GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 351 – 356

FINITE p-GROUPS ALL OF WHOSE PROPER SUBGROUPS HAVE ITS DERIVED SUBGROUP OF ORDER AT MOST p

Zvonimir Janko
University of Heidelberg, Germany

Abstract. We give in Theorem 7 a complete characterization of the title groups.

Here we give a complete characterization of the title groups. This result is important for the structure theory of finite p-groups and also it solves the Problem 39 stated by Y. Berkovich in [1]. In the proofs we use partly some ideas of J. Q. Zhang and X. H. Li ([5, Proposition 3]) and V. Ćepulić and O. Pylyavska ([4, Proposition 5]). To facilitate the proof of the main result (Theorem 7), we shall first prove some auxiliary results.

Our notation is standard (see [1]) and we consider here only finite p-groups.

Proposition 1. *Let G be a title group. Then for all $x, y \in G$ such that $\langle x, y\rangle < G$ we have $\mathrm{o}([x, y]) \leq p$ and $[x, y] \in \mathrm{Z}(G)$.*

Proof. Suppose that $[x, y] \neq 1$. Let X be a maximal subgroup of G containing $\langle x, y\rangle$. Then $X' = \langle [x, y]\rangle \trianglelefteq G$ with $\mathrm{o}([x, y]) = p$ and so $[x, y] \in \mathrm{Z}(G)$. ∎

Proposition 2. *If G is a title group, then G' is abelian of order $\leq p^3$.*

Proof. We may assume that G is nonabelian. Let $X \neq Y$ be two maximal subgroups of G. Then $|X'| \leq p$ and $|Y'| \leq p$. By a result of A. Mann (Exercise 1.69(a) in [1]), $|G' : (X'Y')| \leq p$ and so $|G'| \leq p^3$. If G' would be nonabelian, then $|G'| = p^3$ and $\mathrm{Z}(G')$ (being of order p) is cyclic and so (by an elementary result of W. Burnside, see Lemma 1.4 in [1]) G' is also cyclic, a contradiction. Hence G' is abelian. ∎

2010 *Mathematics Subject Classification.* 20D15.
Key words and phrases. Finite p-groups, minimal nonabelian p-groups, commutator subgroups, nilpotency class of p-groups.

Proposition 3 (Zhang and Li). *If G is a title group and $|G'| \geq p^2$, then* $\mathrm{d}(G) \leq 3$.

Proof. Assume that $|G'| \geq p^2$. Then G is not minimal nonabelian and so there exists a maximal subgroup A with $|A'| = p$ and we have $A' \trianglelefteq G$. Suppose that $M' \leq A'$ for each maximal subgroup M of G. Then G/A' is minimal nonabelian. But then $\mathrm{d}(G/A') = 2$, $A' \leq \Phi(G)$ and so $\mathrm{d}(G) = 2$ and we are done in this case.

We may assume that G has a maximal subgroup B such that $B' \not\leq A'$. We get $|A'| = |B'| = p$ and $A' \cap B' = \{1\}$. Let $a_1, a_2 \in A$ and $a_3, a_4 \in B$ be such that $A' = \langle [a_1, a_2] \rangle$ and $B' = \langle [a_3, a_4] \rangle$. Since $|\langle a_1, a_2, a_3, a_4 \rangle'| \geq p^2$, we get $\langle a_1, a_2, a_3, a_4 \rangle = G$ and so $\mathrm{d}(G) \leq 4$.

We assume, by a way of contradiction, that $\mathrm{d}(G) = 4$. By Proposition 1, for any $x, y \in G$ we have $\mathrm{o}([x, y]) \leq p$ and $[x, y] \in \mathrm{Z}(G)$. This implies that G' is elementary abelian and $G' \leq \mathrm{Z}(G)$. In particular, G is of class 2.

For any $k \in \{1, 2\}$ and $l \in \{3, 4\}$, we have $\langle a_1, a_2, a_l \rangle < G$ and $\langle a_k, a_3, a_4 \rangle < G$ and so $\langle a_1, a_2, a_l \rangle' = \langle [a_1, a_2] \rangle$ and $\langle a_k, a_3, a_4 \rangle' = \langle [a_3, a_4] \rangle$. It follows that

$$[a_k, a_l] \in \langle [a_1, a_2] \rangle \cap \langle [a_3, a_4] \rangle = \{1\}.$$

This implies

$$[a_1, a_2a_3] = [a_1, a_2][a_1, a_3] = [a_1, a_2] \text{ and } [a_2a_3, a_4] = [a_2, a_4][a_3, a_4] = [a_3, a_4].$$

But then $\langle a_1, a_2a_3, a_4 \rangle$ is a proper subgroup of G and we have $|\langle a_1, a_2a_3, a_4 \rangle'| \geq p^2$, a contradiction. Our proposition is proved. ∎

