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TRANSLATION SURFACES IN THE GALILEAN SPACE

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Abstract. In this paper we describe, up to a congruence, translation surfaces in the Galilean space having constant Gaussian and mean curvatures as well as translation Weingarten surfaces. It turns out that, contrary to the Euclidean case, there exist translation surfaces with constant Gaussian curvature K that are not cylindrical surfaces, and translation surfaces with constant mean curvature $H \neq 0$ that are not ruled.

1. Introduction

In this paper we describe translation surfaces in the special ambient space – the Galilean space. We are specially interested in the analogues of the results from the Euclidean space concerning translation surfaces having constant Gaussian K and mean curvature H, and translation surfaces that are Weingarten surfaces as well.

A *translation surface* is a surface that can locally be written as the sum of two curves α, β

$$\mathbf{x}(u,v) = \alpha(u) + \beta(v).$$

Translation surfaces in the Euclidean and Minkowski space having constant Gaussian curvature are described in [5]:

Theorem 1.1. *Let S be a translation surface with constant Gaussian curvature K in 3-dimensional Euclidean space or 3-dimensional Minkowski space. Then S is congruent to a cylindrical surface (i.e., generalized cylinder), so $K = 0$.*

Translation surfaces having constant mean curvature, in particular zero mean curvature, are described in [5] as well:

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Theorem 1.2. *Let S be a translation surface with constant mean curvature $H \neq 0$ in 3-dimensional Euclidean space. Then S is congruent to the following surface or a part of it (class of cylindrical surfaces)*

$$z = \frac{-\sqrt{1+a^2}}{2H}\sqrt{1-4H^2x^2} - ay, \quad a \in \mathbf{R},$$

and for $H = 0$ to a plane or to the Scherk minimal surface

$$z = \frac{1}{a}\log(\cos(ax)) - \frac{1}{a}\log(\cos(ay)), \quad a \in \mathbf{R} \setminus \{0\}.$$

Similar results hold in Minkowski space ([5]).

Additionally, we state a general result from Euclidean space where the following theorem holds ([4], p. 254):

Theorem 1.3 (S. Lie). *A surface is a minimal surface if and only if it can be represented as a translation surface whose generators (i.e., translated curves) are isotropic (minimal) curves (i.e., curves having arc-length 0).*

Weingarten surfaces are surfaces whose Gaussian and mean curvature satisfy a functional relationship (of class C^0 at least). The class of Weingarten surfaces contains already mentioned surfaces of constant curvatures K, H, as well as surfaces having both curvatures as functions of a single parameter. In the Euclidean and Minkowski space the latter surfaces are helicoidal surfaces, i.e., surfaces obtained by a rotation of a profile curve around an axis and its simultaneous translation along the axis so that the speed of translation is proportional to the speed of rotation.

For the translation Weingarten surfaces in Euclidean space the following theorem holds ([1]):

Theorem 1.4. *A translation surface in $\mathbf{R}^3$ is a Weingarten surface if and only if it is either (a part of)*

1. *a plane,*
2. *a cylindrical surface,*
3. *the minimal surface of Scherk,*
4. *an orthogonal elliptic paraboloid parametrized by $\mathbf{x}(s,t) = (s, t, a(s^2 + t^2))$.*

Counterparts of these results for surfaces in Minkowski space can be found in [1]. Furthermore, in [3] Weingarten quadric surfaces in Euclidean 3-space are studied and in [8] resp. [11] polynomial translation Weingarten surfaces resp. polynomial translation surfaces of Weingarten types. Polynomial translation surfaces are surfaces parametrized by $\mathbf{x}(u,v) = (u, v, f(u) + g(v))$, where f and g are polynomials.

Translation surfaces have been extensively studied in some other ambient spaces, see for example [7] on affine translation surfaces and [6] on minimal translation surfaces in hyperbolic geometry.

Ruled Weingarten surfaces in the Galilean space have been studied in [9].

