := \|T(x)\|$, $x \in E$, $T \in L(E)$. As usual, by *pointwise convergence* in $L(E)$ we mean ρ-convergence.

Then we have the following:

Let K be not spherically complete. Let E be a normed space of countable type without the MAP (*e.g. Example* 3.3). *Then $G := (L(E), \rho)$ is a locally convex space of countable type and $C := \{T \in FR(E) : \|T\| \leq 1\}$ is an absolutely convex edged set in $L(E)$ such that $\overline{C}^\rho$ is not edged.*

In fact, it suffices to prove that $(L(E), \rho)$ is of countable type; the rest follows from [8, 5.2]. Observe that the map $(L(E), \rho) \to E^E$, $T \mapsto (T(x))_{x \in E}$ is a linear homeomorphism onto the image. Since E is of countable type then, by the stability properties for locally convex spaces of countable type ([9, 4.2.13]), we get that $(L(E), \rho)$ is of countable type.

We conclude this section by proving a stronger-looking, yet equivalent formulation of the MAP (Theorem 3.6). To this end we give two preparatory lemmas.

LEMMA 3.4 (Extension lemma). *Let E be pseudoreflexive, let D be a finite-dimensional subspace and let $0 < \varepsilon_1 < \varepsilon_2$. Then each $A \in L(D, E)$ with $\|A\| \leq \varepsilon_1$ can be extended to a $B \in FR(E)$ for which $\|B\| \leq \varepsilon_2$ and $B(E) = A(D)$.*

PROOF. By pseudoreflexivity, there is a projection P of E onto D with $\|P\| \leq \varepsilon_1^{-1} \varepsilon_2$ (apply [13, 4.35] in the case when K is spherically complete, and [9, 4.4.6] for non-spherically complete K). One verifies directly that $B := A \circ P$ satisfies the requirements. □

LEMMA 3.5 (Taking $\varepsilon = 0$ in the definition of the MAP). *Let E have the* MAP. *Then for each finite set $X \subset E$ there is a $T \in FR(E)$ with $\|T\| \leq 1$ and $T(x) = x$ for all $x \in X$.*

PROOF. We may assume that $X \neq \emptyset$. The space $[X]$ is finite-dimensional, so it has (Theorem 2.1) a 1/2-orthogonal base $x_1, \ldots, x_n$. By scalar multiplication we can arrange that $\|x_i\| \geq 1$ for each i. By assumption there is a $T_1 \in FR(E)$ with $\|T_1\| \leq 1$ and $\|T_1(x_i) - x_i\| \leq 1/4$ for each i. Now put $A := (I_E - T_1) \,|\, [X]$. We next prove that $\|A\| \leq 1/2$. In fact, let $x \in [X]$, $x = \lambda_1 x_1 + \ldots + \lambda_n x_n$, where $\lambda_i \in K$. Then

$$\begin{aligned} \|A(x)\| &\leq \max_i |\lambda_i| \; \|A(x_i)\| = \max_i |\lambda_i| \; \|x_i - T_1(x_i)\| \leq \frac{1}{4} \max_i |\lambda_i| \leq \\ & \frac{1}{4} \max_i \|\lambda_i x_i\| \leq \frac{1}{4} \, 2 \, \|\sum_i \lambda_i x_i\| = 1/2 \, \|x\|, \end{aligned}$$

and we are done.

E is pseudoreflexive (Theorem 3.2), so by the extension lemma 3.4, A can be extended to a $B \in FR(E)$ with $\|B\| \leq 1$. Now put $T := T_1 + B$. We see that $T \in FR(E)$ and $T(x) = x$ for all $x \in X$. Finally, observe that $\|T\| \leq \max(\|T_1\|, \|B\|) \leq 1$, which completes the proof. ◻

Now we arrive at the key result of this section.

THEOREM 3.6. *Let E have the* MAP. *Then every finite-dimensional subspace is contained in a finite-dimensional orthocomplemented subspace.*

PROOF. Throughout the proof we fix a finite-dimensional subspace $D \neq \{0\}$ and prove that D is contained in a finite-dimensional orthocomplemented subspace, using a few steps.