Proposition 4 (Y. Berkovich). *Suppose that G is a nonabelian p-group. If $\mathrm{d}(G) = 2$, then $H' < G'$ for each $H < G$.*

Proof. Let $R < G'$ be a G-invariant subgroup of index p in G'. Then $|(G/R)'| = p$ and $\mathrm{d}(G/R) = 2$. This implies that G/R is minimal nonabelian. For each maximal subgroup H of G, $H' \leq R < G'$ and we are done. ∎

Proposition 5 (Ćepulić and Pylyavska). *Let G be a title p-group with $p > 2$. Then for any $a, b \in G$, we have $[a^p, b] = [a, b^p] = [a, b]^p$.*

Proof. We set $g = [a, b]$. If g commutes with a, then for each $n \geq 1$ we prove by induction that $[a^n, b] = [a, b]^n$. Indeed, for $n > 1$,

$$\begin{aligned}[a^n, b] &= [aa^{n-1}, b] = [a, b]^{a^{n-1}}[a^{n-1}, b] \\ &= [a, b][a^{n-1}, b] = [a, b][a, b]^{n-1} = [a, b]^n.\end{aligned}$$

In particular, we have $[a^p, b] = [a, b]^p$.

We assume now that $[g, a] = z \neq 1$. Since $\langle g, a \rangle < G$, Proposition 1 implies that $\mathrm{o}(z) = p$ and $z \in \mathrm{Z}(G)$. We note that

$$g^a = a^{-1}ga = g(g^{-1}a^{-1}ga) = g[g, a] = gz \text{ and so } g^{a^i} = gz^i$$

for all $i \geq 1$. We have

$$[a^p, b] = [a \cdot a^{p-1}, b] = [a, b]^{a^{p-1}}[a^{p-1}, b] = [a, b]^{a^{p-1}}[a \cdot a^{p-2}, b]$$
$$= [a, b]^{a^{p-1}}[a, b]^{a^{p-2}}[a^{p-2}, b]$$

and so continuing we get finally:

$$\begin{aligned}[a^p, b] &= [a, b]^{a^{p-1}}[a, b]^{a^{p-2}}...[a, b]^a[a, b] \\ &= g^{a^{p-1}}g^{a^{p-2}}...g^a g = (gz^{p-1})(gz^{p-2})...(gz)g \\ &= g^p z^{(p-1)+(p-2)+...+1} = g^p z^{(p-1)\frac{p}{2}} = g^p = [a, b]^p,\end{aligned}$$

where we have used the fact that $p > 2$. We have proved that in any case we get $[a^p, b] = [a, b]^p$. Now, $[a, b^p] = [b^p, a]^{-1} = [b, a]^{-p} = [a, b]^p$, and so our proposition is proved. ∎

Proposition 6. *Suppose that G is a p-group which has one of the following properties:*

(a) $|G'| \leq p$;
(b) $\mathrm{d}(G) = 2$, $|G'| = p^2$;
(c) $p > 2$, $\mathrm{d}(G) = 2$, $\mathrm{cl}(G) = 3$, $G' \cong \mathrm{E}_{p^3}$, $\mho_1(G) \leq \mathrm{Z}(G)$;
(d) $\mathrm{d}(G) = 3$, $\mathrm{cl}(G) = 2$, $G' \cong \mathrm{E}_{p^3}$ or E_{p^2}.

Then G has the title property, i.e., $|H'| \leq p$ for each proper subgroup H of G.

Proof. If G is a p-group in (a), then obviously G has the title property. Suppose that G is a p-group in (b). By Proposition 4, for each $H < G$ we have $H' < G'$ and so G has the title property.

Now assume that G is a p-group in (c). Since $\mathrm{cl}(G) = 3$, we have $\{1\} \neq \mathrm{K}_3(G) \leq \mathrm{Z}(G)$. But $\mathrm{d}(G) = 2$ and so $\{1\} \neq G'/\mathrm{K}_3(G)$ is cyclic and therefore $\mathrm{K}_3(G) \cong \mathrm{E}_{p^2}$. We have $\mho_1(G) \leq \mathrm{Z}(G)$ and so $\Phi(G) = \mho_1(G)G'$ is abelian and $G/\Phi(G) \cong \mathrm{E}_{p^2}$. Also, $\mho_1(G)\mathrm{K}_3(G) \leq \mathrm{Z}(G)$ and in fact $\mho_1(G)\mathrm{K}_3(G) = \mathrm{Z}(G)$. Indeed, if $\mho_1(G)\mathrm{K}_3(G) < \mathrm{Z}(G)$, then $G/\mathrm{Z}(G) \cong \mathrm{E}_{p^2}$. But in that case G has $p+1$ abelian maximal subgroups and this implies (Exercise P1 in [3]) $|G'| = p$, a contradiction. Let M be any maximal subgroup of G so that $|M : \Phi(G)| = p$. Then M is either abelian or $\mathrm{Z}(G) = \mathrm{Z}(M)$ and $M/\mathrm{Z}(M) \cong \mathrm{E}_{p^2}$. In the second case we may use Lemma 1.1 in [1] since $\Phi(G)$ is an abelian maximal subgroup of M. From $|M| = p|\mathrm{Z}(M)||M'|$, we get $|M'| = p$. We have proved that in this case G has the title property.