2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space defined from a 3-dimensional projective space $\mathcal{P}(\mathbf{R}^3)$ with the absolute figure that consists of an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f the line (absolute line) in ω and I the fixed elliptic involution of points of f. We introduce homogeneous coordinates in G_3 in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : -x_2)$. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, distance between points $P_i = (x_i, y_i, z_i)$, $i = 1, 2$, is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2. \end{cases} \tag{2.1}$$

The group of motions of G_3 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \bar{x} &= a + x \\ \bar{y} &= b + cx + y\cos\varphi + z\sin\varphi \\ \bar{z} &= d + ex - y\sin\varphi + z\cos\varphi. \end{aligned}$$

With respect to the absolute figure, there are two types of lines in the Galilean space – isotropic lines which intersect the absolute line f and non-isotropic lines which do not. A plane is called Euclidean if it contains f, otherwise it is called isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, whereas Euclidean planes are of the form $x = k$, $k \in \mathbf{R}$. The induced geometry of a Euclidean plane is Euclidean and of an isotropic plane isotropic (i.e., 2-dimensional Galilean or flag-geometry).

More about the Galilean geometry can be found in [10].

A C^r-surface S, $r \geq 1$, immersed in the Galilean space, $\mathbf{x} : U \to S$, $U \subset \mathbf{R}^2$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, has the following first fundamental form

$$I = (g_1 du + g_2 dv)^2 + \varepsilon(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2),$$

where the symbols $g_i = x_i$, $h_{ij} = \tilde{\mathbf{x}}_i \cdot \tilde{\mathbf{x}}_j$ stand for derivatives of the first coordinate function $x(u, v)$ with respect to u, v and for the Euclidean scalar product of the projections $\tilde{\mathbf{x}}_k$ of vectors $\mathbf{x}_k$ onto the yz-plane, respectively. Furthermore,

$$\varepsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is non-isotropic}, \\ 1, & \text{if direction } du : dv \text{ is isotropic}. \end{cases}$$

In every point of a surface there exists a unique isotropic direction defined by $g_1 du + g_2 dv = 0$. In that direction, the arc length is measured by

$$ds^2 = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2 = \frac{h_{11}g_2^2 - 2h_{12}g_1g_2 + h_{22}g_1^2}{g_1^2}dv^2,$$

$$= \frac{W^2}{g_1^2}dv^2, \quad g_1 \neq 0.$$

A surface is called admissible if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$ or $g_2 \neq 0$ holds. An admissible surface can always locally be expressed as

$$z = f(x, y).$$

The Gaussian K and mean curvature H are C^{r-2}-functions, $r \geq 2$, defined by

$$K = \frac{LN - M^2}{W^2}, \quad H = \frac{g_2^2\, L - 2g_1g_2\, M + g_1^2 N}{2W^2},$$

where

$$L_{ij} = \frac{x_1 \mathbf{x}_{ij} - x_{ij}\mathbf{x}_1}{x_1} \cdot \mathbf{N}, \quad x_1 = g_1 \neq 0.$$

We will use $L_{ij},\ i, j = 1, 2$, for L, M, N if more convenient. The vector $\mathbf{N}$ defines a normal vector to a surface

$$\mathbf{N} = \frac{1}{W}(0, -x_2z_1 + x_1z_2, x_2y_1 - x_1y_2),$$

where $W^2 = (x_2\mathbf{x}_1 - x_1\mathbf{x}_2)^2$.

3. Translation surfaces in the Galilean space

For counterparts of Euclidean results, we will consider translation surfaces that are obtained by translating two planar curves. In order to obtain an admissible surface, translated curves can be, with respect to the absolute figure, either

Type 1. a non-isotropic curve (having its tangents non-isotropic) and an isotropic curve or,

Type 2. non-isotropic curves.

There are no motions of the Galilean space that carry one type of a curve into another, so we will treat them separately.

