FINITE GROUPS HAVING AT MOST 27 NON-NORMAL PROPER SUBGROUPS OF NON-PRIME-POWER ORDER

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Abstract. We prove that any finite group having at most 27 non-normal proper subgroups of non-prime-power order is solvable except for $G \cong A_5$, the alternating group of degree 5.

1. Introduction

All groups are considered to be finite. Note that a group of non-prime-power order in which every non-trivial subgroup has prime-power order is a minimal group of non-prime-power order. In [3], Gallian and Moulton obtained a complete classification of non-abelian minimal groups of non-prime-power order. Obviously any non-abelian minimal group of non-prime-power order is solvable. In [6] and [7], we showed that if a group $G$ has either less than three conjugacy classes of proper subgroups of non-prime-power order or less than three classes of proper subgroups of the same non-prime-power order then $G$ is solvable, and $G$ is a non-solvable group having exactly either three conjugacy classes of proper subgroups of non-prime-power order or three classes of proper subgroups of the same non-prime-power order if and only if $G \cong A_5$. Moreover, we proved that a non-solvable group $G$ has exactly four conjugacy classes of proper subgroups of non-prime-power order if and only if $G \cong PSL(2,8)$, and a non-solvable group $G$ has exactly four classes of proper subgroups of the same non-prime-power order if and only if $G \cong PSL(2,7)$ or $PSL(2,8)$, see [8].

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Note that there always exists at least one solvable group \( G \) such that \( G \) has exactly \( n \) proper subgroups of non-prime-power order for any positive integer \( n \geq 1 \). For example, let \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_{p^{n+1}} \), where \( p \geq 3 \) is a prime. Then \( G \) is a solvable group having exactly \( n \) proper subgroups of non-prime-power order.

From the above fact, a natural question arises:

**Question 1.1.** Does there always exist a non-solvable group \( G \) such that \( G \) has exactly \( n \) proper subgroups of non-prime-power order for any positive integer \( n \geq 1 \)?

As an answer to the above question, we have the following result, the proof of which is given in Section 3.

**Theorem 1.1.** Any group \( G \) having at most 27 proper subgroups of non-prime-power order is solvable except for \( G \cong A_5 \), the alternating group of degree 5.

As an extension of Theorem 1.1, we further obtain the following result, the proof of which is given in Section 4.

**Theorem 1.2.** Any group \( G \) having at most 27 non-normal proper subgroups of non-prime-power order is solvable except for \( G \cong A_5 \).

2. Preliminaries

In this section, we prove two essential lemmas needed in the sequel.

**Lemma 2.1.** Suppose that \( G \) is a minimal non-abelian simple group.

1. If \( G \cong A_5 \), then \( G \) has exactly 21 proper subgroups of non-prime-power order.

2. If \( G \not\cong A_5 \), then \( G \) has at least 50 proper subgroups of non-prime-power order.

**Proof.** It is obvious that \( G \cong A_5 \) has exactly 21 proper subgroups of non-prime-power order by [1]. Next, suppose that \( G \) is a minimal non-abelian simple group that is not isomorphic to \( A_5 \). By [10], \( G \) might be isomorphic to one of the following groups: \( \text{PSL}(2, p) \), \( p > 5 \) is a prime such that \( 5 \nmid p^2 - 1 \); \( \text{PSL}(2, 2^q) \), \( q \) is an odd prime; \( \text{PSL}(2, 3^q) \), \( q \) is an odd prime; \( \text{PSL}(3, 3) \); \( \text{Sz}(2^q) \), \( q \) is an odd prime.