Suppose that G is a p-group in (d). For any $x, y \in G$ we have $[x^p, y] = [x, y]^p = 1$ and so $\mho_1(G) \leq \mathrm{Z}(G)$. It follows that $\Phi(G) = \mho_1(G)G' \leq \mathrm{Z}(G)$ and $G/\Phi(G) \cong \mathrm{E}_{p^3}$. Let M be any maximal subgroup of G so that $M/\Phi(G) \cong \mathrm{E}_{p^2}$. It follows that $p+1$ maximal subgroups of M which contain $\Phi(G)$ are abelian. This implies that $|M'| \leq p$ and we are done. ∎

Theorem 7. *A p-group G has the property that each proper subgroup of G has its derived subgroup of order at most p if and only if one of the following holds:*

(a) $|G'| \leq p$;
(b) $\mathrm{d}(G) = 2$, $|G'| = p^2$;
(c) $p > 2$, $\mathrm{d}(G) = 2$, $\mathrm{cl}(G) = 3$, $G' \cong \mathrm{E}_{p^3}$, $\mho_1(G) \leq \mathrm{Z}(G)$
(note that such p-groups exist. See for example A_2*-groups of order* p^5*, $p > 2$, in Proposition 71.5(b) in [2]);*
(d) $\mathrm{d}(G) = 3$, $\mathrm{cl}(G) = 2$, $G' \cong \mathrm{E}_{p^3}$ or E_{p^2}. *Here we have* $\Phi(G) = \mathrm{Z}(G)$.

Proof. If G is a p-group in (a), (b), (c) or (d), then Proposition 6 implies $|H'| \leq p$ for each subgroup $H < G$.

Suppose that G is a p-group all of whose proper subgroups have its derived subgroup of order $\leq p$. If $|G'| \leq p$, then we have the groups in part (a) of our theorem. In what follows we assume that $|G'| \geq p^2$. By Proposition 2, G' is abelian of order p^2 or p^3. By Proposition 3, we have $\mathrm{d}(G) \leq 3$.

(i) First assume $\mathrm{d}(G) = 2$. If $|G'| = p^2$, then we have obtained the groups in part (b) of our theorem. In the sequel we shall assume here $|G'| = p^3$. By a result of A. Mann (Exercise 1.69(a) in [1]), all $p + 1$ maximal subgroups M_i $(i = 1, 2, ..., p + 1)$ of G are nonabelian, $|M_i'| = p$ and for any $i \neq j$ we have $M_i' \cap M_j' = \{1\}$ so that $M_i' \times M_j' \cong \mathrm{E}_{p^2}$ and $M_i' \times M_j' \leq \mathrm{Z}(G)$. If $\mathrm{cl}(G) = 2$, then $\mathrm{d}(G) = 2$ would imply that G' is cyclic, contrary to the existence of the subgroup $M_i' \times M_j' \cong \mathrm{E}_{p^2}$. Hence $\mathrm{cl}(G) \geq 3$. But $\{1\} \neq \mathrm{K}_3(G) = [G, G'] \leq M_i' \times M_j' \leq \mathrm{Z}(G)$ and so $\mathrm{cl}(G) = 3$. We set $E = M_i' \times M_j' = G' \cap \mathrm{Z}(G) \cong \mathrm{E}_{p^2}$. Whenever $a, b \in G$ are such that $\langle a, b\rangle = G$, then $[a, b] \in G' - E$. Indeed, if $1 \neq [a, b] \in E$, then $\mathrm{o}([a, b]) = p$ and $[a, b] \in \mathrm{Z}(G)$. But then $G/\langle [a, b]\rangle$ is abelian and so $G' = \langle [a, b]\rangle$ is of order p, a contradiction. Let $g = [a, b]$, where $\langle a, b\rangle = G$ and $g \in G' - E$. For any $x \in G$ we have $g^x = ge$ with some $e \in E$. Then $g^{x^i} = ge^i$ and so $g^{x^p} = g$. It follows that $\mho_1(G)$ centralizes G' and so $\Phi(G) = \mho_1(G)G'$ centralizes G'.

