FINITE GROUPS WITH FEW VANISHING ELEMENTS

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Abstract. Let $G$ be a finite group, and $\text{Irr}(G)$ the set of irreducible complex characters of $G$. We say that an element $g \in G$ is a vanishing element of $G$ if there exists $\chi$ in $\text{Irr}(G)$ such that $\chi(g) = 0$. Let $\text{Van}(G)$ denote the set of vanishing elements of $G$, that is, $\text{Van}(G) = \{ g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \text{Irr}(G) \}$. In this paper, we investigate the finite groups $G$ with the following property: $\text{Van}(G)$ contains at most four conjugacy classes of $G$.

1. Introduction

Let $G$ be a finite group and $v(\chi) := \{ g \in G \mid \chi(g) = 0 \}$, where $\chi$ is an irreducible complex character of $G$. A classical theorem of Burnside asserts that $v(\chi)$ is non-empty for all $\chi \in \text{Irr}_1(G)$, where $\text{Irr}_1(G)$ denotes the set of non-linear irreducible complex characters of $G$. Our aim in this paper is to analyze a particular subset of $\{ x^G : x \in G \}$, which we denote by $\text{Van}(G)$ and which encodes information coming from the set $\text{Irr}(G)$ of irreducible complex characters of $G$. We say that an element $g \in G$ is a vanishing element of $G$ if there exists $\chi$ in $\text{Irr}(G)$ such that $\chi(g) = 0$ and otherwise we call $x$ a non-vanishing element. Let $\text{Van}(G)$ denote the set of vanishing elements of $G$, that is

$$\text{Van}(G) = \{ g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \text{Irr}(G) \}.$$ 

Clearly, $\text{Van}(G)$ is a proper normal subset of $G$. We denote $k_G(N)$ the number of conjugacy classes of $G$ contained in $N$, where $N$ is a normal subset of $G$. In this paper, we investigate the finite groups $G$ with the following property: $\text{Van}(G)$ contains at most four conjugacy classes of $G$. Generally, we define

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Definition 1.1. A group $G$ is called a $VnC$-group if $\text{Van}(G)$ contains at most $n$ conjugacy classes of $G$.

Clearly, if $G$ is a $VnC$-group, then, in particular every irreducible character of $G$ vanishes on at most $n$ conjugacy classes of $G$. However, it turns out (see [20] and Theorem 1.4 below) that for $n = 3$ the two classes, in the case of solvable groups, in fact coincide. The aim of this paper is the classification of $V4C$-groups and the main results are the following.

Theorem 1.2. Let $G$ be a finite non-abelian and solvable group. If $G$ is a $V4C$-group but not a $V3C$-group, then one of the following is true:

1. $G$ is a Frobenius group with complement $Q_8$,
2. $G = G'P$, where $G'$ is a normal abelian 2-complement of $G$, $P \in \text{Syl}_2(G)$, $|P| = 8, |Z(G)| = 4$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,
3. $G = (G'(u)) \times (t)$, where $t$ is an involution and $G'(u)$ is a Frobenius group with kernel $G'$ and complement of order 3,
4. $G = G'P$, where $G'$ is a normal abelian 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$, $F(G) = G'$, $G\backslash G'$ is a union $x^G \cup y^G \cup z^G \cup h^G$ of four conjugacy classes satisfying $|C_G(x)| = 4, |C_G(y)| = 4, |C_G(z)| = 6$ and $|C_G(h)| = 12$,
5. $G$ is a Frobenius group with abelian kernel $G'$ and complement of order 5.

Theorem 1.3. Let $G$ be finite non-solvable group. If $G$ is a $V4C$-group, then $G$ is isomorphic to $A_5$.

Clearly, $A_5$ is not a $V3C$-group. Hence by [20, Theorem A] and Theorem 1.3 above we easily get the following result.

Theorem 1.4. Let $G$ be a finite non-abelian group. Then $G$ is a $V3C$-group if and only if $G$ is one of the following groups:

1. $G$ is a Frobenius group with abelian kernel $G'$ and complement of order 2
2. $G$ is a Frobenius group with abelian kernel $G'$ and complement of order 3,
3. $G \cong D_8$ or $Q_8$,
4. $G$ is a Frobenius group with kernel $G'$ and cyclic complement of order 4,
5. $G = G'P$, where $G'$ is a normal and abelian 2-complement of $G$, $P \in \text{Syl}_2(G)$, $|P| = 4, |Z(G)| = 2$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,
6. $G \cong S_4$,
7. $G = (G'(t)) \times \langle u \rangle$, where $\langle u \rangle$ is a cyclic group of order 3 and $G'(t)$ is a Frobenius group with kernel $G'$ and complement of order 2.
In this paper, \( G \) always denotes a finite group. Notation is standard and taken from [8]. In particular, denote \( \pi_e(G) \) the set of all element orders of \( G \), \( C(n) \) a cyclic group of order \( n \), \((H, N)\) a Frobenius group with a complement \( H \) and kernel \( N \). we write \( G = [F]H \) to denote a semidirect product of a normal subgroup \( F \) and a subgroup \( H \) of \( G \). For \( N \triangleleft G \), set \( \text{Irr}(G|N) = \text{Irr}(G) \setminus \text{Irr}(G/N) \).

2. NILPOTENT V4C-GROUPS

In this paper, we shall freely use the following facts. Let \( N \triangleleft G \) and write \( G = G/N \).

(1) For any \( x \in G \), \( x^G \) (when viewed as a subset of \( G \), that is, the set \( \bigcup_{g \in G} x^gN \)) is a union of conjugacy classes of \( G \). Note that \( |C_G(x)| = |C_G(x^G)| + \sum \{|\chi(x)|^2 \mid \chi \in \text{Irr}(G|N)\} \) and that \( k_G(x^G) = 1 \) if and only if \( |C_G(x)| = |C_G(x^G)| \), and then \( k_G(x^G) = 1 \) if and only if \( \chi(x) = 0 \) for all \( \chi \in \text{Irr}(G|N) \).

(2) If \( G \) is a \( VnC \)-group, then so is \( G/N \).

(3) Any character of \( G \) can be viewed, by inflation, as a character of \( G \).

In particular, if \( xN \in \text{Van}(G) \), then \( xN \subseteq \text{Van}(G) \).

(4) Let \( M \) be a normal subgroup of \( G \) contained in \( N \). If \( \psi \in \text{Irr}(N) \) vanishes on \( N \setminus M \), then by Clifford’s theorem, every irreducible constituent of \( \psi^G \) also vanishes on \( N \setminus M \). In particular, \( N \setminus M \subseteq \text{Van}(G) \).

We will use frequently the following lemma (see [16, Theorem 2.1]).

**Lemma 2.1.** Let \( G \) be non-abelian, and let \( \chi \in \text{Irr}(G) \). Assume that \( N \) is a normal subgroup of \( G \) such that \( G' \leq N < G \). If \( \chi_N \) is not irreducible, then the following two statements hold:

1. There exists a normal subgroup \( H \) of \( G \) such that \( N \leq H < G \) and \( G - H \subseteq v(\chi) \).
2. \( (G\setminus G') \cap v(\chi) \) consists of \( n \) conjugacy classes of \( G \), then \( |H : G'| (|G : H| - 1) \) \leq \( n \).

The following result gives a lower bound on the number of conjugacy classes of zeros of irreducible characters of \( p \)-groups.

**Lemma 2.2.** [13, Theorem C] Let \( \chi \) be a non-linear irreducible character of a \( p \)-group \( P \) of degree \( p^n \). Then \( k_G(v(\chi)) \) is a multiple of \( p - 1 \) bigger than or equal to \( p - 1 \). In particular, \( k_G(v(\chi)) \geq p^2 - 1 \).

Next, we classify the non-abelian nilpotent groups in which every irreducible character vanishes on at most four conjugacy classes of \( G \).

**Theorem 2.3.** Suppose that \( G \) is a non-abelian nilpotent group. If every irreducible character of \( G \) vanishes on at most four conjugacy classes of \( G \), then \( G \) is one of the following groups:
(1) \( G \cong D_8 \) or \( Q_8 \),
(2) \( G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^{-1} \rangle \),
(3) \( G \cong \langle a, b \mid a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1} \rangle \),
(4) \( G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle \).

