THE GRAPH OF EQUIVALENCE CLASSES OF ZERO-DIVISORS OF A POSET

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Abstract. In this paper, we give the definition of the graph of equivalence classes of zero-divisors of a poset \( P \). We prove that if \([a]\) has maximal degree in \( V(\Gamma_E(P)) \), then \( \text{ann}(a) \) is maximal in \( \text{Anih}(P) \). Also, we give some other properties of the graph \( \Gamma_E(P) \). Moreover, we characterize the cut vertices of \( \Gamma_E(P) \) and study the cliques of these graphs.

1. Introduction

The concept of zero-divisor graph was first introduced by Beck in [7] to investigate the interplay between ring-theoretic properties and graph-theoretic properties. The concept of zero-divisor graph has also been extended to many algebraic structures such as rings, semigroups, semirings (see [4–11,16]). Haláš and Jukl ([13]) introduced the zero-divisor graph of a poset. Since then, many authors continued to study the zero-divisor graphs of posets, see [1,15,16,20].

Let \( R \) be a ring and \( r, s \in R \). Define \( r \sim s \) if and only if \( \text{ann}(r) = \text{ann}(s) \). Write \( [r] = \{ s \in R \mid r \sim s \} \) and \( R_E = \{ [r] \mid r \in R \} \). Denote by \( \Gamma_E(R) \) the graph of equivalence classes of zero-divisors of \( R \). The set of vertices \( V(\Gamma_E(R)) \) is \( R_E \setminus \{ [0], [1] \} \) and two vertices are adjacent if and only if \( [r][s] = [0] \), if and only if \( rs = 0 \). Motivated by ideas in paper [18], Spiroff and Wickham ([19]) studied the graph of equivalence classes of zero-divisors of a commutative Noetherian ring. Anderson and LaGrange ([2]) continued to study these graphs. In [2], the graph is called the compressed zero-divisor graph. In this paper, we will extend the graph of equivalence classes of zero-divisors to a poset \( P \) and study the properties of these graphs.

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The paper is constructed as follows: In Section 2, we give some relevant definitions and notations of graphs and posets. In Section 3, we give the definition of the graph of equivalence classes of zero-divisors of a poset $P$ and study the basic properties of these graphs. In Section 4, we investigate the cut vertices and clique number of the graph $\Gamma_E(P)$.

Throughout, all posets $P$ will be a poset with 0 and 1 and all graphs will be simple graphs.

2. Preliminaries

Let $(P, \leq)$ be a partially ordered set (abbreviated as a poset) and $X \subseteq P$. Let $L(X) = \{ y \in P \mid y \leq x \text{ for all } x \in X \}$ denote the lower cone of $X$. Dually, let $U(X) = \{ y \in P \mid y \geq x \text{ for all } x \in X \}$ denote the upper cone of $X$. If $X = \{x_1, \ldots, x_n\}$, we shall write $L(x_1, \ldots, x_n)$ or $U(x_1, \ldots, x_n)$ instead of $L(X)$ or $U(X)$.

Let $P$ be a poset and $\emptyset \neq I \subseteq P$. Then $I$ is called an ideal of $P$ if $x \in I$ and $y \leq x$, then $y \in I$. A proper ideal $I$ of $P$ is called prime if for all $x, y \in P, L(x, y) \subseteq I$ implies $x \in I$ or $y \in I$.

For $x \in P$, the set $\text{ann}(x) = \{ y \in P | L(x, y) = \{0\} \}$ is called the annihilator of $x$.

For $x \in P$, $x$ is called a zero-divisor of $P$ if there exists $0 \neq y \in P$ such that $L(x, y) = \{0\}$. Denote by $Z(P)$ the zero-divisors of $P$ and write $Z(P)^\times = Z(P) \setminus \{0\}$.

The zero-divisor graph of $P$, denoted by $\Gamma(P)$, is as follows: the set of vertices is $V(\Gamma(P)) = Z(P)^\times$ and distinct vertices $x$ and $y$ are adjacent if and only if $L(x, y) = \{0\}$ ([1]).

Let $G$ be a graph. For $k \geq 2$, a graph is called a $k$-partite graph if the vertices of the graph are partitioned into $k$ disjoint sets such that there is no edge between two vertices in the same set. A 2-partite graph is usually called a bipartite graph. It is well known that a graph is bipartite if and only if it contains no cycle of odd length. A complete bipartite graph is a bipartite graph such that every vertex in one set is connected to every vertex in the other set. The complete graph $K_n$ is a graph with $n$ vertices in which each vertex is connected to each of the others. The diameter of a graph $G$ is the largest distance between two vertices in $G$, denoted by $\text{diam}(G)$. A clique of a graph $G$ is a subset of its vertices such that there exists an edge between each pair of vertices in the subset. The clique number $\text{cl}(G)$ of a graph $G$ is the number of vertices in a maximum clique in $G$.