(I) For a finite-dimensional subspace F of E and a $T \in FR(E)$ we say that (F, T) is a *proper pair* if: (i) $T(E) \subset F$, (ii) $\|T\| = 1$, (iii) $T(x) = x$ for all $x \in D$. (Notice that $D \subset F$).

Straightforward computation shows:

If (F, T) is a proper pair then so is $(T(F), T^2)$.

(II) A proper pair (F, T) is called *minimal* if there do not exist proper pairs (F_1, T_1) with $\dim F_1 < \dim F$.

By taking in Lemma 3.5 for X a base of D, we obtain the existence of proper pairs. Then obviously:

There exist minimal proper pairs.

From now on in this proof we fix a minimal proper pair (F, T); we will prove that F is orthocomplemented (completing the proof of Theorem 3.6) as follows:

(III) $T(F) = F$. PROOF. We have $T(E) \subset F$, so certainly $T(F) \subset F$. Now by (I), $(T(F), T^2)$ is a proper pair, so by minimality $\dim T(F) \geq \dim F$, and we get (III).

(IV) *$T|F$ is an isometry.* PROOF. Suppose not; we derive a contradiction. There is an $y \in F$ with $\|T(y)\| \neq \|y\|$. But, as $\|T\| = 1$, we must have $\|T(y)\| < \|y\|$. We first prove that $Ky \perp D$. For that it suffices to see that $\|y - x\| \geq \|x\|$ for all $x \in D$. This is clear if $\|y\| \neq \|x\|$, so suppose $\|y\| = \|x\|$ $(> \|T(y)\|)$. Then $\|y - x\| \geq \|T(y) - T(x)\| = \|T(y) - x\| = \|x\|$, and we are done.

Next, consider the map $A : D+Ky \mapsto KT(y)$ given by $A(x+\lambda y) = \lambda T(y)$ ($x \in D$, $\lambda \in K$). Then from orthogonality (i.e. $\|x+\lambda y\| \geq \|\lambda y\|$) one arrives easily at $\|A\| = \|T(y)\|/\|y\| < 1$. Since E is pseudoreflexive (Theorem 3.2), by the extension lemma (Lemma 3.4) we can extend A to a $B \in FR(E)$ with $\|B\| < 1$ and $B(E) = A(D + Ky) = KT(y)$.

Now define $U := T-B$. From (i) $U(E) \subset T(E)+B(E) \subset F+KT(y) \subset F$, (ii) $\|U\| = \|T-B\| = \max(\|T\|, \|B\|) = 1$, (iii) $U(x) = T(x)-B(x) = T(x) = x$ for all $x \in D$, we infer that (F, U) is a proper pair. Then, by (I), $(U(F), U^2)$ is also a proper pair, so by minimality, $\dim U(F) \geq \dim F$. On the other hand, $U(y) = T(y) - B(y) = T(y) - A(y) = 0$, so by finite-dimension considerations we have $\dim U(F) < \dim F$, a contradiction.

(V) *F is orthocomplemented.* Proof. (i) $\operatorname{Ker} T \perp F$: let $x \in \operatorname{Ker} T$, $y \in F$. To show $\|x-y\| \geq \|y\|$ we may assume $\|x\| = \|y\|$. Then, using (IV), we obtain $\|x-y\| \geq \|T(x)-T(y)\| = \|T(y)\| = \|y\|$.