\textbf{Proof.} Take \( \varphi \in \text{Irr}(G) \) such that \( \varphi \chi \) is not irreducible. It follows from the hypothesis and Lemma 2.1 that \( G \) has a proper subgroup \( H \) such that \( G' \leq H < G \), \( G/H \) and \( [H : G'] ([G : H] - 1) \leq 4 \).

Since \( G \) is nilpotent, it follows by Lemma 2.1 that \( G \) is a 2-group. Note that \( [H : G'] ([G : H] - 1) \leq 4 \) and that \( |G/G'| \geq 2^2 \); then we obtain that \( |G/G'| = 4 \) or \( 8 \). Suppose that \( |G/G'| = 8 \). Let \( N \) be normal in \( G \) with \( |G : N| = 2 \). Now consider the group \( G/N \) of order 16. Note that \( |G/N : (G/N)'| = 8 \), by the hypothesis, we easily conclude that it is impossible (see [10, P. 300]). Hence we may suppose that \( |G/G'| = 4 \). If \( |G'| = 2 \), then \( G \) satisfies (1) of the Theorem. So we may assume that \( |G'| \geq 4 \). Now let \( N \) be normal in \( G \) with \( [G : N] = 4 \). Consider the group \( \overline{G} := G/N \) of order 16. Applying [10, P. 300], \( G/N \) satisfies (2), (3) or (4) of the Theorem.

Now we show that \( N = 1 \). Suppose that \( N > 1 \). Set \( M/N = Z(G/N) \). Take the irreducible character \( \xi \) of \( \overline{G} \) with \( k_{\overline{G}}(v(\xi)) = 4 \). Thus the hypothesis yields that \( k_{G/N}(v(\xi)) = k_G(v(\xi)) = 4 \), and so \( \chi \) vanishes only on \( v(\xi) \) for every \( \chi \in \text{Irr}(G/N) \). Hence it follows by [9, Theorem A] that \( M = Z(G) \). Consequently, \( |G : Z(G)| = 8 \).

Take an irreducible character \( \rho \) of \( G \) with \( k_G(v(\rho)) = 3 \). Choose \( g \in G \) such that \( \overline{g} = v(\rho) \setminus v(\xi) \). We easily see that \( \overline{g} G' = G' \setminus Z(G) \). Hence \( k_{G/G'}(Z(G)) = 6 \).

On the other hand, set \( G \setminus Z(G) = n_1 G' \cdots + n_s G' \). Then, we get
\[
|G \setminus Z(G)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|}.
\]
It follows that
\[
1 \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|} = 1 - \frac{1}{|G : Z(G)|} = \frac{7}{8}.
\]
Recall that all the centralizers of the elements in \( G \setminus Z(G) \) have order greater than 4, then \( G \setminus Z(G) \) has at least 7 conjugacy classes, a contradiction. The proof is complete.

\textbf{Remark 2.4.} Clearly, a \( V4C \)-group is such a group whose irreducible characters vanish on at most four conjugacy classes. But the converse is not true (see the types (2), (3) or (4) in Theorem 2.3).

\textbf{Corollary 2.5.} Suppose that \( G \) is a non-abelian nilpotent group. If \( G \) is a \( V4C \)-group, then \( G \) is isomorphic to \( D_8 \) or \( Q_8 \)
3. Vanishing elements and the nilpotent normal subgroup

In the following, we consider the situation in which a vanishing element of a group $G$ lies in a nilpotent normal subgroup.

**Lemma 3.1.** Let $N$ be a nilpotent subgroup of $G$ such that $|G : N| = 2$. Assume that $G$ is non-nilpotent. If $N$ is non-abelian, then $k_G(N \cap \text{Van}(G)) \geq 3$.

**Proof.** Consider $\theta \in \text{Irr}_1(N)$. Note that $|G : N| = 2$, thus there exists $\chi \in \text{Irr}(G)$ such that either $\chi_N = \theta$ or $\chi = \theta^G$. If $\chi_N = \theta$, then it follows by the nilpotency of $N$ and Lemma 2.1 that $k_N(\psi(\theta)) \geq 6$, and so $k_G(\psi(\theta)) \geq 3$. Thus $k_G(N \cap \text{Van}(G)) \geq 3$. Hence we may assume that $\chi = \theta^G$.

Now let $M$ be a normal subgroup of $N$ maximal with respect to $N/M$ being non-abelian. Then $N/M$ is a $q$-group for some prime $q$ and $(N/M)'$ is the unique minimal normal subgroup of $N/M$. Let $Z > M$ with $Z/M = Z(N/M)$. Consider $\phi \in \text{Irr}(N/M)$ with $\phi(1) = q^n > 1$. The choice of $M$ implies that $\phi$ is a faithful irreducible character of $N/M$, and thus $\phi$ satisfies $\phi(1)^2 = |N/M : Z/M|$ by [8, Theorem 2.31]. Applying [8, Corollary 2.30], $\phi$ vanishes on $N \setminus Z$.

From the first paragraph of the proof, we have that $\phi^G$ is an irreducible character of $G$. Take $\psi = \phi^G$. Observe that $\psi$ vanishes on $N \setminus (Z \cup Z^x)$, where $x \in G \setminus N$. Set $N \setminus (Z \cup Z^x) = n_1^G + \cdots + n_s^G$. Then, we get

$$|N \setminus (Z \cup Z^x)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|},$$

and

$$|N \setminus (Z \cup Z^x)| > |N| - 2|Z|.$$ 

It follows that

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{q^{2n}}.$$ 

Assume that $q = 2$. Then $2 ||N|$. Recall that $N$ is not a 2-group; thus $|C_G(n_i)| \geq 12$. Note that

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$ 

Hence, $s \geq 4$. Now, we assume that $q$ is odd.

Since $q$ is odd, we get have $|C_G(n_i)| \geq 9$. On the other hand, we have

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{9} = \frac{7}{18}.$$ 

Therefore, $s \geq 4$ and we are done. □

Arguing as in Lemma 3.1 we easily see that the following two results.
Lemma 3.2. Let \( N \) be a nilpotent normal subgroup of \( G \) such that \(|G:N|=3\). Assume that \( G \) is non-nilpotent. If every element of \( N \) is a non-vanishing element of \( G \), then \( N \) is abelian.

Lemma 3.3. Assume that \( G = NP \), where \( N \) is a nilpotent normal 2-complement of \( G \) and \( P \in \text{Syl}_2(G) \). Assume that \(|P|=4\). If every element of \( N \) is a non-vanishing element of \( G \), then \( N \) is abelian.

Theorem 3.4. Let \( G \) be Frobenius group with kernel \( N \) and complement of order 3. If \( N \) is non-abelian, then \( k_G(N \cap \text{Van}(G)) \geq 3 \).

Proof. Let \( G \) be a minimal counter-example. Then \( N \) is a \( q \)-group for some prime \( q \), also \( N' \) is minimal normal in \( G \) and \( N' \leq Z(N) \). Let \( C = \langle x \rangle \) be a complement of \( N \) in \( G \).

Observe that any irreducible \( C \)-invariant subgroup of \( N \) is of order at most \( q^3 \), also that if \( q=2 \) then the bound is \( 2^2 \). In particular, \(|N'|=q^e \leq q^3\).

Let \( \psi \in \text{Irr}(N) \) be of maximal degree. Observe that there exists a subgroup \( Z \) of \( N \) such that \( \psi \) vanishes on \( N \setminus Z \) and that \( Z \geq N' \), \(|N/Z|=q^{2m}\).

Suppose that \( q > 2 \) or \( m > 1 \). Then there exists an irreducible character \( \chi \) of \( G \) such that \( \chi \) vanishes on \( G \setminus \Delta \), where \( \Delta := Z \cup Zx \cup Zx^2 \). Write \(|Z/N'|=q^f \). Observe that the centralizer of any element in \( N \setminus N' \) has order at least \(|N/N'|\).