Translation surfaces of the Type 1 in the Galilean space can be locally represented by

$$z = f(x) + g(y),$$

which yields the parametrization

$$\mathbf{x}(x, y) = (x, y, f(x) + g(y)).$$

One translated curve is a non-isotropic curve in the plane $y = 0$

$$\alpha(x) = (x, 0, f(x))$$

and the other is an isotropic curve in the plane $x = 0$

$$\beta(y) = (0, y, g(y)).$$

The Gaussian curvature is then given by

$$K = \frac{f''(x)g''(y)}{(1 + g'^2(y))^2}$$

where by $'$ we have denoted derivatives with respect to corresponding variables. Contrary to the Euclidean case, since variables x, y in the function K can be separated, K is constant if and only if either

$$f''(x) = const. \neq 0 \text{ and } \frac{g''(y)}{(1 + g'^2(y))^2} = const. \neq 0,$$

or

$$f''(x) = 0 \text{ or } \frac{g''(y)}{(1 + g'^2(y))^2} = 0.$$

Therefore, in the first case we obtain $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ with $g(y)$ which is the solution of the ordinary differential equation

$$\frac{g'(y)}{1 + g'^2(y)} + \arctan g'(y) = Ay + B, \quad A, B \in \mathbf{R}.$$

By reparametrizing the translated curve β by the arc-length u, $\beta(u) = (0, h(u), k(u))$, $h'^2(u) + k'^2(u) = 1$, we get $K = f''(x)k''(u)$ and we can solve the obtained ordinary differential equation. We get

$$\begin{aligned} k(u) &= \frac{1}{2}Au^2 + Bu + C, \\ h(u) &= \frac{Au + B}{2A}\sqrt{1 - (Au + B)^2} + \frac{1}{2A}\arcsin(Au + B) + C_1, \end{aligned} \tag{3.1}$$

for $A, B, C, C_1 \in \mathbf{R}$. The obtained surface is a special translation surface having parabolas (i.e., parabolic circles in the Galilean geometry) as one family of translated curves.

In the second case the obtained surface is a cylindrical surface

$$z(x, y) = ax + b + g(y),$$

or (for $g(y) = cy + d$, $c, d \in \mathbf{R}$)

$$z(x, y) = f(x) + cy + d.$$

We have proved in Galilean space the counterpart of Theorem 1.1:

Theorem 3.1. *If S is a translation surface of Type 1 of constant Gaussian curvature in the Galilean space, then S is congruent to a special surface with $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ and k, h given by (3.1) ($K \neq 0$), or to a cylindrical surface ($K = 0$) having either non-isotropic or isotropic rulings.*

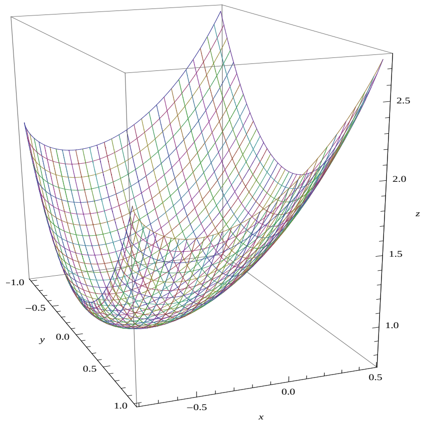

FIGURE 1. A surface of Type 1 with constant Gaussian curvature $K \neq 0$

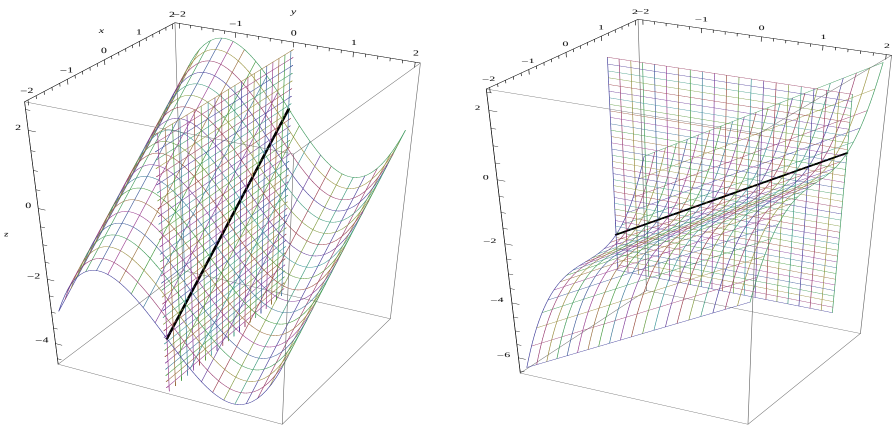

FIGURE 2. A cylindrical surface with non-isotropic and isotropic rulings which lie in an isotropic and a Euclidean plane, respectively