(ii) $E = \operatorname{Ker} T + F$: let $z \in E$. Then $T(z) \in F = T(F)$ by (III), so there is an $y \in F$ with $T(z) = T(y)$. Therefore, $z = (z-y)+y \in \operatorname{Ker} T + F$. ∎

Corollary 3.7. *The following are equivalent.*

- (α) *E has the* MAP.
- (β) *Each finite-dimensional subspace is contained in a finite-dimensional orthocomplemented subspace.*
- (γ) *There is a net $(P_i)_{i\in I}$ of finite rank orthoprojections $E \to E$ such that, for each $x \in E$, $P_i(x) = x$ for large i.*
- (δ) *There is a net $(P_i)_{i\in I}$ of finite rank operators $E \to E$ with $\|P_i\| \leq 1$ for all i, such that $P_i \to I_E$ pointwise.*

Proof. (α) $\Longrightarrow$ (β) is Theorem 3.6, (γ) $\Longrightarrow$ (δ) is obvious. For (β) $\Longrightarrow$ (γ), let I be the set of all finite-dimensional subspaces of E, directed by inclusion. By (β) we can choose, for every $D \in I$, an orthoprojection P_D of E onto some finite-dimensional subspace $F \supset D$. Clearly $(P_D)_{D\in I}$ satisfies (γ).

(δ) $\Longrightarrow$ (α). Let $\varepsilon > 0$ and $X \subset E$ be finite. By (δ) there is a $j \in I$ such that $\|P_j(x) - x\| \leq \varepsilon$ for all $x \in X$, so E has the MAP. ∎

4. Hereditary aspects of the MAP

Theorem 4.1. *The* MAP *is stable for orthocomplemented subpaces.*

Proof. Let D be an orthocomplemented subspace of a normed space E with the MAP. Let $\varepsilon > 0$ and $X \subset D$ be finite. By assumption there is a $T_1 \in FR(E)$ with $\|T_1\| \leq 1$ and $\|T_1(x) - x\| \leq \varepsilon$ for all $x \in X$. Now let P be an orthoprojection of E onto D and put $T := (P \circ T_1)|D$. Then clearly $T \in FR(D)$ and $\|T\| \leq 1$. Also, for each $x \in X$, $\|T(x) - x\| = \|(P \circ T_1)(x) - P(x)\| \leq \|P\|\,\|T_1(x) - x\| \leq \varepsilon$. Hence D has the MAP. ∎

To describe the stability of the MAP for dense subspaces we need a general lemma.

LEMMA 4.2. *Let D be a finite-dimensional subspace of E, let F be a dense subspace of E. Then, for each $\varepsilon > 0$, there is a $T \in L(D, F)$ with $\|x - T(x)\| \leq \varepsilon \|x\|$ for all $x \in D$.*

PROOF. Let $x_1, \ldots, x_n$ be a 1/2-orthogonal base of D (Theorem 2.1). By density there are $y_1, \ldots, y_n \in F$ such that $\|x_i - y_i\| \leq (\varepsilon/2) \|x_i\|$ for all i. Define $T : D \to F$ by $T(x_i) := y_i$ ($i \in \{1, \ldots, n\}$) and linearity. Then $T \in L(D, F)$. To get the conclusion, let $x = \sum_i \lambda_i x_i \in D$. Then we have $\|x - T(x)\| = \|\sum_i \lambda_i (x_i - y_i)\| \leq \max_i |\lambda_i| \, \|x_i - y_i\| \leq (\varepsilon/2) \max_i |\lambda_i| \, \|x_i\|$, which by 1/2-orthogonality, is $\leq \varepsilon \|x\|$, completing the proof. ∎

THEOREM 4.3. (Stability of the MAP for dense subspaces and closures) *Let E_1 be a dense subspace of a normed space E_2. Then E_1 has the* MAP *if and only if E_2 has the* MAP. *In particular, the completion of a normed space with the* MAP *has the* MAP.