3. Basic properties of the graph $\Gamma_E(P)$

In this section, we will define the graph of equivalence classes of zero-divisors of a poset $P$ and investigate the properties of this graph.
An element $0 \neq p$ of a poset $P$ is called an atom if there exists no element $x \in P$ such that $0 < x < p$. The set of atoms of $P$ is denoted by $\text{Atom}(P)$. If $p \in P$, set $\text{atom}(p) = \{ \alpha \in \text{Atom}(P) \mid \alpha \leq p \}$.

For any elements $a, b \in P$, define a relation on $P$ by $a \sim b$ if and only if $\text{ann}(a) = \text{ann}(b)$. Then $\sim$ is an equivalence relation on $P$.

For any $a \in P$, let $[a] = \{ r \in P \mid r \sim a \}$. It is easy to get the following statements.

**Lemma 3.1.** Let $P$ be a poset. Then:
1) $\text{ann}(1) = \{0\}$ and $\text{ann}(0) = P$. Moreover, if $a \neq 0$, then $[a] \neq [0]$.
2) $[a] \subseteq Z(P)$, for all $a \in P \setminus \{0, 1\}$.

Let $\overline{P} = \{ [a] \mid a \in P \}$. Define a partial order relation on $\overline{P}$ by $[a] \leq [b]$ if and only if $\text{ann}(b) \subseteq \text{ann}(a)$. It is clear that this partial order relation is well-defined and $(\overline{P}, \leq)$ is a poset. $[0]$ is the least element in $\overline{P}$ and $[1]$ is the largest element in $\overline{P}$. Without causing confusion, we will let $\leq$ represent the partial order relation on both $P$ and $\overline{P}$ in the following.

Now, we give the definition of the graph of equivalence classes of zero-divisors of a poset $P$.

**Definition 3.2.** The graph of equivalence classes of zero-divisors of a poset $P$ is the graph $\Gamma_E(P) = \Gamma(\overline{P})$ whose vertices are the elements in $\overline{P}\setminus\{[0], [1]\}$, such that two distinct vertices $[a]$ and $[b]$ are adjacent if and only if $L([a], [b]) = \{0\}$.

Let $P$ be a poset as below. Then one can check that $\text{diam}(\Gamma(P)) = 2$ while $\text{diam}(\Gamma_E(P)) = 1$. The properties of the graph $\Gamma_E(P)$ and the graph $\Gamma(P)$ are different.

![Diagram](image-url)

**Lemma 3.3.** Let $P$ be a poset and $a, b \in P$. Then
1) If $a \leq b$, then $\text{ann}(b) \subseteq \text{ann}(a)$ and $[a] \leq [b]$ in $\overline{P}$.
2) If $[a] \neq [1]$, $[b] \neq [1]$, and $L([a], [b]) = \{0\}$, then $L(a, b) = \{0\}$.
3) If $L([a], [b]) = \{0\}$, then $L([a], [b]) = \{0\}$.

**Proof.** 1) Obvious.

2) Suppose $x \in L(a, b)$. Then $x \leq a$ and $x \leq b$. It follows that $\text{ann}(a) \subseteq \text{ann}(x)$ and $\text{ann}(b) \subseteq \text{ann}(x)$. Hence, $[x] \leq [a]$ and $[x] \leq [b]$. Therefore, we have $[x] = [0]$, and so $x = 0$ by 1) in Lemma 3.1. Hence, $L(a, b) = \{0\}$. 
3) Suppose \([c] \in L([a], [b]). Then \([c] \leq [a] and [c] \leq [b]. Hence, \text{ann}(a) \subseteq \text{ann}(c) and \text{ann}(b) \subseteq \text{ann}(c). By \(L(a, b) = \{0\}\), we have \(b \in \text{ann}(a) \subseteq \text{ann}(c), and so \(L(b, c) = \{0\}\). Thus \(c \in \text{ann}(b) \subseteq \text{ann}(c)\). It follows that \(c = 0\), and so \([c] = [0]\). Therefore, \(L([a], [b]) = \{[0]\}\).