Now
\[
k_G(G \setminus \Delta) - 2 = k_G(N \setminus \Delta)
\]
\[
> (q^{2m+2} + q^{q^e} - 3q^{q+e})/3q^e
\]
\[
= q^f(q^{2m} - 3)/3 > 3,
\]
hence, we obtain a contradiction.

Suppose that \( q = 2 \) and \( m = 1 \). As \( \psi \) is of maximal degree, \( \text{cd}(N) = \{1, 2\} \). It follows that either \(|N:Z(N)|=8\) or \( N \) has an abelian subgroup of index 2. Observe that if \( N/Z(N) \) has order 8, then \( N/Z(N) \) has a \( C \)-invariant subgroup of order 2, which is impossible because \( G \) is a Frobenius group. Assume now that \( N \) has an abelian subgroup \( E \) of index 2 and set \( D = \cap_{g \in G} E^g \). Then all non-linear \( \psi \in \text{Irr}(N) \) vanishes on \( N \setminus E \). Take \( \chi \in \text{Irr}(G) \) of degree 6. We see that \( \chi \) vanishes on \( N \setminus E \), and it follows that \( \chi \) vanishes on every element of \( N \setminus D \). It follows by [9, Theorem A] that \( Z(N) \leq D \). Then we get
\[
N' \leq Z(N) \leq D \leq N.
\]
Observe that \(|N/D| \geq 4\), thus we conclude
\[
4 \leq |N'| \leq |Z(N)| \leq |D| \leq \frac{|N|}{4}.
\]
If all are equalities above, then \( N \) is of order 16, \( N' = Z(N) \) and \(|N:N'|=4\). However, it is impossible (see [10, P. 300]). Hence \(|N| \geq 64\). Set \( N \setminus D =
Thus, we have

\[ |N \setminus D| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|}. \]

It follows that

\[ \frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} = \frac{1}{3} \frac{1}{3|N : D|} \geq \frac{1}{4}. \]

Observe that all the centralizers of the elements in \( N \setminus D \) have order greater than 4, also that the centralizer of some element, say \( g \), has order at least 32. So we conclude by the above equality (*) that \( N \setminus D \) consists of at least 3 conjugacy classes, a contradiction.

Arguing as in Theorem 3.4, we easily see that the following lemma.

**Lemma 3.5.** [1, Proposition 4.1] Let \( G \) be Frobenius group with kernel \( N \) and complement of order \( p \), where \( p \leq 5 \). If \( N \cap \text{Van}(G) = \emptyset \), then \( N \) is abelian.

**4. Conjugacy classes outside a normal subgroup**

We will use frequently the following result.

**Lemma 4.1.** Let \( G \) be non-abelian. Assume that \( N \) is a normal subgroup of \( G \) such that \( G' \leq N < G \) and that \( G \setminus N \subseteq \text{Van}(G) \). If \( \text{Van}(G) \) consists of at most \( n \) conjugacy classes of \( G \), then \( |N : G'| (|G : N| - 1) \leq n \).

**Proof.** Assume that \( |N : G'| = m \) and \( |G : N| = r \). Then we have

\[ G = N + N x_1 + \ldots + N x_{r-1}, \quad x_i \not\in N, \]

and

\[ N = G' + G'y_1 + \ldots + G'y_{m-1}, \quad y_j \not\in G'. \]

It follows that

\[ G \setminus N = \sum_{i=1}^{r-1} \sum_{j=1}^{m-1} G'y_j x_i + \sum_{i=1}^{r-1} G'x_i. \]

For \( x \not\in G' \), \( G'x \) is a normal subset of \( G \), and so we conclude by the above equality (*) that \( G \setminus N \) consists of at least \( m(r-1) \) conjugacy classes. Bearing in mind that \( G \setminus N \subseteq \text{Van}(G) \), by the hypothesis we obtain that \( m(r-1) \leq n \), that is \( |N : G'| (|G : N| - 1) \leq n \).

We will also make use of the following result, which is [9, Theorem 4.3].

As usual, we denote by \( F(G) \) the Fitting subgroup of a group \( G \).

**Lemma 4.2.** Let \( G \) be a solvable group and let \( x \) be an element of \( G \) such that \( \chi(x) \neq 0 \) for every \( \chi \in \text{Irr}(G) \). Then the image of \( x \) modulo \( F(G) \) has 2-power order.
Lemma 4.3. [4, Lemma 2.6] Let $G$ be a solvable group, and $N$ a normal subgroup of $G$. If $N/F(N)$ is abelian, then $N\backslash F(N) \subseteq \text{Van}(G)$.

The following lemma is the key to the proof of Theorem 1.2.

Lemma 4.4. Suppose that $G = KP$, where $K$ is a normal 2-complement of $G$ and $P \in \text{Syl}_2(G)$. If $G$ is a $V4C$-group and $P$ is non-abelian, then one of the following statements holds:

1. $K = 1$ and $G \cong D_8$ or $Q_8$.
2. $G$ is a Frobenius group with complement $Q_8$.

Proof. By Corollary 2.5, we may assume that $K > 1$. Since $P$ is non-abelian, it follows by Corollary 2.5 that $G/K \cong P \cong Q_8$ or $D_8$. Hence we have that $|G/K : G'K/K| = 4$ and that $G' \backslash G \subseteq \text{Van}(G/K)$. We then in a position to apply Lemma 4.1 to $G$, with $G'K$ playing the role of $N$, obtaining that $K \leq G'$ and so $G' \backslash G \subseteq \text{Van}(G)$. Observe that $k_G(G' \backslash G') = 3$ or 4.

Assume that $k_G(G' \backslash G') = 3$. Set $G' \backslash G' = xG' + yG' + zG'$. Thus, we get

$$|G' \backslash G'| = \frac{G}{|C_G(x)|} + \frac{G}{|C_G(y)|} + \frac{G}{|C_G(z)|}.$$ 

Then we conclude


Hence every element in $P \backslash P'$ acts fixed point freely on $K$. Therefore, since $P$ is a 2-group of class 2, we conclude by [11, Lemma 19.1] that $P \cong Q_8$ and $G$ is a Frobenius group with complement $Q_8$.

Assume that $k_G(G' \backslash G') = 4$. The hypothesis implies that $K \cap \text{Van}(G) = \emptyset$. Since $K$ is of odd order, it follows by Lemma 4.2 that $K \leq F(G)$. Consequently, $F(G) = K \times O_2(G)$. If $O_2(G) = 1$, then, as $F(G') = F(G) = K$ and $|G' : K| = 2$, an application of Lemma 4.3 yields that $G' \backslash G \subseteq \text{Van}(G)$. Thus we obtain a contradiction. Therefore, $O_2(G) > 1$.

Since $O_2(G) > 1$, we get that $G'$ is nilpotent, and also that $Z(G) = O_2(G)$. Set $G' = Q$. We now show that $Q$ is abelian. Otherwise, Suppose that $Q$ is non-abelian. Consider $\theta \in \text{Irr}_1(Q)$. Let $\chi \in \text{Irr}(\theta^G)$. The hypothesis implies that $\chi_Q$ is not irreducible. Then by Clifford’s theorem, $\chi_Q = \epsilon(\theta x_1 + \cdots + \theta x_t)$, where $x_1, \ldots, x_t$ is a transversal of $I_G(\theta)$ in $G$ and $\epsilon = [\chi_Q, \theta]$. Clearly, $t = 2$ or 4. Let $M$ be a normal subgroup of $Q$ maximal with respect to $Q/M$ being non-abelian. Then $Q/M$ is a $q$-group for some prime $q$ and $(Q/M)'$ is the unique minimal normal subgroup of $Q/M$. Let $Z > M$ with $Z/M = Z(Q/M)$. Consider $\theta \in \text{Irr}(Q/M)$ with $\theta(1) = q^n > 1$. Then, by [8, Corollary 2.30 and Theorem 2.31], $\theta$ vanishes on $Q/Z$ and $|Q/Z| = q^{2n}$. Notice that $\chi_Q$ vanishes on $Q \backslash (Z^{x_1} \cup \cdots \cup Z^{x_t})$. The hypothesis yields that $Q = Z^{x_1} \cup \cdots \cup Z^{x_t}$ and thus $|Q| \leq 4|Z|$. Hence $|Q/Z| \leq 4$, thus we have reached a contradiction (note that $q$ is odd), and so $G'$ is abelian.
Set $G \setminus G' = zG' + yG' + zG'$, $x, y, z \in G \setminus G'$. Since $k_G(G \setminus G') = 4$, we may assume that both $xG' \cap yG'$ and $zG'$ are a union of two conjugacy classes of $G$, respectively. Set $zG' = z^G + hG$. Recall now that $G/G' \cong P/P'$ is an elementary abelian group of order 4, there exists a normal subgroup $N$ of $G$ such that $G' < N < G$, also that $|G : N| = 2$ and $G \setminus N = xG' + yG'$. Thus we have

$$|G \setminus G'| = \frac{|G|}{|C_G(x)|} + \frac{|G|}{|C_G(y)|} + \frac{|G|}{|C_G(z)|} + \frac{|G|}{|C_G(h)|},$$

and

$$|C_G(x)| = |C_G(y)| = 4.$$ 

It follows that

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{C_G(z)} + \frac{1}{C_G(h)}.$$ 

Therefore, we easily see that $|C_G(z)| = |C_G(h)| = 8$ or $|C_G(z)| = 6$ and $|C_G(h)| = 12$.