PROOF. (i) Suppose E_1 has the MAP. To prove that E_2 has the MAP, let $\varepsilon > 0$ and $X := \{x_1, \ldots, x_n\} \subset E_2$; we construct a $T_2 \in FR(E_2)$ with $\|T_2\| \leq 1$ and $\|T_2(x_i) - x_i\| \leq \varepsilon$ for all i. By density there are $y_1, \ldots, y_n \in E_1$ such that $\|x_i - y_i\| \leq \varepsilon$ for each i. By assumption there is a $T_1 \in FR(E_1)$ with $\|T_1\| \leq 1$ and $\|T_1(y_i) - y_i\| \leq \varepsilon$ for each i. T_1 extends uniquely to a $T_2 \in L(E_2)$. As $T_1(E_1)$ is finite-dimensional, hence complete, we have $T_2(E_2) \subset \overline{T_1(E_1)} = T_1(E_1)$, so that $T_2 \in FR(E_2)$. Clearly $\|T_2\| \leq 1$. Finally, $\|T_2(x_i) - x_i\| = \|(T_2(x_i) - T_2(y_i)) + (T_2(y_i) - y_i) + (y_i - x_i)\| \leq \max(\|x_i - y_i\|, \|T_1(y_i) - y_i\|, \|y_i - x_i\|) \leq \varepsilon$ for each i, showing that E_2 has the MAP.

(ii) Suppose E_2 has the MAP. To prove that E_1 has the MAP, let $\varepsilon > 0$ and $\emptyset \neq X \subset E_1$ be finite. By assumption there is a $T_2 \in FR(E_2)$ with $\|T_2\| \leq 1$ and $\|T_2(x) - x\| \leq \varepsilon$ for all $x \in X$. Now let $\delta \in (0, 1)$ with $\delta \max\{\|T_2(x)\| : x \in X\} \leq \varepsilon$. By Lemma 4.2, there is a $S \in L(T_2(E_2), E_1)$ such that $\|z - S(z)\| \leq \delta \|z\|$ for all $z \in T_2(E_2)$. Finally, put $T_1 := (S \circ T_2)|E_1$. Then $T_1 \in FR(E_1)$ and $\|T_1\| \leq 1$ (as S is an isometry). Also, for each $x \in X$,

$$\|T_1(x) - x\| = \|((S \circ T_2)(x) - T_2(x)) + (T_2(x) - x)\| \leq$$
$$\max(\|S(T_2(x)) - T_2(x)\|, \|T_2(x) - x\|) \leq \max(\delta \|T_2(x)\|, \varepsilon) \leq \varepsilon,$$

proving that E_1 has the MAP. ∎

As a next step we consider algebraic orthogonal direct sums.

THEOREM 4.4. *Let $(E_i)_{i \in I}$ be a collection of normed spaces. Then its algebraic orthogonal direct sum $\bigoplus_i^a E_i$ has the* MAP *if and only if each E_i has the* MAP.

PROOF. Each E_i is orthocomplemented in $E := \bigoplus_i^a E_i$ (we identify each E_i with its image under the natural injection $E_i \to E$). Thus, if E has the MAP then so has each E_i (Theorem 4.1).

Now assume that each E_i has the MAP. For each i, let P_i be the canonical orthoprojection $E \to E_i$, $x = (x_i)_{i\in I} \mapsto x_i$. Let $\varepsilon > 0$ and $X \subset E$ be finite. There is a finite set $J \subset I$ for which $X \subset \sum_{j\in J} P_j(E_j)$. Then $X \subset \sum_{j\in J} P_j(X)$. By assumption there is, for each $j \in J$, a $T_j \in FR(E_j)$ with $\|T_j\| \le 1$ and $\|T_j(z) - z\| \le \varepsilon$ for all $z \in P_j(X)$. Now define $T : E \to E$ by the formula $(T(x))_i = T_i(P_i(x))$ if $i \in J$; $(T(x))_i = 0$ otherwise. Then $T \in FR(E)$, $\|T\| \le 1$ and, for each $x \in X$, we have $\|T(x)-x\| = \max_{i\in J} \|(T(x))_i - x_i\| = \max_{i\in J} \|T_i(x_i) - x_i\| \le \varepsilon$, and we are done. ∎

The step towards orthogonal direct sums is now easy:

COROLLARY 4.5. *Let $(E_i)_{i\in I}$ be a collection of normed spaces. Then $\bigoplus_i E_i$ has the* MAP *if and only if each E_i has the* MAP*.*

PROOF. Combine Theorem 4.3 and Theorem 4.4. ∎

As finite-dimensional spaces trivially have the MAP, the next result follows directly.