**Proposition 3.4.** Let \(P\) be a poset. If \([x] = [x_1]\) and \([y] = [y_1]\), then \(L(x, y) = \{0\}\) if and only if \(L(x_1, y_1) = \{0\}\).

**Proof.** \(\Rightarrow\): Suppose \([x] = [x_1]\) and \([y] = [y_1]\). Then \(\text{ann}(x) = \text{ann}(x_1)\) and \(\text{ann}(y) = \text{ann}(y_1)\). Since \(L(x, y) = \{0\}\), we have \(y \in \text{ann}(x) = \text{ann}(x_1)\), and hence \(L(x_1, y_1) = \{0\}\). That is, \(x_1 \in \text{ann}(y) = \text{ann}(y_1)\). Thus \(L(x_1, y_1) = \{0\}\).

\(\Leftarrow\): The proof is similar to that of “\(\Rightarrow\).”

**Remark 3.5.** By Definition 3.2, Lemma 3.3, and Proposition 3.4, we know that the graph \(\Gamma_E(P)\) is isomorphic to a subgraph of \(\Gamma(P)\).

Let \(a\) be a vertex of a graph \(G\). The degree of \(a\) is the number of edge ends at \(a\), denoted by \(\text{deg}(a)\). Let \(N(a)\) be the set of vertices which are adjacent to \(a\), then \(|N(a)| = \text{deg}(a)\). For any two vertices \(u\) and \(v\) of a graph \(G\), define \(u \approx v\) if and only if \(N(u) = N(v)\). Let \(\Gamma(P)\) be the zero-divisor graph of a poset \(P\) and \(u, v \in P\). Note that \(N(u) = \text{ann}(u) \setminus \{0\}\). Then \(u \approx v\) if and only if \(\text{ann}(u) = \text{ann}(v)\), if and only if \([u] = [v]\). Let \(\bar{u} = \{r \in G \mid r \approx u\}\) and \(G/\approx = \{u \mid u \in G\}\). Then \(G/\approx\) becomes a graph in the natural way with \([u]\) and \([v]\) are adjacent in \(G/\approx\) if and only if \(u\) and \(v\) are adjacent in \(G\). Using Lemma 3.3, we get the following analog of [2, Theorem 2.4].

**Theorem 3.6.** Let \(P\) be a poset. Then \(\Gamma_E(P) \cong \Gamma(P)/\approx\).

**Proof.** Suppose \(a \in P\). Define a map \(\phi : \Gamma_E(P) \to \Gamma(P)/\approx\) by \([a] \mapsto \bar{a}\). By the above comments, the map \(\phi\) is well-defined. One can easily check that \(\phi\) is also bijective. If \([a] - [b]\) is an edge in \(\Gamma_E(P)\), then \(L([a], [b]) = \{0\}\), and hence \(L(a, b) = \{0\}\) by Lemma 3.3. Therefore, \(\bar{a} - \bar{b}\) is an edge in \(\Gamma(P)/\approx\).

Conversely, if \(\bar{a} - \bar{b}\) is an edge in \(\Gamma(P)/\approx\), then \(a\) and \(b\) are adjacent in \(\Gamma(P)\), and hence \(L(a, b) = \{0\}\). By Lemma 3.3, we get \(L([a], [b]) = \{0\}\). Therefore, \([a] - [b]\) is an edge in \(\Gamma_E(P)\).

The diameter of the graph \(\Gamma_E(R)\) is less or equal to 3, where \(R\) is a commutative ring with identity (Proposition 1.4 in [19]). The following statement gives a similar result for the graph \(\Gamma_E(P)\), where \(P\) is a poset.

**Theorem 3.7.** Let \(P\) be a poset. Then \(\Gamma_E(P)\) satisfies the following conditions.

1) \(\Gamma_E(P)\) is connected.
2) \(\text{diam}(\Gamma_E(P)) \leq 3\).
Proof. By the definition of $\Gamma_E(P)$, we know that it is also a zero-divisor graph of the poset $\overline{P}$. Using [1, Theorem 3.3], we have that $\Gamma_E(P)$ is connected and $\text{diam}(\Gamma_E(P)) \leq 3$.

In [19], Spiroff and Wickham investigated infinite graphs of equivalence classes of zero-divisors of a ring $R$ and associated primes of $R$, where $R$ is a commutative Noetherian ring with identity. We shall study the corresponding problems in poset settings.

Proposition 3.8. Let $P$ be a poset and $a, b \in P$. Then $\text{ann}([a]) = \text{ann}([b])$ if and only if $[a] = [b]$.