Furthermore, the mean curvature of a translation surface in the Galilean space is given by

$$H = \frac{g''(y)}{2(1+g'^2(y))^{\frac{3}{2}}}.$$

THEOREM 3.2. *If S is a translation surface of Type 1 of constant mean curvature $H \neq 0$ in the Galilean space, then S is congruent to a surface*

$$z = f(x) - \frac{1}{2H}\sqrt{1-(2Hy+c_1)^2} + c_2, \;\; c_1, c_2 = const. \tag{3.2}$$

PROOF. One should solve the ordinary differential equation

$$h'(y) = 2H(1 + h^2(y))^{\frac{3}{2}}, \;\; H = const.$$

where $h(y) = g'(y)$. The solution is given by $h(y) = \frac{2Hy+c_1}{\sqrt{1-(2Hx+c_1)^2}}$, $c_1 \in \mathbf{R}$. Therefore, the result follows. □

Notice that contrary to the Euclidean situation, this surface need not be ruled.

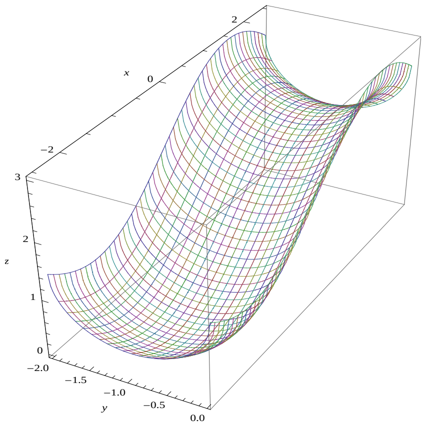

FIGURE 3. A translation surface of Type 1 with constant mean curvature with $f(x) = \sin x$

THEOREM 3.3. *If S is a translation surface of Type 1 of zero mean curvature in the Galilean space, then S is congruent to a cylindrical surface with isotropic rulings (and therefore $K = 0$)*

$$z = f(x) + ay + b, \;\; a, b \in \mathbf{R}.$$

In other words, the obtained surface is a ruled surface with rulings having the constant isotropic direction $(0, 1, a)$. This theorem agrees with the following theorem which describes minimal surfaces in the Galilean space (see [10]):

THEOREM 3.4. *Minimal surfaces in G_3 are cones whose vertices lie on the absolute line and the ruled surfaces of type C. They are all conoidal ruled surfaces having the absolute line as the directional line in infinity.*

Let now consider a surface of Type 2, i.e., a surface having both translated curves non-isotropic

$$x(u, v) = (u + v, g(v), f(u)),$$

where $\alpha(u) = (u, 0, f(u))$ is a curve in the isotropic plane $y = 0$, and $\beta(v) = (v, g(v), 0)$ is a curve in the isotropic plane $z = 0$.

The Gaussian curvature of a translation surface of Type 2 is given by

$$K = \frac{1}{W^4} f'(u) f''(u) g'(v) g''(v), \tag{3.3}$$

and the mean curvature

$$H = \frac{1}{2W^3}(f''(u)g'(v) + f'(u)g''(v)) \tag{3.4}$$

where $W^2 = f'^2(u) + g'^2(v)$. Derivatives are taken with respect to corresponding variables.

The Gaussian curvature is equal to 0 if and only if $f''(u) = 0$ or $g''(v) = 0$. Then $f(u) = au + b$ or $g(v) = cv + d,\ a, b, c, d \in \mathbf{R}$. Therefore the obtained surface is a cylindrical surface with non-isotropic rulings.