Suppose that $|C_G(z)| = 6$ and $|C_G(h)| = 12$. As $|G : G'| = 4$, the order of $z$ is even. Note that $|Z(G)| = 2$, thus it is easy to see that 4 divides $|C_G(z)|$, a contradiction. Hence $|C_G(z)| = |C_G(h)| = 8$, then every element in $P \setminus P'$ acts fixed point freely on $K$. Observe that $G$ is a Frobenius group with kernel $K$ and complement $Q_8$, and thus $k_G(G \setminus G') = 3$, a contradiction. The contradiction completes the proof.

The following result, which appears as [15, Theorem 2.2], will turn out to be useful in handling the case that $k_G(G \setminus N) = 2$ and that $G \setminus N \subseteq \text{Van}(G)$, where $N$ is normal in $G$.

**Lemma 4.5.** Let $N$ be a normal subgroup of a non-abelian solvable group $G$. Then $k_G(G \setminus N) = 2$ if and only if $G$ is one of the following solvable groups.

1. $N = 1$ and $G \cong S_3$.
2. $|G/N| = 3$ and $G$ is a Frobenius group with kernel $N$.
3. $|G/N| = 2$ and $|C_G(x)| = 4$ for all $x \in G \setminus N$. In particular, $P \in \text{Syl}_2(G)$ has a cyclic subgroup of order $|P|/2$; furthermore, one of the following holds:
   - (3.a) $G$ has a normal and abelian 2-complement.
   - (3.b) $G$ has a normal 2-complement and $P$ is a quaternion group.
   - (3.c) $G$ has an abelian 2-complement and $P \cong D_8$, the dihedral group of order 8.

**Proposition 4.6.** Let $N$ be a normal subgroup of a non-abelian solvable group $G$. Assume that $k_G(G \setminus N) = 2$ and that $G \setminus N \subseteq \text{Van}(G)$. If $G$ is a $V4C$-group, then $G$ is one of the following solvable groups:

1. $G$ is a Frobenius group with kernel $N$ and complement of order 3,
2. $G \cong D_8$ or $Q_8$,
3. $G$ is a Frobenius group with complement $Q_8$. 

(4) $G = G' P$, where $G' < N$ is a normal 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$.

(5) $G' = N$, $|G : G'| = 2$ and $G/O_2(G) \cong S_4$.

**Proof.** By the hypothesis and Lemma 4.5, $G$ satisfies the condition of Lemma 4.1. Then we conclude by Lemma 4.1 that one of the following three cases occurs: (i) $|G : G'| = 3$, $G' = N$; (ii) $|N : G'| = 2$, $|G : N| = 2$; (iii) $G' = N$, $|G : G'| = 2$.

Assume that $|G : G'| = 3$, $G' = N$. Then by Lemma 4.5, we have that $G$ is a Frobenius group with kernel $G'$ and complement of order 3.

Assume that $|N : G'| = 2$, $|G : N| = 2$. Then it follows by Lemma 4.5 that $G = K P$, where $K \leq G'$ is a normal 2-complement of $G$ and $P \in \text{Syl}_2(G)$. If $P$ is non-abelian, then by Lemma 4.4 we conclude that $G$ satisfies (2) or (3) of the Proposition. Hence we may assume that $P$ is abelian, then we easily see that $G = G' P$, where $G' < N$ is a normal 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$.

Assume that $G' = N$, $|G : G'| = 2$. Recall that $|C_G(g)| = 4$ for any $g \in G \setminus G'$. Take $y \in G$ with $T = C_G(y)$ of order 4. Clearly $T \subseteq C_G(T) \subseteq C_G(y) = T$. Let $O_2(G)$ be the largest normal subgroup of odd order in $G$. Notice that $G$ is solvable, we use [19, Theorem 1, 2] to conclude that $G/O_2(G) \cong S_4$. The proof is complete. □

The following proposition, which comes from [15, Theorem 3.6] and [17, Theorem 1.1], will turn out to be useful in handling the case that $G$ has a normal subgroup $N$ such that $k_G(G,N) = 3$ and that $G \setminus N \subseteq \text{Van}(G)$, where $N$ is normal in $G$.

**Proposition 4.7.** Let $N$ be a normal subgroup of a non-abelian solvable group $G$. Then $G \setminus N$ is a union $x^G \cup y^G \cup z^G$ of three conjugacy classes satisfying $|x^G| \geq |y^G| \geq |z^G|$ if and only if one of the following is true:

1. $N = 1$ and $G \cong A_4$ or $D_{10}$,
2. $G/N \cong S_3$ and $G \cong S_4$,
3. $G$ is a Frobenius group with kernel $N$ and cyclic complement of order 4,
4. $G \cong D_8$ or $Q_8$,
5. $|G/N| = 4$ and $G$ is a Frobenius group with complement $Q_8$,
6. $|G/N| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$. In this case, $N$ is of odd order and $N$ has a normal and abelian 3-complement,
7. $|G/N| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ and $|C_G(z)| = 12$ and in this case, either $G$ has a normal 2-complement or $G/O_2(G) \cong S_4$,
8. $|G/N| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = |C_G(z)| = 8$ and in this case, either $G/O_2(G) \cong GL(2, 3)$ with abelian $O_2(G)$, or $G/O_2(G)$ is isomorphic to a non-abelian group of order 16.
LEMMA 4.8 ([20, Lemma 2.7]). Let $G$ be a meta-abelian group. If $[G : G'] = p$, then $G$ is a Frobenius group with kernel $G'$ and complement of order $p$.

The following result will turn out to be useful in handling the case that $G/F(G)$ is abelian.

THEOREM 4.9. Let $G$ be a non-nilpotent group such that $G/F(G)$ is abelian. Assume that $k_G(G\setminus F(G)) = 4$. If $G$ is a V4C-group, then $G$ is one of the following groups:

1. $G = G'P$, where $G'$ is a normal abelian 2-complement of $G$, $P \in \text{Syl}_2(G)$, $|P| = 8$, $|Z(G)| = 4$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,

2. $G = (G'\langle u \rangle) \times (t)$, where $t$ is an involution and $G'\langle u \rangle$ is a Frobenius group with kernel $G'$ and complement of order 3,

3. $G = G'P$, where $G'$ is a normal abelian 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$, $F(G) = G'$, $G'G'$ is a union $xG \cup yG \cup zG \cup hG$ of four conjugacy classes satisfying $|C_G(x)| = 4$, $|C_G(y)| = 4$, $|C_G(z)| = 6$ and $|C_G(h)| = 12$.

4. $G$ is a Frobenius group with abelian kernel $G'$ and complement of order 5.

PROOF. Since $G/F(G)$ is abelian, it follows the hypothesis and by Lemma 4.3 that $G\setminus F(G) = \text{Van}(G)$, and so $[G : F(G)] \leq 5$.

Case 1. $[G : F(G)] = 2$.