Proof. $\Rightarrow$: Let $a, b \in P$ and $\text{ann}([a]) = \text{ann}([b])$. Suppose $z \in \text{ann}(a)$. By Lemma 3.3, we have $[z] \in \text{ann}([a]) = \text{ann}([b])$, and so $L([z], [b]) = \{[0]\}$. Using Lemma 3.3 again, we have $L(z, b) = \{0\}$. This proves that $z \in \text{ann}(b)$, and hence $\text{ann}(a) \subseteq \text{ann}(b)$. Similarly, one can prove that $\text{ann}(b) \subseteq \text{ann}(a)$. Therefore, $[a] = [b]$.

$\Leftarrow$: Obvious.

A poset $P$ is atomic if for all $0 < b \in P$, there exists an atom $a \in P$ such that $0 < a \leq b$. Let $P$ be a poset. Let $\text{Anih}(P) = \{\text{ann}(a) \mid a \in P, \text{ann}(a) \neq P\}$. If $a \in P$ and $\text{ann}(a)$ is maximal among $\text{Anih}(P)$, then $\text{ann}(a)$ is a prime ideal of $P$ ([13], Lemma 2.2).

Proposition 3.9. Let $P$ be a poset. If $a$ is an atom of $P$, then $\text{ann}(a)$ is maximal in $\text{Anih}(P)$. Moreover, $\text{ann}(a)$ is prime. Conversely, if $P$ is atomic and $\text{ann}(b)$ is maximal in $\text{Anih}(P)$, then there exists an atom $a$ such that $\text{ann}(a) = \text{ann}(b)$.

Proof. Suppose there exists an element $0 \neq c \in P$ with $\text{ann}(a) \subset \text{ann}(c)$. Then there exists $x \in \text{ann}(c) \setminus \text{ann}(a)$, that is, $L(x, c) = \{0\}$, but $L(x, a) \neq \{0\}$. Assume $0 \neq z \in L(x, a)$. Since $a$ is an atom, we must have $z = a$. Hence $a \leq x$. Thus $L(a, c) = \{0\}$, and so $c \in \text{ann}(a)$. Therefore $c \in \text{ann}(c)$. This is impossible. Thus $\text{ann}(a)$ is maximal. By Lemma 2.2 in [13], it follows that $\text{ann}(a)$ is prime.

Conversely, suppose $\text{ann}(b)$ is maximal in $\text{Anih}(P)$ and $a$ is an atom such that $0 < a \leq b$. We have $\text{ann}(b) \subseteq \text{ann}(a)$, and so $\text{ann}(b) = \text{ann}(a)$ by the maximality of $\text{ann}(b)$.

The following proposition is similar to Proposition 2.2 in [19].

Proposition 3.10. Let $P$ be a poset and $|\text{Atom}(P)| < \infty$. Then $|V(\Gamma_E(P))| = \infty$ if and only if there exists $x \in P$ such that $\text{ann}(x)$ is maximal in $\text{Anih}(P)$ and $\text{deg}(x) = \infty$.

Proof. $\Rightarrow$: Suppose $\text{Atom}(P) = \{a_1, a_2, \ldots, a_n\}$. By Proposition 3.9, we know that $\text{ann}(a_1), \text{ann}(a_2), \ldots, \text{ann}(a_n)$ are maximal in $\text{Anih}(P)$. If
deg([a_1]) < \infty$, there exist infinitely many vertices [x] such that $L([x], [a_1]) \neq \{[0]\}$. By Lemma 3.3, we have $L(x, a_1) \neq \{[0]\}$. If $[v] \neq [x]$ and $L([x], [v]) = \{[0]\}$, then $L(x, v) = \{0\} \subseteq \text{ann}(a_1)$. Since $\text{ann}(a_1)$ is prime and $x \notin \text{ann}(a_1)$, we have $v \in \text{ann}(a_1)$, and so $[v]$ is adjacent to $[a_1]$. If there exist infinitely many distinct vertices $[v]$ which are adjacent to $[a_1]$, then $\text{deg}([a_1]) = \infty$. This is a contradiction. Hence, the set of $[v]$’s is finite. Note that $[x]$ is adjacent to $[v]$ and the set of $[x]$’s is infinite. We have $\text{deg}([v]) = \infty$ for some $v$. If $\text{ann}(v)$ is maximal, we get the desired result. If $\text{ann}(v) \subseteq \text{ann}(a_i)$ for some $i \neq 1$, we have $\text{deg}([a_i]) = \infty$, and we also get the desired result.