COROLLARY 4.6. *A Banach space with the* FDDP *has the* MAP*.*

The converse of Corollary 4.6 does not hold.

EXAMPLE 4.7. There exists a Banach space E having the MAP but not the FDDP.

PROOF. In [7, 3.6], for non-spherically complete K, a closed subspace E of ℓ^∞ was constructed that has no orthogonal base but whose finite-dimensional subspaces are orthocomplemented. Then certainly E has the MAP (e.g. Corollary 3.7) and it is also easily seen that finite-dimensional subspaces of E have orthogonal bases. Then E, having no orthogonal base, cannot have the FDDP. ∎

However, for spaces of countable type we do have a converse.

LEMMA 4.8. *Let E be a normed space of countable type having the* MAP*. Then there exists an orthogonal sequence $(D_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces such that $\sum_n D_n$ is dense in E.*

PROOF. Let $x_1, x_2, \ldots \in E$ be such that $[x_1, x_2, \ldots]$ is dense in E. We will construct inductively an orthogonal sequence $D_1, D_2, \ldots$ of finite-dimensional subspaces, and subspaces $H_1, H_2, \ldots$ such that, for each n, (i) $[x_1, \ldots, x_n] \subset D_1 + \ldots + D_n$, (ii) H_n is an orthocomplement of $D_1 + \ldots + D_n$. (This will prove the lemma). To this end, we first apply Corollary 3.7 to conclude that Kx_1 is contained in an orthocomplemented finite-dimensional subspace, say, D_1. Let

H_1 be an orthocomplement of D_1. For the step $n \to n+1$, suppose we have constructed $D_1, \ldots, D_n$ and $H_1, \ldots, H_n$ in the above fashion. Then x_{n+1} has a unique decomposition $x_{n+1} = y_n + h_n$, where $y_n \in D_1 + \ldots + D_n$, $h_n \in H_n$. Now by Theorem 4.1 H_n has the MAP, so h_n lies in a finite-dimensional subspace D_{n+1} of H_n that is orthocomplemented in H_n. Let H_{n+1} be such an orthocomplement. Then H_{n+1} is trivially an orthocomplement of $D_1 + \ldots + D_{n+1}$ in E and $x_{n+1} = y_n + h_n \in D_1 + \ldots + D_n + D_{n+1}$, which proves the step $n \to n+1$. □

We can now formulate the following result.

THEOREM 4.9. *A Banach space of countable type has the* MAP *if and only if it has the* FDDP.

REMARK 4.10. Throughout this remark, let K be not spherically complete. Let E have the MAP and let D be a subspace of E.

1. (Subspaces) Does D have the MAP?

We know that the answer is yes if D is finite-dimensional, or orthocomplemented (Theorem 4.1) or dense (Theorem 4.3), but the general question remains open. Notice that, for Banach spaces E of countable type, the above question is by Theorem 4.9 equivalent to:

Let E have the FDDP. Do subspaces have the FDDP?

Observe that the related problem: Let E have an orthogonal base. Do subspaces have an orthogonal base?, is solved affirmatively ([9, 2.3.22]).

2. (Quotients) Let D be closed. Does E/D have the MAP?

The answer is "no" in general: it suffices to take a Banach space F without the MAP and observe that, thanks to [9, 2.5.6], F is a quotient of some Banach space with an orthogonal base (which has the MAP by Theorem 3.1(ii)). If we choose for F a space of countable type (see Example 3.3) we can even conclude by [9, 2.3.28] that F is a quotient of c_0.

However, quotients of E by finite-dimensional subspaces have the MAP, as we show in the next result.