The Gaussian curvature is constant if by differentiating the expression for K with respect to u and v we get

$$\frac{\partial K}{\partial u} = h(v)\frac{W^2 e'(u) + 4e^2(u)}{W^6} = 0,$$
$$\frac{\partial K}{\partial v} = e(u)\frac{W^2 h'(v) + 4h^2(v)}{W^6} = 0,$$

where we put $e(u) = f'(u)f''(u)$, $h(v) = g'(v)g''(v)$. Previous equation are simultaneously equal to 0 if and only if $e(u) = 0$ or $h(v) = 0$. Therefore, either $f(u) = au + b,\ a, b \in \mathbf{R}$ or $g(v) = cv + d,\ c, d \in \mathbf{R}$. A translated curves in both cases is a non-isotropic line. The following theorem holds:

THEOREM 3.5. *If S is a translation surface of Type 2 of constant Gaussian curvature, then S is congruent to a cylindrical surface with non-isotropic rulings (and therefore $K = 0$).*

The mean curvature H is equal to 0 if and only if $f''(u)g'(v) + f'(u)g''(v) = 0$, i.e.,

$$\frac{f''(u)}{f'(u)} = -\frac{g''(v)}{g'(v)}$$

which implies there exists a constant $c \in \mathbf{R}$ such that

$$\frac{f''(u)}{f'(u)} = -\frac{g''(v)}{g'(v)} = c.$$

If $c = 0$, then $f''(u) = 0$, $g''(v) = 0$ which generates an isotropic plane. If $c \neq 0$ then

$$f(u) = \frac{1}{c}e^{cu} + c_1,\ \ g(v) = -\frac{1}{c}e^{-cv} + c_2,\ \ c, c_1, c_2,$$

and the surface is

$$\mathbf{x}(u, v) = (u + v, -\frac{1}{c}e^{-cv} + c_2, \frac{1}{c}e^{cu} + c_1). \tag{3.5}$$

Let us notice that by reparametrizing the obtained surface by $\bar{u} = u + v$, $\bar{v} = -\frac{1}{c}e^{-cv} + c_2$, we obtain

$$\mathbf{x}(\bar{u}, \bar{v}) = (\bar{u}, \bar{v}, -e^{c\bar{u}}\bar{v} + d), \quad d \in \mathbf{R},$$

which is a ruled surface with isotropic rulings of non-constant direction

$$\bar{v} \mapsto (\bar{u_0}, 0, d) + \bar{v}(0, 1, e^{-cu_0}).$$

Therefore, the following theorem agrees again with Theorem 3.4.

THEOREM 3.6. *If S is a translation surface of Type 2 of zero mean curvature, then S is congruent to a ruled surface of type C with isotropic rulings, which is not a cylindrical surface.*

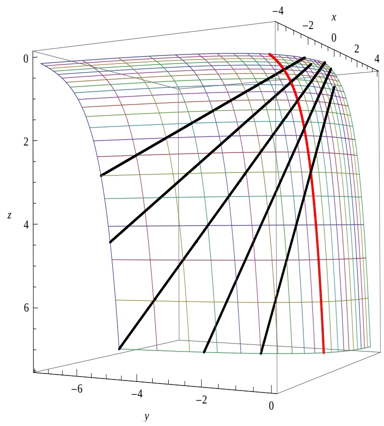

FIGURE 4. A translation minimal surface of Type 2 with traced rulings

EXAMPLE 3.7. Notice that there is an one-parametric family of translation minimal surfaces of Type 2 with a curve α lying in the plane $y = 0$, and a curve β in a plane $y\cos\varphi - z\sin\varphi = 0$ which forms the angle φ to the plane $y = 0$,

$$\beta(v) = (v, g(v)\cos\varphi, g(v)\sin\varphi).$$

In the same way as before, we can calculate

$$f(u) = \frac{1}{C}e^{Cu} + C_1, \;\; g(v) = -\frac{1}{C}e^{-Cv} + C_2, \;\; C, C_1, C_2 \in \mathbf{R}.$$

Finally, we investigate the case when the mean curvature H is constant but not equal to zero. Differentiating (3.4) with respect to u, and then the obtained expression with respect to v, we obtain

$$f'''g' + f''g' = 6Wf'f''H,$$

$$(f'''g'' + f''g'')W = 6Hf'f''g'g''.$$

The assumption $f''(x) \neq 0$ and $g''(y) \neq 0$ implies

$$6H = (\frac{f'''}{f''} + \frac{g'''}{g''})\sqrt{\frac{1}{f'^2} + \frac{1}{g'^2}}.$$

Differentiating the previous expression with respect to u, we get

$$(\frac{f'''}{f''})'\sqrt{\frac{1}{f'^2} + \frac{1}{g'^2}} - (\frac{f'''}{f''})\frac{f''}{f'^3\sqrt{\frac{1}{f'^2} + \frac{1}{g'^2}}} = 0,$$

or

$$(\frac{f'''}{f''})'\sqrt{(\frac{1}{f'^2} + \frac{1}{g'^2})^3} = 6H\frac{f''}{f'^3}. \tag{3.6}$$

Now, if we differentiate (3.6) with respect to v and if $f''(u) \neq 0$ and $g''(v) \neq 0$, we have $(\frac{f'''}{f''})' = 0$, which implies $H = 0$. That contradicts the assumption $H \neq 0$.