$\Leftarrow$: It is obvious.

**Theorem 3.11.** Let $P$ be a poset and $a \in P$. If $[a]$ has maximal degree in $V(\Gamma_E(P))$, then $\text{ann}(a)$ is maximal in $\text{Ani}(P)$.

**Proof.** Suppose $\text{ann}(a) \subseteq \text{ann}(b)$. It is easy to show $N([a]) \subseteq N([b])$. By the maximality of the degree of $[a]$, we have $N([a]) = N([b])$. If there exists $z \in \text{ann}(b) \setminus \text{ann}(a)$, by Lemma 3.3 we get $[z]$ is adjacent to $[b]$, but not adjacent to $[a]$. That is, $[z] \in N([b])$, but $[z] \notin N([a])$. This is a contradiction. Therefore, $\text{ann}(a) = \text{ann}(b)$.

The following example proves that the converse of the preceding theorem is not true.

**Example 3.12.** Let $P_A$ be the poset in Figure (A). Then $\text{ann}(b)$ is maximal in $\text{Ani}(P_A)$. One can check that $\text{deg}([b]) = 2$ and $\text{deg}([a]) = 3$. Hence, the degree of $[b]$ is not maximal.

\[\begin{array}{c}
\text{(A)} & \text{($\Gamma_E(P_A)$)} & \text{(B)} & \text{($\Gamma_E(P_B)$)} \\
\text{0} & \text{0} & \text{1} & \text{1} \\
a & \text{[a]} & a_1 & \text{[a_1]} \\
b & \text{[b]} & b & \text{[b]} \\
c & \text{[c]} & c & \text{[c]} \\
d & \text{[d]} & \text{[e]} & \text{[y]} \\
d \rightarrow a \rightarrow c \\
d \rightarrow b \rightarrow c \\
d \rightarrow a \rightarrow b \rightarrow c
\end{array}\]

4. **Cut vertices, cliques of the graph $\Gamma_E(P)$**

In this section, we will give a characterization of the cut vertices of the graph $\Gamma_E(P)$ and also study the cliques of these graphs.

Let $G$ be a graph. A vertex $a$ is called a cut vertex of $G$ if the removal of $a$ along with edges through $a$ leads to more components than $G$. That is, a vertex $a$ is called a cut vertex if there exist distinct vertices $b$ and $c$ such that $a$ is in every $b \rightarrow c$ path, where both $b$ and $c$ are different from $a$. Axtell et al. ([6]) studied cut vertices in zero-divisor graphs of commutative rings with identity and proved that if $x$ is a cut vertex of the graph $\Gamma(R)$, then
the annihilator of $x$ is properly maximal (see [6, Proposition 2.7]). In the following, we investigate cut vertices in the graph $\Gamma_E(P)$.

**Proposition 4.1.** Let $P$ be a poset. If $[a]$ is a cut vertex in $\Gamma_E(P)$, then $[a]$ is an atom in $\mathcal{T}$.

**Proof.** Suppose $[x] - [a]$ is an edge in $\Gamma_E(P)$ and $[0] \neq [b] < [a]$. Then $[x] - [b]$ is also an edge in $\Gamma_E(P)$. Using this fact, one can prove that if $[a]$ is not an atom in $\mathcal{T}$, then $[a]$ is not a cut vertex in $\Gamma_E(P)$.

Let $P$ be a poset and $0 \neq x, 0 \neq y \in P$. By Lemma 3.3, $[x] - [y]$ is an edge in $\Gamma_E(P)$ if and only if $x - y$ is an edge in $\Gamma(P)$. Hence, we have the following lemma.

**Lemma 4.2.** Let $P$ be a poset and $a \in P$. If $a$ is a cut vertex in $\Gamma(P)$, then $[a]$ is also a cut vertex in $\Gamma_E(P)$.

The following example shows that the converse of Lemma 4.2 is not true.

**Example 4.3.** Let $P_B$ be the poset in Figure (B). In $\Gamma_E(P_B)$, $[a_1] = [a_2]$ is a cut vertex, since $[b] - [a_1] - [y]$ is the only path from $[b]$ to $[y]$. While, both $b - a_1 - y$ and $b - a_2 - y$ are paths from $b$ to $y$ in $\Gamma(P_B)$. Hence, $a_1$ is not a cut vertex.

**Proposition 4.4.** Let $P$ be a poset and $a \in P$. If $[a]$ is a cut vertex in $\Gamma_E(P)$, then $[a] \cup \{0\}$ is an ideal of $P$.