THEOREM 4.11. *Let E have the* MAP*, let D be a finite-dimensional subspace. Then E/D has the* MAP*.*

PROOF. Let M be a finite-dimensional subspace of E/D. We construct (Lemma 3.5) a $S \in FR(E/D)$ with $\|S\| \leq 1$ and $S(z) = z$ for all $z \in M$. Let $\pi : E \to E/D$ be the canonical quotient map. Then $\pi^{-1}(M)$ is finite-dimensional, so by assumption and Lemma 3.5, there is a $T \in FR(E)$ with $\|T\| \leq 1$ and $T(x) = x$ for all $x \in \pi^{-1}(M)$, in particular, $T(x) = x$ for all $x \in D$, as $D \subset \pi^{-1}(M)$. Now, let $S : E/D \to E/D$ be the map given by $S(\pi(x)) = \pi(T(x))$ $(x \in E)$. Then S is a well-defined finite rank operator with $\|S\| \leq 1$. Also, for each $z \in M$ there is an $x \in \pi^{-1}(M)$ for which $\pi(x) = z$. Thus, $S(z) = S(\pi(x)) = \pi(T(x)) = \pi(x) = z$, so S meets the requirements. □

We conclude this section with the following for tensor products.

THEOREM 4.12. *Let E, F have the* MAP*. Then the tensor product $E \otimes F$ and its completion $E \hat{\otimes}_\pi F$ have the* MAP*.*

PROOF. By Theorem 4.3 we only need to consider $E \otimes F$. Let $\varepsilon > 0$ and $\{0\} \neq Z \subset E \otimes F$ be finite. There are non-empty finite sets $\{0\} \neq X \subset E$, $\{0\} \neq Y \subset F$ such that every $z \in Z$ can be written as a finite sum, $z = \sum_i x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$. Let $M_X := \max\{\|x\| : x \in X\}$, $M_Y := \max\{\|y\| : y \in Y\}$. By assumption there exist $T \in FR(E)$, $S \in FR(F)$ with $\|T\| \leq 1$, $\|S\| \leq 1$ and

$$\|T(x) - x\| \leq \tfrac{\varepsilon}{M_Y} \text{ for all } x \in X, \quad \|S(y) - y\| \leq \tfrac{\varepsilon}{M_X} \text{ for all } y \in Y.$$

Then $T \otimes S \in FR(E \otimes F)$ and $\|T \otimes S\| \leq 1$. Now, for each $x \in X$, $y \in Y$,

$$\|(T \otimes S)(x \otimes y) - x \otimes y\| = \|T(x) \otimes S(y) - T(x) \otimes y + T(x) \otimes y - x \otimes y\|$$

$$\leq \max(\|T(x)\| \, \|S(y) - y\|, \|T(x) - x\| \, \|y\|) \leq \max(M_X \frac{\varepsilon}{M_X}, \frac{\varepsilon}{M_Y} M_Y) = \varepsilon.$$

Then it is easily seen that $\|(T \otimes S)(z) - z\| \leq \varepsilon$ for all $z \in Z$, and we are done. ∎

PROBLEM Let $E \otimes F$ have the MAP, and suppose $E \neq \{0\}$, $F \neq \{0\}$. Does it follow that E and F have the MAP?

5. Comparison with the classical case

Finally we compare the results given in this paper with their classical (or archimedean) counterparts, for Banach spaces over $\mathbb{R}$ or $\mathbb{C}$.

Since every space over a spherically complete K has the MAP (Theorem 3.1(ii)), in this section we assume that E is a non-archimedean Banach space over a non-spherically complete K. Also, we assume that $\mathcal{E}$ is a Banach space over $\mathbb{R}$ or $\mathbb{C}$.

The notion of the MAP for $\mathcal{E}$ is just a translation of the one given in Definition 1.1.

I. The classical approximation theory was initiated in the Grothendieck's memoir [5], where among other things, he studied the MAP. At that moment all known classical Banach spaces had the MAP. He conjectured that every space $\mathcal{E}$ had this property. It was not until 1973, when Enflo proved in [3] that the conjecture of Grothendieck was false. He gave an *example of a separable reflexive space $\mathcal{E}$ without the* MAP. For more examples of classical Banach spaces with and without the MAP see e.g. [2] and its references on the subject.