Therefore either $f''(u) = 0$ or $g''(v) = 0$. Let us take $f''(u) = 0$, which implies $f'(u) = a$, $a \in \mathbf{R}$. Then the function g satisfies the following ordinary differential equation

$$ag''(v) = 2H(a^2 + g'^2(v))^{3/2}.$$

By substituting $h(v) = g'(v)$, we get

$$h(v) = \frac{2aHv + c}{\sqrt{1 - (2aHv + c)^2}}, \quad c \in \mathbf{R},$$

which implies

$$g(v) = -\frac{1}{2aH}\sqrt{1 - (2aHv + c)^2} + c_1, \quad c, c_1 \in \mathbf{R}. \tag{3.7}$$

The obtained surfaces form a special class of cylindrical surfaces (ruled surface of type C) having non-isotropic rulings translated along the curve $\beta(v) = (v, g(v), 0)$ where $g(v)$ is given by (3.7).

THEOREM 3.8. *If S is a translation surface of Type 2 of constant mean curvature $H \neq 0$ in the Galilean space, then S is congruent to a cylindrical surface*

$$\mathbf{x}(u, v) = (u + v, -\frac{1}{2aH}\sqrt{1 - (2aHv + c)^2} + d, au + b), \quad a, b, c, d \in \mathbf{R}, \tag{3.8}$$

i.e.,

$$x = \frac{1}{a}(z - b) + \frac{1}{2aH}\sqrt{1 - 4a^2H^2(y - d)^2}.$$

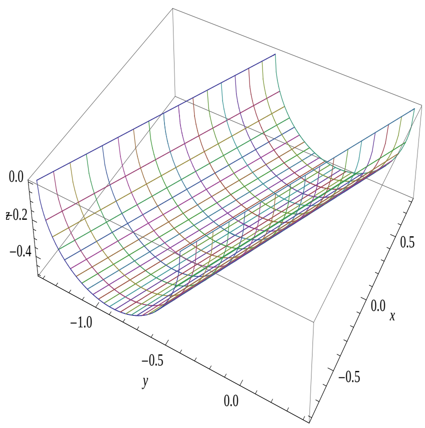

FIGURE 5. A translation surface of Type 2 with constant mean curvature

4. Translation Weingarten surfaces

In the case of translation surfaces of Type 1 the following theorem holds:

THEOREM 4.1. *A translation surface of Type 1 in the Galilean space is a Weingarten surface if and only if it is either (a part of)*

1. *an isotropic plane,*
2. *a cylindrical surface with isotropic or non-isotropic rulings,*
3. *a translation surface of constant Gaussian curvature of Theorem 3.1,*
4. *a translation surface (3.2) of constant mean curvature of Theorem 3.2,*
5. *a surface* $z = ax^2 + bx + c + g(y)$, $a, b, c \in \mathbf{R}$.

PROOF. A C^3-surface is Weingarten if and only $K_x H_y - H_x K_y = 0$. Since the mean curvature H is a function of y only, the previous equation reduces to $K_x H_y = 0$. Therefore, either $K_x = 0$ or $H_y = 0$. Obviously, an isotropic plane satisfies these conditions since its Gaussian and mean curvatures are $K = H = 0$.