In this case, the hypothesis together with Lemma 3.1 yields that $F(G)$ is abelian. Notice that $G = KP$, where $K$ is a normal abelian 2-complement of $G$ and $P \in \text{Syl}_2(G)$. Then, as $k_G(G\setminus F(G)) = 4$ and $G/F(G)$ is abelian, it follows by Lemma 4.4 that $P$ is abelian. Consequently, $G' \leq K$. Applying Lemma 3.1, we conclude that $|P| \leq 8$ and that one of the following two cases occurs: (i) $|F(G) : G'| = 1, 2$ or 4; (ii) $|F(G) : G'| = 3$.

Case 1.1. $|F(G) : G'| = 1, 2$ or 4.

First, assume that $F(G) = G'$. Since $G'$ is abelian, it follows by Lemma 4.8 that $G$ is a Frobenius group with abelian kernel $G'$ and complement of order 2, and so $k_G(G\setminus F(G)) = 1$, a contradiction.

Second, assume that $|F(G) : G'| = 2$. Then $[G : G'] = 4$, and thus $G = G'P$, where $G'$ is a normal 2-complement of $G$ and $|P| = 4$. Observe that $G/O_2(G)$ is a Frobenius group with Frobenius kernel $(G/O_2(G))' \cong G'$ and complement of order 2. Thus $k_G(G\setminus F(G)) = 2$, a contradiction.

Finally, assume that $|F(G) : G'| = 4$. Thus $[G : G'] = 8$ and so $G = G'P$, where $G'$ is a normal 2-complement of $G$ and $|P| = 8$. Clearly, $|O_2(G)| = 4$, and thus $[G/O_2(G) : (G/O_2(G))'] = 2$. Furthermore, applying again Lemma 4.8, we see that $G/O_2(G)$ is a Frobenius group with Frobenius kernel $(G/O_2(G))' \cong G'$ and complement of order 2. Thus $G$ satisfies (1) of the theorem.
CASE 1.2. \(|F(G) : G'| = 3\).

Recall now that \(G' \leq K \leq F(G)\), then \(K = F(G)\) and so \(|P| = 2\).
Observe that \(G' \setminus F(G) = xG' + yG' + zG'\), where \(x, y, z \in G' \setminus F(G)\).
Since \(k_{G'}(G') = 4\), we may assume that both \(xG' \cup yG'\) and \(zG'\) are a union of two conjugacy classes of \(G\), respectively. Set \(zG' = z^G + hG'\). We have
\[
\frac{|G|}{|C_G(x)|} + \frac{|G|}{|C_G(y)|} + \frac{|G|}{|C_G(z)|} + \frac{|G|}{|C_G(h)|},
\]
and
\[
\frac{|C_G(x)|}{|C_G(y)|} = \frac{|C_G(y)|}{|C_G(z)|} = 6.
\]
It follows that
\[
\frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(h)|}.
\]
Hence \(|C_G(x)| = |C_G(y)| = 6\). Again, \(|C_G(z)| = |C_G(h)| = 8\) and \(|C_G(h)| = 24\). As \(|P| = 2\), we reach a contradiction.

CASE 2. \(|G : F(G)| = 3\).

In this case, by Lemma 3.2, we get that \(F(G)\) is abelian. It follows by Lemma 4.1 and Lemma 4.8 that \(|F(G) : G'| = 2\). Observe that \(G' \setminus F(G) = x_1G' + x_2G' + x_3G' + x_4G'\), where \(x_1\) is a 3-element of \(G\).
We easily see that \(|C_G(x_1)| = 6\). As \(x_1\) is a 3-element of \(G\), \(|C_G(x)| = 6\) implies that \(|G_3| = 3\). Set \(P \in \text{Syl}_3(G)\). Then \(|P| = 3\). Note that \(F(G)\) is abelian. By Fitting lemma, we get \(F(G) = C_{F(G)}(P) \times [F(G), P]\). Obviously, \(C_{F(G)}(P) = Z(G)\) and \(|Z(G)| = 2\) since \(|C_G(g)| = 6\) for every \(g \in G' \setminus F(G)\).
So, \(G = B \times Z(G)\), where \(B = [F(G), P]P\). Observe that \(B\) is a Frobenius group with kernel \(B' = [F(G), P]\) and complement of order 3. Hence, \(G\) satisfies (2) of the theorem.

CASE 3. \(|G : F(G)| = 4\).

In this case, applying again Lemma 4.1, we get that \(G' = F(G)\). Clearly, \(G = KF\), where \(K \leq G'\) is a normal 2-complement of \(G\) and \(P \in \text{Syl}_3(G)\).
If \(P\) is non-abelian, then by Lemma 4.4, it is impossible (note that if \(G\) has the structure described in Lemma 4.4(2), then \(K = F(G) < G'\), but we have \(G' = F(G)\)). In the following, we suppose that \(P\) is abelian and so \(|P| = 4\). Then it follows from the hypothesis and Lemma 3.3 that \(G'\) is abelian.

We may assume that \(G' \setminus G' = xG' + yG' + zG'\), \(x, y, z \in G' \setminus G'\). Since \(k_{G'}(G') = 4\), we may assume that both \(xG' \cup yG'\) and \(zG'\) are a union of two conjugacy classes of \(G\), respectively. Set \(zG' = z^G + hG'\). We have
\[
\frac{|G \setminus F(G)|}{|C_G(x)|} + \frac{|G \setminus F(G)|}{|C_G(y)|} + \frac{|G \setminus F(G)|}{|C_G(z)|} + \frac{|G \setminus F(G)|}{|C_G(h)|},
\]
and
\[
\frac{|C_G(x)|}{|C_G(y)|} = \frac{|C_G(y)|}{|C_G(z)|} = 4.
\]
It follows that
\[
\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(h)|}.
\]
Hence $|C_G(z)| = 6$ and $|C_G(h)| = 12$ (note that $|P| = 4$). Thus $G$ satisfies (3) of the theorem.

**Case 4.** $|G:F(G)| = 5$.

Clearly, $G' = F(G)$. It follows by Lemma 4.8 that $G$ is a Frobenius group with kernel $G'$ and complement of order 5. Then by Lemma 3.5, we have that $G'$ is abelian, and so $G$ satisfies (4) of the theorem. The proof is completed.

5. **Proof of Theorem 1.2**

For a finite solvable group $G$, we define characteristic subgroup $F_i(G)$ by letting $F_1(G) = F(G)$, the unique largest nilpotent normal subgroup of $G$, and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$. The nilpotent length (or the Fitting height) of a group $G$, denoted by $nl(G)$, is the smallest number $l$ for which $F_l(G) = G$.

The following two results, which can be found in [14], will turn out to be useful in proof of Theorem 1.2.

**Lemma 5.1.** Let $G$ be a solvable group. Then $nl(G) \leq (2m(G) + 5)/3$, where $m(G)$ denotes the maximal number of conjugacy classes of $G$ on which some $\chi \in \text{Irr}(G)$ vanishes.

**Lemma 5.2.** Suppose that $nl(G) \geq 2$. If $|F_2(G):F(G)|$ is not prime, then there exists $\chi \in \text{Irr}(G)$ such that $\chi$ vanishes on at least two conjugacy classes of $G$ contained in $F_2(G) - F(G)$.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Since $G$ is solvable, the hypothesis together with Lemma 5.1 yield that $nl(G) \leq 4$. If $nl(G) = 1$, then by Corollary 2.5, $G \cong D_8$ or $Q_8$. Thus $G$ is a $V_3C$-group. Hence we may assume that $nl(G) = 2, 3$ or 4.

**Case 1.** $nl(G) = 2$.

In this case, if $G/F(G)$ is non-abelian, then by Corollary 2.5, $G/F(G) \cong D_8$ or $Q_8$. Set $N/F(G) = (G/F(G))'$. We have that $[G : N] = 4$ and that $G/N \subseteq \text{Van}(G)$. Then it follows by Lemma 4.1 that $N = G'$, and thus $G = KP$, where $K < G'$ is a normal 2-complement of $G$ and $P \in \text{Syl}_2(G)$. Thus Lemma 4.4 implies that $G$ is a Frobenius group with complement $Q_8$ (note that $nl(G) = 2$). Thus $G$ satisfies (1) of the Theorem. Hence we assume that $G/F(G)$ is abelian.