1. Suppose that \( G \cong \text{PSL}(2, p) \), \( p > 5 \) is a prime such that \( 5 \nmid p^2 - 1 \). Let \( p = 7 \). By [1], \( \text{PSL}(2, 7) \) has exactly 22 maximal subgroups that have non-prime-power order. Note that \( S_3 \) is also a proper subgroup of \( \text{PSL}(2, 7) \) of non-prime-power order. The number of conjugates of \( S_3 \) in \( \text{PSL}(2, 7) \) is equal to \( |\text{PSL}(2, 7) : N_{\text{PSL}(2,7)}(S_3)| = |\text{PSL}(2, 7) : S_3| = 28 \). Therefore, \( \text{PSL}(2, 7) \) has at least 50 proper subgroups of non-prime-power order. If \( p > 7 \), then by the hypothesis, \( p \geq 13 \). By [2], \( \text{PSL}(2, p) \) has a maximal subgroup \( A \) that
is isomorphic to a dihedral group of order $p + 1$ and a maximal subgroup $B$ that is isomorphic to a dihedral group of order $p - 1$. Obviously $p + 1$ and $p - 1$ cannot be a 2-power at the same time. If $p + 1$ is not a 2-power, then $\text{PSL}(2, p)$ has at least $|\text{PSL}(2, p) : N_{\text{PSL}(2, p)}(A)| = |\text{PSL}(2, p) : A| = \frac{2(p+1)}{2 \cdot 2^3} = 78$ proper subgroups of non-prime-power order. If $p - 1$ is not a 2-power, then $\text{PSL}(2, p)$ has at least $|\text{PSL}(2, p) : N_{\text{PSL}(2, p)}(B)| = |\text{PSL}(2, p) : B| = \frac{2(p+1)}{2 \cdot 2^3} = 91$ proper subgroups of non-prime-power order. Therefore, whenever $p > 5$ is a prime such that $5 \nmid p^2 - 1$, $\text{PSL}(2, p)$ has at least 50 proper subgroups of non-prime-power order.

(2) Suppose that $G \cong \text{PSL}(2, 2^q)$, $q$ is an odd prime. By [2], $G$ has a maximal subgroup $C$ that is isomorphic to a dihedral group of order $2 \cdot (2^q + 1)$ and a maximal subgroup $D$ that is isomorphic to a dihedral group of order $2 \cdot (2^q - 1)$. Since $|G : N_G(C)| = |G : C| = 2^{q-1}, (2^q - 1) \geq 2^2 \cdot (2^2 - 1) = 28$ and $|G : N_G(D)| = |G : D| = 2^{q-1} \cdot (2^q + 1) \geq 2^3 \cdot (2^3 + 1) = 36$, $G$ has at least 64 proper subgroups of non-prime-power order.

(3) Suppose that $G \cong \text{PSL}(2, 3^q)$, $q$ is an odd prime. By [2], $G$ has a maximal subgroup $E$ that is isomorphic to a dihedral group of order $3^q - 1$. Since $|G : N_G(E)| = |G : E| = \frac{3^q(3^q + 1)}{2} \geq \frac{3^3(3^3 + 1)}{2} = 378$, $G$ has at least 378 proper subgroups of non-prime-power order.

(4) Suppose that $G \cong \text{PSL}(3, 3)$, By [1], $G$ has a maximal subgroup $F$ of order 39 that is isomorphic to the normalizer of a Sylow 13-subgroup of $G$. Since $|G : N_G(F)| = |G : F| = 144$, $G$ has at least 144 proper subgroups of non-prime-power order.

(5) Suppose that $G \cong S_6(2^q)$, $q$ is an odd prime. By [9], $G$ has a maximal subgroup $S$ of order $2^{2q}(2^q - 1)$ that is isomorphic to a Frobenius group. Since $|G : N_G(S)| = |G : S| = 2^{2q} + 1 \geq 2^6 + 1 = 65$, $G$ has at least 65 proper subgroups of non-prime-power order. Therefore, $G$ has at least 50 proper subgroups of non-prime-power order whenever $G$ is a minimal non-abelian simple group that is not isomorphic to $A_5$.

**Lemma 2.2.** Suppose that $G$ is a group such that $G/\Phi(G) \cong A_5$. If $\Phi(G) \neq 1$, then $G$ has at least 37 proper subgroups of non-prime-power order.

**Proof.** (1) Suppose that $|\Phi(G)|$ is not a prime-power. Since $A_5$ has exactly 58 proper subgroups, $G$ has exactly 58 proper subgroups $H$ such that $\Phi(G) \leq H$. Obviously $H$ has non-prime-power order. It follows that $G$ has at least 58 proper subgroups of non-prime-power order.

(2) Suppose that $|\Phi(G)|$ is a prime-power. Since $G/\Phi(G) \cong A_5$, $|\Phi(G)|$ might only be a 2-power or a 3-power or a 5-power. Let $|\Phi(G)|$ be a 2-power. Since $A_5$ has exactly 37 non-trivial subgroups of non-2-power order, $G$ has at least 37 proper subgroups of non-prime-power order. Let $|\Phi(G)|$ be a 3-power. Since $A_5$ has exactly 47 non-trivial subgroups of non-3-power order, $G$ has at least
least 47 proper subgroups of non-prime-power order. If \(|\Phi(G)|\) is a 5-power, since \(A_5\) has exactly 51 non-trivial subgroups of non-5-power order, \(G\) has at least 51 proper subgroups of non-prime-power order.