The first condition $K_x = 0$ describes translation surfaces that are either of constant curvature K, i.e., a special translation surface or cylindrical surfaces with non-isotropic or isotropic rulings (Theorem 3.1), and more generally, surfaces that satisfy

$$\frac{\partial K}{\partial x} = f'''(x)\frac{g''(y)}{(1+g'^2(y))^2} = 0.$$

Therefore either $f'''(x) = 0$ or $g''(y) = 0$, which implies

$$f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbf{R},$$

or

$$g(y) = Ay + B, \quad A, B \in \mathbf{R}.$$

Surfaces $z = ax^2 + bx + c + g(y)$ are obtained by translating a parabolic circle $\alpha(x) = (x, 0, ax^2 + bx + c)$ along an isotropic curve $\beta(y) = (0, y, g(y))$. Among them, there are also (parabolic) spheres in the Galilean space.

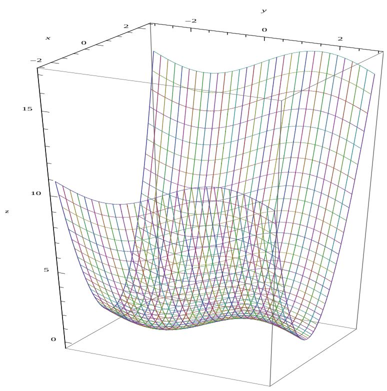

FIGURE 6. A translation Weingarten surface of Type 1 for $g(y) = \sin y$

Surfaces $z = f(x) + Ay + B$ are cylindrical surfaces with isotropic rulings.

The second equation describes the surfaces of constant mean curvature, including minimal surfaces (Theorem 3.2, Theorem 3.3). □

For the translation surfaces of Type 2, the condition $K_u H_v - K_v H_u = 0$ can be written as

$$\begin{aligned} &f'^2 f''(3f''^2 - f'f''')g'^2 g''' + f'^2 f''' g'^2 g''(-3g''^2 + g'g''') \\ &\quad + f'f''f''' g'^3 (g''^2 - g'g''') + g'g''g''' f'^3(-f''^2 + f'f''') \\ &\quad + 2f''^3 g''^2 (-g')^3 + 2f'^3 f''^2 g''^3 = 0. \end{aligned} \tag{4.1}$$

Obviously, if $f''(u) = 0$ or $g''(v) = 0$, then the relation is fulfilled. By these functions, cylindrical (ruled) surfaces with non-isotropic rulings are generated.

The expression (4.1) is analyzed with the method as in [1]. We write

$$\sum_{i=1}^{6} f_i(u) g_i(v) = 0,$$

where

$$f_1(u) = f'^2(u) f''(u) \left(3f''^2(u) - f'(u) f'''(u)\right), \qquad g_1(v) = g'^2(v) g'''(v)$$

etc. Because of the symmetry of the problem, we investigate only the possibilities for the functions f_i. First we treat the cases when one of the functions f_i is equal to 0.

If $f_6 = 0$ or $f_5 = 0$, then $f''(u) = 0$ and the equation (4.1) is fulfilled for any function g. The same happens if the first factor of f_1 or f_2 is equal to 0. Similarly for f_3.

Let us now consider the second factor of f_1,

$$3f''(u)^2 - f'(u)f'''(u) = 0. \tag{4.2}$$

Since $f'' \neq 0$, by integrating (4.2) written as $3\frac{f''}{f'} = \frac{f'''}{f''}$ we get

$$f''(u) = -\frac{1}{a^2}(f')^3(u), \ \ a \in \mathbf{R} \setminus \{0\}, \tag{4.3}$$

and therefore

$$f(u) = a\sqrt{2u+b} + c, \ \ a, b, c \in \mathbf{R}. \tag{4.4}$$

Substitution of $f'''(u)$ from the expression (4.2) in (4.1) leads to

$$f''^2 \left(3f'g_2 + 3f''g_3 + 2f'^3 g_4 + 2f''g_5 + 2f'^3 g''^3\right) = 0$$

which by using the expression (4.3) turns to

$$(f'(u))^2(3cg_3 + 2cg_5 + 2g_4 + 2g_6)(v) + 3g_2(v) = 0.$$

Now we get $g_2(v) = 0$ i.e., $-3g''(v)^2 + g'(v)g'''(v) = 0$. Therefore, if $g''(v) \neq 0$,

$$g(v) = A\sqrt{2v+B} + C, \ \ A, B, C \in \mathbf{R}. \tag{4.5}$$

By substituting (4.4), (4.5) in (4.1) we get that the relation (4.1) is fulfilled if and only if $a = A$. Therefore, the obtained surface is parametrized by