Since $G/F(G)$ is abelian, it follows by Lemma 4.3 that $G/F(G) \subseteq \text{Van}(G)$, and so $k_G(G/F(G)) \leq 4$. If $k_G(G/F(G)) = 1$, then $G$ is a Frobenius group with kernel $G'$ and complement of order 2, and thus $G$ is a $V_3C$-group. Therefore, $k_G(G/F(G)) = 2, 3$ or 4.

**Case 1.1.** $k_G(G/F(G)) = 2, 3$ or 4.
In the case, note that $G' \setminus F(G) \subseteq \text{Van}(G)$, then by the hypothesis and Proposition 4.6, we have to the following three cases.

Case 1.1.1. $G$ is a Frobenius group with kernel $G'$ and complement of order 3.

In this case, by Theorem 3.4, $G'$ is abelian. Thus $G$ is a $V3C$-group.

Case 1.1.2. $G = G'P$, where $G' \leq F(G)$ and $G'$ is a normal 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$.

In this case, observe that $F(G) = G' \times O_2(G)$ and $O_2(G) = Z(G)$. Observe that $k_{G/Z(G)}[G/Z(G) \setminus F(G)/Z(G)] = 1$, then $G/Z(G)$ is a Frobenius group with Frobenius kernel $(G/Z(G))' \cong G'$ and complement of order 2. Then $G$ is of type (5) of Theorem 1.4, so a $V3C$-group.

Case 1.1.3. $G = KP$, where $K \leq G'$ is a normal abelian 2-complement of $G$, $P \in \text{Syl}_2(G)$, $P \cong D_8$ or $Q_8$, $|G : G'| = 4$, and $G'$ is abelian, $G' \setminus G$ is a union $x^G \cup y^G \cup z^G \cup h^G$ of four conjugacy classes satisfying $|C_G(x)| = 4$, $|C_G(y)| = 4$, $|C_G(z)| = 6$ and $|C_G(h)| = 12$.

As $k_{G(G')}(F(G)) = 2$, we get that $|G : F(G)| = 2$ and $|F(G) : G'| = 2$.

Clearly, $F(G) = K \times O_2(G)$ and so $F(G)$ is abelian. Note that $12 | |F(G)|$ and hence $12 | |C_G(g)|$ for any $g \in F(G)$ yielding a contradiction.

Case 1.2. $k_{G(G')}(F(G)) = 3$. Thus $G' \setminus F(G)$ is a union $x^G \cup y^G \cup z^G$ of three conjugacy classes of $G$.

Assume that $G$ is of type (8) of Proposition 4.7. Recall that $\text{Van}(GL(2,3))$ contains 6 conjugacy classes of $GL(2,3)$ (see [6, P.161]), hence we may assume that $G'/O_2(G)$ is isomorphic to a non-abelian group of order 16. It follows by Corollary 2.5 that the case also does not occur.

Assume that $G$ is of type (5) of Proposition 4.7. Then $|G : F(G)| = 4$ and $G$ is a Frobenius group with complement $Q_8$, which is impossible (as $F(G)$ is the Frobenius kernel). From the hypothesis and Proposition 4.7, we only need to consider the following three cases:

Case 1.2.1. $G$ is a Frobenius group with abelian kernel $G'$ and cyclic complement of order 4.

In this case, $G$ is of type (4) of Theorem 1.4, so a $V3C$-group.

Case 1.2.2. $|G : F(G)| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$, and $F(G)$ is of odd order and has an abelian 3-complement.

In this case, applying the hypothesis and Lemma 3.1, we get that $F(G)$ is abelian. If $G' = F(G)$, then Lemma 4.8 yields that $G$ is a Frobenius group with kernel $G'$ and cyclic complement of order 2, and thus $G$ is of type (1) of Theorem 1.4, so a $V3C$-group. Hence $G' < F(G)$. Furthermore, applying again by Lemma 4.1, we have that $|F(G) : G'| = 3$.

Recall that $G' \setminus F(G) = xG' + yG' + zG'$, where $x, y, z \in G' \setminus F(G)$. Then by the second orthogonality relation we have

$$6 = |C_G(g)| = |G'/G'| + \sum \{ |\chi(g)|^2 \mid \chi \in \text{Irr}_1(G) \},$$

for all $g \in G \setminus F(G)$. Hence $\chi(g) = 0$ for all $g \in G \setminus F(G)$ and all $\chi \in \text{Irr}_1(G)$. 


Set $P \in \text{Syl}_2(G)$. Note that $|G : F(G)| = 2$ and that $F(G)$ is of odd order. Let $P = \langle t \rangle$, where $t$ is an involution. By Fitting Lemma, we have $F(G) = C_{F(G)}(P) \times [F(G), P]$. Obviously, $C_{F(G)}(t) = C_{F(G)}(P) = Z(G)$. Since $|C_G(g)| = 6$ for every $g \in G \setminus F(G)$, we conclude that $|Z(G)| = 3$. So, $G = B \times Z(G)$, where $B = [F(G), P]P$. Observe that $B$ is a Frobenius group with kernel $B' = G' = [F(G), P]G'$ and complement of order 2. Then $G$ is of type (7) of Theorem 1.4, so a V3C-group.

Case 1.2.3. $|G : F(G)| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ and $|C_G(z)| = 12$.

In this case, it follows by Lemma 3.1 that $F(G)$ is abelian. Recall now that $G = KP$, where $K \leq G'$ is a normal 2-complement of $G$ and $P \in \text{Syl}_2(G)$. It follows by Lemma 4.4 that $P$ is abelian (note that $|G : F(G)| = 2$), and so $K = G'$ and $|P| = 4$.

Notice that $|G : F(G)| = 2$ and $F(G)$ is abelian. It easily follows that $F(G) = G' \times Z(G)$ and that $Z(G)$ is of order 2. Then we have that $[G/Z(G) : (G/Z(G))'] = [G/Z(G) : F(G)/Z(G)] = 2$, and thus it follows by Lemma 4.8 that $G$ is of type (5) of Theorem 1.4, so a V3C-group.

Case 1.3. $k_G(G \setminus F(G)) = 4$.

Note that $G/F(G)$ is abelian and $G/F(G) = \text{Van}(G)$. Then applying Theorem 4.9, $G$ satisfies (2), (3), (4) or (5) of Theorem 1.2.

Case 2. $nl(G) = 3$.

Form the proof of the case when $nl(G) = 2$, we see that if $nl(G) = 2$ then $G/F(G) \subseteq \text{Van}(G)$ and $F(G)$ is abelian. So, if $nl(G) = 3$, then $G/F(G) \setminus F_2(G) \subseteq \text{Van}(G/F(G))$ and $F_2(G)/F(G)$ is abelian. It follows by Lemma 4.3 that $F_2(G)/F(G) \subseteq \text{Van}(G)$, and thus $G/F(G) \subseteq \text{Van}(G)$ (note that $G/F_2(G) \subseteq \text{Van}(G)$). Hence the hypothesis yields that $k_{G/F(G)}(G/F(G)) \leq 5$. Since $nl(G/F(G)) = 2$, it follows by [18] that $G/F(G)$ is isomorphic to one of the following groups: $S_3$, $D_{10}$, $A_4$, $D_{14}$, $(C_4, C_5)$ or $(C_3, C_7)$.

From the proof of paragraph above, we have that both $G/F_2(G)$ and $F_2(G)/F(G)$ are contained in $\text{Van}(G)$. Since $F_2(G)/F(G)$ contains at least one conjugacy class of $G$, it follows by the hypothesis that $k_G(G \setminus F_2(G)) = 2$ or 3 (if $k_G(G \setminus F_2(G)) = 1$, then $G$ is a Frobenius group with kernel $G'$ and complement of order 2, and thus $G$ is a V3C-group.).

Case 2.1. $k_G(G \setminus F_2(G)) = 2$.

In this case, by Proposition 4.6, we have to discuss the following two cases:

Case 2.1.1. $|F_2(G) : G'| = 2$, $|G : F_2(G)| = 2$, $G = G'P$, where $G'$ is a normal 2-complement of $G$, and $P \in \text{Syl}_2(G)$, $|P| = 4$.