In the non-archimedean setting, the space E of Example 3.3 plays the role of the classical example given by Enflo: it is a reflexive Banach space of countable type for which the non-archimedean version of the conjecture of Grothendieck is false.

II. In the classical case one verifies:

(i) ([2, 3.10]) c_0 *has the* MAP *in every equivalent norm.*
(ii) ([4, VI.3]) *There exists a closed subspace* $\mathcal{E}$ *of* c_0 *such that* $\mathcal{E}$ *has the* 8-BAP *but fails the* MAP.

The non-archimedean counterparts of these classical results are false.

To see that (i) is false, let E be the Banach space of countable type, without the MAP, constructed in Example 3.3. Then E is linearly homeomorphic to c_0 ([9, 2.3.9]), i.e. there is an equivalent norm $\|\,.\,\|$ on c_0, such that $(c_0, \|\,.\,\|)$ is isometrically isomorphic to E, so $(c_0, \|\,.\,\|)$ does not have the MAP.

Falsity of (ii) follows from the fact that every closed subspace of c_0 has an orthogonal base (Remark 4.10.1), so it has the MAP (Theorem 3.1(ii)).

III. In the archimedean theory we have ([2, 3.6]): *If* $\mathcal{E}$ *is a separable dual space such that*

(*) *for every* $\varepsilon > 0$ *and every compact set* $X \subset \mathcal{E}$ *there exists a* $T \in FR(\mathcal{E})$ *with* $\|T(x) - x\| \leq \varepsilon$ *for all* $x \in X$,

then $\mathcal{E}$ *has the* MAP.

The non-archimedean counterpart of this classical result is false.

Indeed, let E be the reflexive (hence dual) space of countable type of Example 3.3. We know that E does not have the MAP. Let us see that E satisfies (*), and we are done. E is reflexive, hence pseudoreflexive, i.e. E has the λ-BAP for all $\lambda > 1$ (Theorem 3.2). Now, as the finite sets in Definition 1.1 can be replaced by compact sets ([8, 3.2]), we derive that E satisfies (*).

IV. Let us discuss the situation in III when we consider the approximation properties (in the archimedean and in the non-archimedean case) obtained from the MAP and (*), by imposing the operator T appearing in their definitions to be compact, instead of finite rank. Let us call CMAP and (C*), respectively, the approximation properties obtained after these replacements. Then, an open problem in the classical theory ([2, 8.7]) is the following:

$\mathcal{E}$ *is a separable dual space with property* (C*) $\Longrightarrow$ $\mathcal{E}$ *has the* CMAP*?*

In the non-archimedean case the answer to this problem is NO.

Indeed, it was proved in [8, 3.2] that E has the MAP if and only if E has the CMAP. Then the non-archimedean result given in III provides the desired negative answer.

V. It is well-known (see e.g. [5, I.5.39]) that $\mathcal{E}$ has the MAP if and only if it has an approximating net, i.e. a net $(P_i)_{i\in I}$ of finite rank operators $\mathcal{E} \to \mathcal{E}$ with $\|P_i\| \leq 1$ for all i, such that $P_i \to I_{\mathcal{E}}$ pointwise (this result is the archimedean version of $(\alpha) \Longleftrightarrow (\delta)$ of Corollary 3.7). But *there exist spaces* $\mathcal{E}$ *with the* MAP *and*:

(i) *having no approximating nets consisting of finite rank projections*,
(ii) *having no finite-dimensional decompositions* ([2, 6.1]).

In fact, it is proved in [2, 5.2] (and the comments before it) that there is a separable reflexive space $\mathcal{E}$ with property (*), hence with the MAP (see III), for which there are not bounded nets of finite rank projections $\mathcal{E} \to \mathcal{E}$ converging pointwise to $I_{\mathcal{E}}$. The non-existence of such bounded nets implies (i) and (ii).

The assertions (i) and (ii) above show, respectively, that *the classical counterparts of* $(\alpha) \Longrightarrow (\gamma)$ *of Corollary* 3.7 *and of the "only if" of Theorem* 4.9 *are false.*

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