**Proof.** Suppose $b \in [a]$ and $y < b$. We have to show that $y \in [a]$. Since $y < b$, we have that $\text{ann}(b)$ is contained in $\text{ann}(y)$. So $N([a]) = N([b])$ is contained in $N([y])$. On the other hand, since $[a]$ is a cut vertex, there exists no vertex $[x]$ distinct from $[a]$ with $N([a])$ containing $N([x])$. Hence, $[y] = [a]$.

Let $P$ be a poset. For $x, y \in P$, if $x$ and $y$ are incomparable, we denote by $y||x$. For $a \in \text{Atom}(P)$, we define $\tilde{U}(\text{Atom}(P) \setminus \{a\}) = \{y \in P \mid y \parallel a \text{ and } \forall b \in \text{Atom}(P), \text{ if } b \neq a, \text{ then } y \geq b\}$.

**Proposition 4.5.** Let $P$ be a poset and $a \in P$. Then $a$ is an atom in $P$ if and only if $[a]$ is an atom in $\mathcal{T}$ and $a$ is a minimal element in $[a]$.

**Proof.** $\Rightarrow$: Suppose $0 \neq [b] \in \mathcal{T}$ and $[b] \leq [a]$. Then we have $\text{ann}(a) \subseteq \text{ann}(b)$. By Proposition 3.9, $\text{ann}(a)$ is maximal in $\text{Anih}(P)$. So we have $\text{ann}(a) = \text{ann}(b)$. That is, $[a] = [b]$. Thus $[a]$ is an atom in $\mathcal{T}$. Obviously, $a$ is a minimal element in $[a]$.

$\Leftarrow$: Suppose $0 \neq b \in P$ such that $b \leq a$. We have $\text{ann}(a) \subseteq \text{ann}(b)$, and so $[b] \leq [a]$. Since $[a]$ is an atom in $\mathcal{T}$, this proves that $[b] = [a]$ or $[b] = [0]$. If $[b] = [0]$, then $b = 0$. This is a contradiction. Therefore, we have $[b] = [a]$. Since $a$ is the minimal element in $[a]$, we have $b = a$, and so $a$ is also an atom in $P$. 

\[\]
Using Proposition 4.5, we have the following theorem characterizing the cut vertices of $\Gamma_E(P)$.

**Theorem 4.6.** Let $P$ be a poset. If $[a] \in \text{Atom}(\overline{P})$ and $a$ is a minimal element in $[a]$, then $[a]$ is a cut vertex in $\Gamma_E(P)$ if and only if $\overline{U}(\text{Atom}(P) \setminus \{a\}) \neq \emptyset$.

**Proof.** $\Rightarrow$: Without loss of generality, let $[x] − [a] − [y]$ be a path of shortest length from $[x]$ to $[y]$. By Lemma 3.3, we have that $x − a − y$ is a path in $\Gamma(P)$. This concludes that $x||a$ and $y||a$. If $\overline{U}(\text{Atom}(P) \setminus \{a\}) = \emptyset$, then we have $u, v \in \text{Atom}(P)$ with $x||u$ and $y||v$. If $u \neq v$, then $x − u − v − y$ is a path in $\Gamma(P)$. Using Lemma 3.3 again, we have that $[x] − [u] − [v] − [y]$ is a path in $\Gamma_E(P)$. If $u = v$, then $[x] − [u] − [y]$ is a path in $\Gamma_E(P)$. In either case, we have a contradiction.

$\Leftarrow$: If $x \in \overline{U}(\text{Atom}(P) \setminus \{a\})$, then $[a]$ is the unique vertex which is adjacent to $[x]$. This proves that $[a]$ is a cut vertex.

In paper [12], Estaji and Khashyarmanesh proved that the clique number of the graph $\Gamma(L)$ is equal to the number of atoms in $L$, where $\Gamma(L)$ is the zero-divisor graph of a lattice $L$ (Theorem 5.13). The following theorem shows that the clique number of the graph $\Gamma_E(P)$ is also equal to the number of atoms in $P$.