Note that $[G : F_2(G)] = 2$ and $k_G(F_2(G)/F(G)) \leq 2$, we have that $G/F(G) \cong S_4$ or $D_{10}$, and thus $|G/F(G) : (G/F(G))'| = 2$. If $O_2(G) = 1$, then $F(G) \leq G'$ and so $[G/F(G) : (G/F(G))'] = |G : G'| = 4$, a contradiction. Hence $|O_2(G)| = 2$ and so $O_2(G) = Z(G)$. Clearly,

$$k_{G/Z(G)}(G/Z(G) \setminus F_2(G)/Z(G)) = 1,$$
so $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2. Then $G'$ is abelian and thus $G' \leq F(G)$, a contradiction (note that $\text{nl}(G) \geq 3$).

**Case 2.1.2.** $G' = F_2(G)$, $|G : G'| = 2$ and $G/O_{2'}(G) \cong S_4$.

Write $G = G/O_{2'}(G)$. Then $F_2(G)/O_{2'}(G) = F_2(G)$. Observe that $G/F(G) \cong S_3$ or $D_{10}$; thus $|O_{2'}(G)| = 4$ and so $F(G) = F(G)$. Hence we easily see that $G/F(G) \cong S_3$ and that $F(G) = O_{2'}(G) 	imes O_{2'}(G)$.

Assume that $F_2(G)/F(G) = x^G + y^G$, where $x$ is a 3-element of $G$. Observe that $|C_G(x)| = |C_G(y)| = 6$ and so $x$ is an element of order 3. We easily see that $x$ commutes with an involution $t$ in $F(G)$, we obtain a contradiction (note that $F_2(G)$ is a Frobenius group with kernel $F(G)$). The contradiction shows that $k_{G}(F_2(G)/F(G)) = 1$ and so $k_{G}(F_2(G)) = 3$.

Recall that $G/F(G) \cong S_3$; then by Proposition 4.7, $G \cong S_4$ and thus $G$ is of type (6) of Theorem 1.4, so a $V3C$-groups.

**Case 2.2.** $k_{G}(F_2(G)) = 3$.

Recall that $\text{nl}(G) = 3$. We then in a position to apply Proposition 4.7 to $G$, with $F_2(G)$ playing the role of $N$, obtaining that $G$ is of types (6), (7) or (8) of Proposition 4.7. In particular, $[G : F_2(G)] = 2$. On the other hand, it easily see that $k_{G}(F_2(G)/F(G)) = 1$, and so $F_2(G)/F(G)$ contains exactly two conjugacy classes of $G/F(G)$. Hence $G/F(G) \cong S_3$. Set $F_2(G)/F(G) = g^G$ such that $g$ is a 3-element. Observe that $|C_G(g)| = 3$ and thus $|G| = 3$.

Recall that $k_{G}(F_2(G)) = 3$ and $G/F(G) \cong S_3$; thus $|G : F_2(G)| = 2$ and $G$ satisfies (6) or (7) of Proposition 4.7. Then $G$ contains an element $x$ with $|C_G(x)| = 6$, which shows that there exists an element $y$ in $F_2(G)/F(G)$ such that $C_G(y)$ contains an involution, a contradiction.

**Case 3.** $\text{nl}(G) = 4$.

Then $\text{nl}(G/F(G)) = 3$ and thus $G/F(G) \cong S_4$. Consequently, $G/F(G)/F_2(G)/F(G) \subseteq \text{Van}(G/F(G))$.

It follows that $G/F_2(G) \subseteq \text{Van}(G)$. Note that $F_2(G)/F(G)$ is an elementary abelian group of order 4, then Lemma 4.3 together with Lemma 5.2 yield that $F_2(G)/F(G) \subseteq \text{Van}(G)$ and that $k_{G}(F_2(G)/F(G)) \geq 2$, thus we reach a contradiction (recall that $k_{G}(F_2(G)) \geq 3$). The contradiction completes the proof.

**Lemma 5.3 ([1, Theorem 2.3]).** There is some vanishing sum $v_1 + v_2 + \ldots + v_n = 0$ of $n$ $n$-th roots of unity if and only if $n$ is a linear combination, with non-negative integer coefficients, of the prime divisors of $m$.

**Corollary 5.4.** Suppose that $\chi(1) = 2$ or 4 for all $\chi \in \text{Irr}_1(G)$. Then $\text{Van}(G)$ cannot contain elements of odd orders.

Let $\chi$ be an irreducible character of $G$. Note that if $x \in v(\chi)$, $z \in Z(G)$, then $xz \in v(\chi)$. Indeed, if $D$ is a representation of $G$ with character $\chi$, then
\[ D(xz) = D(x)D(z) = (\lambda(z)I)D(x), \] where \( \lambda \) is a linear character of \( Z(G) \) and \( I \) is the identity matrix with degree \( \chi(1) \). So \( \chi(xz) = tr(D(xz)) = \lambda(z)\chi(x) = 0. \)

Remark 5.5. We show that \( G \) is one of types except for (1) and (5) in Theorem 1.2, then \( G \) is a \( V4C \)-group (but not a \( V3C \)-group). Suppose that \( G \) is of type (2) of Theorem 1.2. Then by Corollary 5.4, \( F(G) \) does not contain vanishing elements of \( G \) (note that \( F(G) = G' \times Z(G) \)). Note that \( |G : F(G)| = 2 \), thus by Lemma 4.3, \( G/F(G) = \text{Van}(G) \). Set \( G/F(G) = n_1^G + \cdots + n_s^G \). Then, we get

\[
|G/F(G)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|}.
\]

It follows that

\[
\frac{1}{2} = \frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|}.
\]

Since \( G/Z(G) \) is a Frobenius group with kernel \( (G/Z(G))' \cong G' \) and complement \( P/Z(G) \) of order 2, \( n_i \) acts fixed point freely on \( G' \), and so \( |C_G(n_i)| = 8 \) (note that \( P \) is abelian). Hence \( s = 4 \). If \( G \) satisfies (4) of Theorem 1.2, then by Corollary 5.4, \( G \) is a \( V4C \)-group (but not a \( V3C \)-group). Clearly, the same is true if \( G \) is of type (3) in Theorem 1.2.

It is worth mentioning that there exists Frobenius groups \( G \) with complement of order 5 and abelian kernel \( N \), such that \( N \cap \text{Van}(G) \) is non-empty (see [1, Example 2]), so \( G \) is not a \( V4C \)-group.

In the following, we give three groups \( G_1, G_2 \) and \( G_3 \) satisfying (2), (3) and (4) in Theorem 1.2, respectively (see [18]).

\[
G_1 = [C(3)], \quad C(8) = \langle a, b \mid a^8 = b^3 = 1, a^{-1}ba = b^{-1} \rangle,
\]

\[
G_2 = A_4 \times C(2), \quad G_3 = [C(15)].
\]

\[
C(4) = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, c^{-1}ac = a^2, c^{-1}bc = b^{-1} \rangle.
\]

However, if \( G \) satisfies (1) of Theorem 1.2, that is, \( G \) is a Frobenius group with kernel \( M \) and complement \( Q_8 \), then we do not know whether \( M \cap \text{Van}(G) \) is empty or not.

6. Non-solvable \( V4C \)-groups

In this section, we study the non-solvable \( V4C \)-groups.

Let \( p \) be a prime number. Recall that a character \( \chi \in \text{Irr}(G) \) is said to be of \( p \)-defect zero if \( p \) does not divide \( |G|/\chi(1) \). By a fundamental result of Brauer (see [8, Theorem 8.17]), if \( \chi \in \text{Irr}(G) \) is of \( p \)-defect zero then, for every element \( g \in G \) such that \( p \) divides \( o(g) \), we have \( \chi(g) = 0 \).

The following Lemma comes from [3, Proposition 2.1].

**Lemma 6.1.** Let \( G \) be a non-abelian simple group and \( p \) a prime number. If \( G \) is of Lie type, or if \( p \geq 5 \), then there exists \( \chi \in \text{Irr}(G) \) of \( p \)-defect zero.
Lemma 6.2. Let $G$ be a non-abelian simple group. If $G$ is a $V4C$-group, then $G$ is isomorphic to $A_5$.