From above arguments, \(G\) has at least 37 proper subgroups of non-prime-power order.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the following three lemmas.

**Lemma 3.1.** Suppose that \(G\) is a group having at most 20 proper subgroups of non-prime-power order. Then \(G\) is solvable.

**Proof.** Let \(G\) be a counterexample of minimal order. It follows that \(G\) is a minimal non-solvable group. Therefore, \(G/\Phi(G)\) is a minimal non-abelian simple group. However, \(G/\Phi(G)\) has at least 21 proper subgroups of non-prime-power order by Lemma 2.1, which implies that \(G\) has at least 21 proper subgroups of non-prime-power order, a contradiction. Therefore, \(G\) is solvable.

**Lemma 3.2.** Suppose that \(G\) is a non-solvable group having exactly 21 proper subgroups of non-prime-power order. Then \(G \cong A_5\).

**Proof.** By the hypothesis, every maximal subgroup of \(G\) has at most 20 proper subgroups of non-prime-power order. Then every maximal subgroup of \(G\) is solvable by Lemma 3.1. It follows that \(G\) is a minimal non-solvable group and then \(G/\Phi(G)\) is a minimal non-abelian simple group. Since \(G\) has exactly 21 proper subgroups of non-prime-power order, by Lemma 2.1, \(G/\Phi(G)\) might only be isomorphic to \(A_5\). Then by Lemma 2.2, one has \(\Phi(G) = 1\). Therefore, \(G \cong A_5\).

**Lemma 3.3.** Suppose that \(G\) has exactly \(n\) proper subgroups of non-prime-power order, where \(22 \leq n \leq 27\). Then \(G\) is solvable.

**Proof.** Let \(G\) be a counterexample of minimal order. That is, for any proper subgroup \(M\) of \(G\) if \(M\) has exactly \(k\) (\(22 \leq k \leq 27\)) proper subgroups of non-prime-power order then \(M\) is solvable.

(1) Suppose that \(G\) has at least one non-solvable maximal subgroup. Let \(N\) be a non-solvable maximal subgroup of \(G\), by Lemmas 3.1 and 3.2, \(N \cong A_5\). We claim that

\[N \leq G.\]

Otherwise, assume that \(N \not\leq G\). By the hypothesis, the number of conjugates of \(N\) in \(G\) is not greater than 6. That is, \(|G : N_G(N)| = |G : N| = k \leq 6\). Then \(G/N_G \leq S_k\), where \(k \leq 6\) and \(N_G\) is the largest normal subgroup of \(G\) that is contained in \(N\). Since \(N\) is simple and \(N \not\leq G\), one has \(N_G = 1\). It follows that \(G \leq S_k\), where \(k \leq 6\). Since \(G\) is non-solvable
and $A_5 \cong N$ is maximal in $G$ but $N \not\trianglelefteq G$. $G$ might only be isomorphic to $A_6$. Obviously $A_6$ has more than 27 proper subgroups of non-prime-power order, a contradiction. Therefore, $N \leq G$.

If $G$ has a normal maximal subgroup $T \neq N$, then $T \cap N = 1$, as $N$ is simple. It follows that $G = T \times N$ and then $N \cong G/T$ is a cyclic group of prime order, a contradiction. Therefore, $N$ is the unique normal maximal subgroup of $G$.