**Theorem 4.7.** Let $P$ be a poset. Then $\text{cl}(\Gamma_E(P)) = |\text{Atom}(P)|$.

**Proof.** By Proposition 4.5, we have $|\text{Atom}(P)| = |\text{Atom}(\overline{P})|$. Since any two atoms in $\overline{P}$ are adjacent, we have $\text{cl}(\Gamma_E(P)) \geq |\text{Atom}(P)|$. Suppose $|\text{cl}(\Gamma_E(P))| > |\text{Atom}(P)|$. Let $\text{cl}(\Gamma_E(P)) = m$ and $|\text{Atom}(P)| = n$. Then $\Gamma_E(P)$ has a complete subgraph with vertices $\{p_1, p_2, \ldots, p_m\}$. Since $p_i$ and $p_j$ are adjacent in $\Gamma_E(P)$, then atom($p_i$) \cap atom($p_j$) = $\emptyset$, for all $i \neq j$. This is impossible, since $m > n$. Hence, $\text{cl}(\Gamma_E(P)) = |\text{Atom}(P)|$.

Let $G$ be a graph and $a, b \in V(G)$. Two vertices $a$ and $b$ are called complements in $G$ if $a$ is connected to $b$, and no vertex in $G$ is connected to both $a$ and $b$, denoted by $a \perp b$. We say that a graph $G$ is complemented if each vertex in $G$ has a complement. The set of all complements in $G$ induces a subgraph of $G$, denoted by $G^c$. It is easy to see that $G$ is complemented if and only if $G = G^c$. Complements were studied for the zero-divisor graph $\Gamma(R)$ in [3] and for $\Gamma_E(R)$ in [2]. The next result is the analog of [2, Theorem 4.3].

**Proposition 4.8.** Let $P$ be a poset. Then the following statements are equivalent.

1) $\Gamma_E(P) = \Gamma_E(P)^c$.
2) $\Gamma_E(P)$ is complemented.
3) $\Gamma(P)$ is complemented.
Proposition 4.9. Let $P$ be a poset and $\text{Atom}(P) = \{a_1, a_2, \ldots, a_n\}$. Then

1) $\Gamma(P)$ is an $n$-partite graph.
2) $\Gamma_E(P)$ is an $n$-partite graph.

Proof. 1) Define $V_i = \{x \mid x \geq a_i\}$ and if $j < i$, there exists no $a_j$ such that $x \geq a_j$.

Then $V_1, \ldots, V_n$ are disjoint sets and $P \setminus \{0\} = \bigcup_{i=1}^{n} V_i$. Suppose $x, y \in V_i$, for all $i = 1, 2, \ldots, n$. Since $x \geq a_i$ and $y \geq a_i$, there is no edge between $x$ and $y$. Hence, we get the desired result.

2) Let $\overline{V_i} = \{x \mid x \in V_i\}$. If $[x], [y] \in \overline{V_i}$, for all $i = 1, 2, \ldots, n$, it is easy to see that there is no edge between $[x]$ and $[y]$. So $\Gamma_E(P)$ is an $n$-partite graph.

Remark 4.10. Proposition 4.9 can also be obtained directly from [13, Theorem 4.7 and Theorem 2.9].

Theorem 4.11. Let $P$ be a poset. Then $\Gamma_E(P)$ is a complete bipartite graph if and only if $|\text{Atom}(P)| = 2$.

Proof. $\Rightarrow$: Suppose $\Gamma_E(P)$ is a complete bipartite graph. If $P$ has only one atom, then $\Gamma_E(P)$ is the null graph. Hence, $|\text{Atom}(P)| \geq 2$. If there exist three atoms $a, b, c \in \text{Atom}(P)$, we obviously have a triangle $[a] - [b] - [c] - [a]$. This is impossible, since a complete bipartite graph has no cycle of odd length.

$\Leftarrow$: Suppose $\text{Atom}(P) = \{a, b\}$. Then $\Gamma_E(P)$ is a bipartite graph by Proposition 4.9.

1) If $x \in P$ such that $x \geq a$ and $x \parallel b$, then $\text{ann}(x) = \text{ann}(a)$, i.e., $[x] = [a]$.
2) Similarly, if $x \in P$ such that $x \geq b$ and $x \parallel a$, then $[x] = [b]$.
3) If $x \in P$ such that $x \geq a$ and $x \geq b$, then $\text{ann}(x) = \{0\}$, i.e., $[x] = [1]$.

In all cases, $\Gamma_E(P)$ has two vertices $\{[a], [b]\}$ and so we have $\Gamma_E(P) = K_2$. 

Proof. 2) $\Rightarrow$ 3) Suppose $a \in P$ and $[a]$ has a complement $[b]$. Then $[a] \neq [b], [a] \neq [0], [b] \neq [0]$ and $L([a], [b]) = \{0\}$. Therefore, $a \neq b, a \neq 0, b \neq 0$ and $L(a, b) = \{0\}$ by Lemma 3.3. If there exists a $c \in P$ such that $L(c, a) = L(c, b) = \{0\}$, then $L([c], [a]) = L([c], [b]) = \{0\}$ by Lemma 3.3 and $[c] \notin \{[a], [b]\}$. That is, $[c]$ is adjacent to both $[a]$ and $[b]$. This is a contradiction. Hence $b$ is a complement of $a$ in $\Gamma(P)$.