Proof. Let $G$ be a simple $V4C$-group. Let $G \cong A_n$ for some $n \geq 14$. Then $\{5, 10, 7, 14, 11\} \subseteq \pi_e(G)$ and thus by Lemma 6.1, we obtain a contradiction. Hence the hypothesis implies that $G \cong A_n$ for some $n \leq 13$, and so $G \cong A_5$ from [2].

By [2], $G$ cannot be a sporadic simple groups. By the classification theorem of the finite simple groups we can now suppose that $G$ is a simple group of Lie type. Then, by Lemma 6.1, for each prime factor $p$ of $|G|$ there exists some $\chi \in \text{Irr}_1(G)$ such that $\chi$ is of $p$-defect zero. Hence any non-identity element of $G$ is contained in $\text{Van}(G)$. It follows by the hypothesis that $G$ consists of five conjugacy classes, and then, by [18], $G$ is isomorphic to $A_5$. \qed

Lemma 6.3. Let $G$ be non-solvable group. If $G$ is a $V4C$-group, then $G$ has the unique non-abelian composite factor.

Proof. By induction, we may assume that $\text{Sol}(G)$, the maximal solvable normal subgroup of $G$, is trivial. Let $N$ be a (non-solvable) minimal normal subgroup of $G$. If $N$ is not a non-abelian simple group, then $N = N_1 \times \ldots \times N_s$ is a direct product of isomorphic simple groups $N_i$, where $s \geq 2$. If $p \geq 5$, then there exists $\theta_i \in \text{Irr}(N_i)$ of $p$-defect zero (see Lemma 6.1), and set $\theta = \theta_1 \times \ldots \times \theta_s$.

Let $\chi_0$ be an irreducible constituent of $\theta^G$, let $x_1 \in N_1$ be of a prime order $p$, $x_2 \in N_2$ be of a prime order $q$. Notice that $x_1^G \subseteq N_1 \cup \ldots \cup N_s$.

So $x_1x_2$ is not conjugate to $x_1$. Clearly, $\theta^g$ is of $p$-defect zero for any $g \in G$, then we have

$$\theta^g(x_1) = \theta^g(x_1x_2) = 0.$$ 

This implies that

$$\chi_0(x_1) = \chi_0(x_1x_2) = 0.$$ 

Then the hypothesis yields that $N_1$ has only one prime divisor $p$ greater than 3, that $N_1$ is a simple $K_3$-group (a simple group $G$ is called a simple $K_3$-group if the number of prime factors of $|G|$ is 3), and also that $N_1$ has no irreducible character of 2-defect zero or 3-defect zero. By [2] and [5], we obtain a contradiction. Hence $N$ is a simple group.

Suppose that $G/N$ is non-solvable. Note that $\text{out}(N)$ is solvable by the classification of the finite simple groups, it follows that $C_G(N)$ is non-solvable and hence contains a non-solvable minimal normal subgroup $M$ of $G$ as $\text{Sol}(C_G(N)) = 1$. Set $T = M \times N$. Let $\psi \in \text{Irr}(M)$ be $q$-defect zero, and let $\theta \in \text{Irr}(N)$ be a $p$-defect zero, where $q, p$ are prime divisors of $|M|$ and $|N|$, respectively. Let $x_1, x_2 \in M$ be of order $q, s$, respectively, where $s \neq p$ and
s ≠ q. Let \( y_1, y_2 \in N \) be of order \( p, r \), respectively, where \( r \neq p \) and \( r \neq q \). Then for any irreducible constituent \( \chi \) of \( (\psi \times \theta)^G \), we see that
\[
\chi(x_1) = \chi(x_1 y_1) = \chi(x_1 y_2) = \chi(y_1) = \chi(y_1 x_2) = 0.
\]
The contradiction completes the proof.

**Lemma 6.4.** Let \( G \) be non-solvable group. If \( G \) is a \( V4C \)-group, then \( G \) is perfect (i.e. \( G = G' \)).

**Proof.** Otherwise, we may assume that \( G' < G \). By Lemma 6.3, there exist two normal subgroups \( N \) and \( M \) of \( G \) such that \( N < M \leq G' \) and \( M/N \) is a non-abelian simple group. From the argument of the proof of Lemma 6.3, we get that \( (M/N) \cap \text{Van}(G/N) \) is non-empty, and thus \( G' \cap \text{Van}(G) \) is also non-empty.

Suppose that there exists \( \chi \in \text{Irr}(G) \) such that \( \chi_{G'} \) is not irreducible. It follows by [8, Theorem 6.22] that \( G \) has a proper subgroup \( H \) such that \( G' \leq H < G \) and \( G' \backslash H \subseteq \nu(\chi) \). Since \( G' \cap \text{Van}(G) \) is also non-empty, it follows by the hypothesis that \( k_G(G' \backslash H) \leq 3 \). If \( k_G(G' \backslash H) \leq 2 \), then \( G \) is solvable (see [15, Theorem 2.2]), a contradiction. Hence we may assume that \( k_G(G' \backslash H) = 3 \). Note that \( G \) is non-solvable; it follows by [15, Theorem 3.5] that \( G \) has a normal subgroup \( E \) such that \( G/E \cong S_5 \) or \( M_{10} \), then we obtain a contradiction from [2] and the proof is complete.

**Lemma 6.5 ([12, Theorem]).** Suppose that a group \( G \) contains a subgroup \( X \) of order \( 3 \) such that \( C_G(X) = X \). If for every \( g \in G \), the subgroup \( \langle X, X^g \rangle \) is finite, then one of the following holds:

1. \( G = NN_G(X) \) for a periodic nilpotent subgroup \( N \) of nilpotent class \( 2 \), and \( NX \) is a Frobenius group with kernel \( N \) and complement \( X \).
2. \( G = NA \), where \( A \) is isomorphic to \( A_5 \cong SL_2(4) \) and \( N \) is a normal elementary Abelian \( 2 \)-subgroup, here, \( N \) is a direct product of order \( 16 \) subgroups normal in \( G \) and isomorphic to the natural \( SL_2(4) \)-module of dimension \( 2 \) over a field of order \( 4 \).
3. \( G \) is isomorphic to \( L_2(7) \).

Next we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Clearly our hypothesis is inherited by any factor group. By Lemma 6.2, it suffices to show that \( G \) is a non-abelian simple group. Assume that \( G \) is a minimal counter-example.

First, we show that \( G \) has a minimal normal subgroup \( N \) such that \( G/N \) is non-solvable. Otherwise, we may assume that \( G \) has the unique minimal normal subgroup \( N \) such that \( G/N \) is solvable. Recall that \( (G/N)' = G'N/N \), thus it follows by Lemma 6.4 that \( (G/N)' = G/N \), which yields that \( N = G \), a contradiction (note that \( G \) is a minimal counter-example). Hence \( G/N \) is non-solvable. Now it follows by induction that \( G/N \) is a non-abelian simple.
group. Applying Lemma 6.2, we conclude that $G/N \cong A_5$. Then $G/N$ has exactly one conjugacy class of elements of order 3. Choose a 3-element $a$ of $G$ such that $(aN)^{G/N}$ is the class of elements of order 3 in $G/N$. Set $A = (aN)^{G/N}$.

Recall that $k_{G/N}(\text{Van}(G/N)) = 4$; thus $k_G(A) = 1$. Then each $\chi \in \text{Irr}(G/N)$ vanishes on $A$. By the second orthogonality relation we have

$$|C_G(a)| = |C_{G/N}(aN)| = 3.$$ 

Hence $G$ has an element $a$ with $C_G(a)$ of order 3. Hence $G$ satisfies the hypothesis of Lemma 6.5.

If $G$ is the group in Lemma 6.5(1), then $G$ is solvable, which is a contradiction.

Suppose that $G$ has the structure described in Lemma 6.5(3). Then we obtain a contradiction from [2]. Hence $G$ is the group in Lemma 6.5(2). But $G$ is not a $V4C^*$-group (see [18, p. 310]), which is the final contradiction. □

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