Since $G$ is non-solvable, by [4], $G$ has at least three conjugacy classes of maximal subgroups. If $G$ has a maximal subgroup of prime-power order $H$, by [5, Theorem 10.4.2], $H$ must have 2-power order. That is, $H$ is a Sylow 2-subgroup of $G$. Since all Sylow 2-subgroups of $G$ are conjugate, $G$ has at least one maximal subgroup $K$ of non-prime-power order such that $K \neq N$. By the hypothesis, $|G : N_G(K)| \leq 5$. Since $N$ is the unique normal maximal subgroup of $G$, we have $K \not\trianglelefteq G$. Then $|G : N_G(K)| = |G : K| = t$, where $3 \leq t \leq 5$. If $3 \leq t \leq 4$, one has that $G/K \cong S_t$ is solvable. It follows that $G/K_G$ has at least one normal maximal subgroup, say $A/K_G$. Obviously, $K_G \not\leq N$. Then $A \neq N$, a contradiction. If $|G : K| = 5$, one has $G/K_G \cong S_5$. Note that $G/K_G$ must be non-solvable. Then $G/K_G \cong S_5$ or $A_5$. If $K_G \leq N$, one has $K_G = 1$. Obviously $G \cong A_5$, and $G \cong S_5$ has more than 27 proper subgroups of non-prime-power order, a contradiction. If $K_G \not\leq N$, one has $K_G \cap N = 1$ and then $G = K_G \times N$ also has more than 27 proper subgroups of non-prime-power order, a contradiction.

(2) Suppose that every maximal subgroup of $G$ is solvable. It follows that $G$ is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. By Lemma 2.1, $G/\Phi(G)$ might only be isomorphic to $A_5$. If $\Phi(G) \neq 1$, by Lemma 2.2, $G$ has at least 37 proper subgroups of non-prime-power order, a contradiction. If $\Phi(G) = 1$, then $G \cong A_5$ has exactly 21 proper subgroups of non-prime-power order, also a contradiction.

From arguments (1) and (2), the counterexample does not exist and so $G$ is solvable. 

Lemmas 3.1, 3.2 and 3.3 combined together give Theorem 1.1.

4. PROOF OF THEOREM 1.2

LEMMA 4.1. Suppose that $G$ is group having at most 20 non-normal proper subgroups of non-prime-power order. Then $G$ is solvable.

PROOF. Let $G$ be a counterexample of minimal order. If $G$ has no normal proper subgroups of non-prime-power order, by Lemma 3.1, $G$ is solvable, a contradiction. Suppose $G$ has a normal proper subgroup of non-prime-power order, say $R$. For the group $G/R$, by the minimality of $G$, both $R$ and $G/R$ are solvable. It follows that $G$ is solvable, also a contradiction.
Lemma 4.2. Suppose that $G$ is a non-solvable group having exactly 21 non-normal proper subgroups of non-prime-power order. Then $G \cong A_5$.

Proof. By the hypothesis and Lemma 4.1, every maximal subgroup of $G$ is solvable. Then $G$ is a minimal non-solvable group and so $G/\Phi(G)$ is a minimal non-abelian simple group. By Lemma 2.1, $G/\Phi(G)$ might only be isomorphic to $A_5$. If $\Phi(G) \neq 1$, by Lemma 2.2, $G$ has at least 37 proper subgroups of non-prime-power order, which are non-normal in $G$, a contradiction. Therefore, $\Phi(G) = 1$, and then $G \cong A_5$.

Lemma 4.3. Suppose that $G$ is a group having exactly $n$ non-normal proper subgroups of non-prime-power order, where $22 \leq n \leq 27$. Then $G$ is solvable.

Proof. Let $G$ be a counterexample of minimal order. That is, for any proper subgroup $T$ of $G$, if $T$ has exactly $m$ ($22 \leq m \leq 27$) non-normal proper subgroups of non-prime-power order, then $T$ is solvable.

(1) Suppose that every maximal subgroup of $G$ is solvable. It follows that $G$ is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. Arguing as in Lemma 4.2, we can get a contradiction.

(2) Suppose that $G$ has a non-solvable maximal subgroup, say $M$. By the hypothesis, and Lemmas 4.1 and 4.2, one has $M \cong A_5$. Moreover, arguing as in Lemma 3.3, we can conclude that $M \leq G$. If $G$ has no other normal proper subgroups of non-prime-power order except $M$, by the hypothesis, $G$ has at most 6 proper subgroups of non-prime-power order not contained in $M$. Arguing as in Lemma 3.3, we can get a contradiction. So suppose now that $G$ has a normal proper subgroup $N$ of non-prime-power order such that $N \neq M$. Since $M$ is simple, $N \cap M = 1$. It follows that $G = N \times M \cong N \times A_5$. Obviously, $N \times A_5$ has more than 27 non-normal proper subgroups of non-prime-power order, a contradiction.

Therefore, the counterexample does not exist and so $G$ is solvable.

Lemmas 4.1, 4.2 and 4.3 combined together give Theorem 1.2.

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