3) $\Rightarrow$ 2) Suppose $[a] \in V(\Gamma_E(P))$ and $a \parallel b$. Then we have $L([a], [b]) = \{0\}$. If there exists $[c] \in V(\Gamma_E(P))$ such that $L([c], [a]) = L([c], [b]) = \{0\}$, then $L(c, a) = L(c, b) = \{0\}$ and $c \notin \{a, b\}$. This is a contradiction. Hence $[a]$ has a complement $[b]$. 

Then $\Gamma_E(P)$ is a complete bipartite graph if and only if $|\text{Atom}(P)| = 2$. 

Proof. $\Rightarrow$: Suppose $\Gamma_E(P)$ is a complete bipartite graph. If $P$ has only one atom, then $\Gamma_E(P)$ is the null graph. Hence, $|\text{Atom}(P)| \geq 2$. If there exist three atoms $a, b, c \in \text{Atom}(P)$, we obviously have a triangle $[a] - [b] - [c] - [a]$. This is impossible, since a complete bipartite graph has no cycle of odd length.

$\Leftarrow$: Suppose $\text{Atom}(P) = \{a, b\}$. Then $\Gamma_E(P)$ is a bipartite graph by Proposition 4.9.

1) If $x \in P$ such that $x \geq a$ and $x \parallel b$, then $\text{ann}(x) = \text{ann}(a)$, i.e., $[x] = [a]$.
2) Similarly, if $x \in P$ such that $x \geq b$ and $x \parallel a$, then $[x] = [b]$.
3) If $x \in P$ such that $x \geq a$ and $x \geq b$, then $\text{ann}(x) = \{0\}$, i.e., $[x] = [1]$.

In all cases, $\Gamma_E(P)$ has two vertices $\{[a], [b]\}$ and so we have $\Gamma_E(P) = K_2$. 

Proof. 1) $\Leftrightarrow$ 2) is obvious.
By the proof of Theorem 4.11, we get the following corollary.

**Corollary 4.12.** Let $P$ be a poset. Then $\Gamma_E(P) = K_2$ if and only if $|\text{Atom}(P)| = 2$.

Estaji and Khashyarmanesh ([12]) showed that two vertices $a$ and $b$ are adjacent in a zero-divisor graph of a lattice if and only if $\text{atom}(a) \cap \text{atom}(b) = \emptyset$ (Theorem 5.8). The following statement is similar to Theorem 5.8 in [12].

**Theorem 4.13.** Let $P$ be a poset. Then

1) $x$ and $y$ are adjacent in $\Gamma(P)$ if and only if $\text{atom}(x) \cap \text{atom}(y) = \emptyset$.
2) $x$ and $y$ are not adjacent in $\Gamma(P)$ if and only if $\text{atom}(x) \cap \text{atom}(y) \neq \emptyset$.

**Proof.** 1) $\Rightarrow$: If there exists $a \in \text{Atom}(P)$ such that $a \in \text{atom}(x) \cap \text{atom}(y)$, then $a \leq x$ and $a \leq y$. This contradicts the fact that $L(x, y) = \{0\}$.

$\Leftarrow$: Suppose $z \in L(x, y)$. If $z \neq 0$, then there exists an $a \in \text{Atom}(P)$ such that $a \leq z$. Hence, $a \in \text{atom}(x) \cap \text{atom}(y)$. This is a contradiction.

2) By 1), we obviously get 2).

By Theorem 4.13 and Proposition 3.4, we have the following theorem.

**Theorem 4.14.** Let $P$ be a poset. Then

1) $[x]$ and $[y]$ are adjacent in $\Gamma_E(P)$ if and only if for all $x' \in [x]$ and $y' \in [y]$, we have $\text{atom}(x') \cap \text{atom}(y') = \emptyset$.
2) $[x]$ and $[y]$ are not adjacent in $\Gamma_E(P)$ if and only if for all $x' \in [x]$ and $y' \in [y]$, we have $\text{atom}(x') \cap \text{atom}(y') \neq \emptyset$.

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**References**


THE GRAPH OF EQUIVALENCE CLASSES OF ZERO-DIVISORS OF A POSET


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