$$\mathbf{x}(u, v) = (u + v, a\sqrt{2v+B} + C, a\sqrt{2u+b} + c).$$

By reparametrizing the obtained surface by $\bar{u} = a\sqrt{2u+b} + c$, $\bar{v} = a\sqrt{2v+B} + C$ we get

$$\mathbf{x}(u, v) = \left(\frac{1}{2a^2}(\bar{u}^2 + \bar{v}^2) + C, \bar{v}, \bar{u}\right), \tag{4.6}$$

i.e.,

$$x = \frac{1}{2a^2}(y^2 + z^2) + C. \tag{4.7}$$

The obtained surface is an analogue of an orthogonal elliptic paraboloid which is a translation Weingarten in Euclidean space ([1]).

Another possibility is that a function g in this case satisfies $g''(v) = 0$. This is the case of a surface with constant mean curvature.

Let us consider the second factor of f_4, i.e., the factor $-f''(u)^2 + f'(u)f'''(u) = 0$. Integration of this expression implies that $f''(u) = Cf'(u)$.

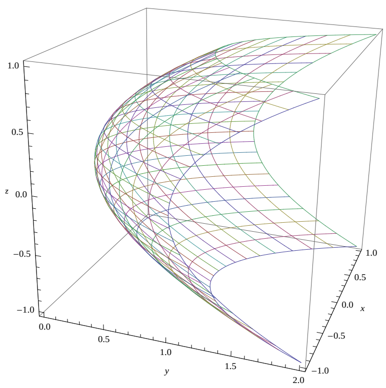

FIGURE 7. A translation Weingarten surface of Type 2 – improper affine sphere

In this case $f(u) = \frac{1}{c}e^{cu+c_1} + c_2$, $c, c_1, c_2 \in \mathbf{R}$, and f satisfies $f''(u) = cf'(u)$ as well. By substituting these conditions in (4.1), the following is obtained

$$c^2 \left(f'(u)^5(cg_1 + g_6)(v) + f'(u)^3(g_2 + cg_3 + 2cg_5)(v)\right) = 0. \tag{4.8}$$

Therefore

$$(cg_1 + g_6)(v) = 0,$$
$$(g_2 + cg_3 + 2cg_5)(v) = 0.$$

The first condition implies $g''(v)^3 + cg'(v)^2 g'''(v) = 0$, which with the second condition substituted in (4.8) gives

$$-cf'(u)^3 g'(v)g''(v)^2(cg'(v) + g''(v))^2 = 0.$$

Therefore, either $g''(v) = 0$ and the obtained surface is a cylindrical surface with non-isotropic rulings, or $cg'(v) + g''(v) = 0$ and the obtained surface is a constant mean curvature surface (3.5).

Finally, we must treat all other possibilities when f_i appears as a linear combination of other functions. For example, if f_1, f_2, f_3 are collinear with the function f_j for some j, $f_1 = a_1 f_j$, $f_2 = a_2 f_j$, $f_3 = a_3 f_j$ then $a_2 f_1 = a_1 f_2$, $a_1 f_3 = a_3 f_1$. Therefore

$$a_1 f''' = a_2 f''(3f''^2 - f'f''')$$
$$a_1 f''' = a_3 f'(3f''^2 - f'f''').$$

Previous equation are satisfied if either $f'' = 0$ or $3f''^2 - f'f'''$. Both cases lead to already obtained surfaces.

The thorough further investigation brings no new types of surfaces. Therefore:

THEOREM 4.2. *A translation surface of Type 2 in the Galilean space is a Weingarten surface if and only if it is either (a part of)*

1. *an isotropic plane,*
2. *a cylindrical surface with non-isotropic rulings,*
3. *a translation surface (3.8) of constant mean curvature of Theorem 3.8 and ruled surfaces of type C of Theorem 3.6,*
4. *a surface* $\mathbf{x}(u,v) = (u+v, a\sqrt{2v+b}+c, a\sqrt{2u+B}+C)$, $a,b,c,B,C \in \mathbf{R}$*, i.e., a surface given by (4.6) or (4.7).*

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