(i1) Now assume $p > 2$. Suppose in addition that G' is not elementary abelian. Then $E = \Omega_1(G')$ and set $\{1\} \neq \mho_1(G') = \langle s\rangle < E$ so that $G'/\langle s\rangle \cong \mathrm{E}_{p^2}$. If $\mathrm{K}_3(G) = [G, G'] = \langle s\rangle$, then $G/\langle s\rangle$ is of class 2 so that $\mathrm{d}(G/\langle s\rangle) = 2$ would imply that $G'/\langle s\rangle = (G/\langle s\rangle)'$ is cyclic, a contradiction. Hence there is an element $c \in G - \Phi(G)$ such that $g^c = gl$ with $l = [g, c] \in E - \langle s\rangle$. Let $d \in G - \Phi(G)$ be such that $\langle c, d\rangle = G$ so that $[c, d] \in G' - E$. By Proposition 5, $[c, d^p] = [c, d]^p = s^j$, where $j \not\equiv 0 \pmod p$. Consider the maximal subgroup $C = \langle \Phi(G), c\rangle$. Since $g, c, d^p \in C$, we have $C' \geq \langle [g, c], [c, d^p]\rangle = \langle l, s^j\rangle = E \cong \mathrm{E}_{p^2}$, a contradiction. We have proved that $G' \cong \mathrm{E}_{p^3}$. For any $x, y \in G$ we get by Proposition 5, $[x^p, y] = [x, y]^p = 1$ and so $\mho_1(G) \leq \mathrm{Z}(G)$. We have obtained the groups given in part (c) of our theorem.

(i2) It remains to consider the case $p = 2$. Assume in addition that $\{1\} \neq \mathrm{K}_3(G) = [G, G'] < E$ and set $[G, G'] = \langle u\rangle$, where u is an involution in $E \leq Z(G)$. Note that $\Phi(G)$ centralizes G' and for each $x \in G - \Phi(G)$ and $y \in G' - E$ we have $y^x = yu'$ with $u' \in \langle u \rangle$. Set $G_0 = C_G(G')$ so that we have $|G : G_0| = |G_0 : \Phi(G)| = 2$. Since $G/\langle u \rangle$ is of class 2 and $d(G/\langle u \rangle) = 2$, we have $G'/\langle u \rangle$ is cyclic. Hence if $g \in G' - E$, then $g^2 = v$ is an involution in $E - \langle u \rangle$ and therefore $E = \Omega_1(G') = \langle u, v \rangle$ and $\mho_1(G') = \langle v \rangle$. Take some elements $a \in G_0 - \Phi(G)$ and $b \in G - G_0$. Then $\langle a, b \rangle = G$ and therefore $[a, b] = h \in G' - E$ with $h^2 = v$, $h^a = h$ and $h^b = hu$. Consider the maximal subgroup $H = \langle \Phi(G), b \rangle$. Since

$$[a^2, b] = [a, b]^a [a, b] = h^a h = h^2 = v \text{ and } [h, b] = u,$$

we get $H' \geq \langle u, v \rangle = E \cong E_4$, a contradiction.

We have proved that $K_3(G) = [G, G'] = E = G' \cap Z(G) \cong E_4$. Let $a, b \in G - \Phi(G)$ be such that $\langle a, b \rangle = G$. Then $g = [a, b] \in G' - E$, $[g, a] = c_1$, $[g, b] = c_2$, where $\langle c_1, c_2 \rangle = E = K_3(G)$. We set $c_3 = c_1 c_2$ and get

$$[g, ab] = [g, b][g, a]^b = [g, b][g, a] = c_2 c_1 = c_3.$$

We compute the commutator subgroups of our three nonabelian maximal subgroups $X_1 = \langle \Phi(G), a \rangle$, $X_2 = \langle \Phi(G), b \rangle$ and $X_3 = \langle \Phi(G), ab \rangle$, where we note that we must have $|X_i'| = 2$ for $i = 1, 2, 3$.

Since $[g, a] = c_1$ and

$$[a, b^2] = [a, b][a, b]^b = gg^b = g \cdot gc_2 = g^2 c_2,$$

we have $X_1' = \langle c_1 \rangle$ and so we must have $g^2 c_2 \in \langle c_1 \rangle$. This forces either $g^2 = c_2$ or $g^2 = c_3$.

Since $[g, b] = c_2$ and

$$[a^2, b] = [a, b]^a [a, b] = g^a g = gc_1 \cdot g = g^2 c_1,$$

we have $X_2' = \langle c_2 \rangle$ and so we must have $g^2 c_1 \in \langle c_2 \rangle$. This forces either $g^2 = c_1$ or $g^2 = c_3$. With the above we get exactly $g^2 = c_3$.

Since $[g, ab] = c_3$ and

$$[a^2, ab] = [a, ab]^a [a, ab] = g^a g = gc_1 \cdot g = g^2 c_1,$$

(where we have used the fact that $[a, ab] = [a, b]$) we have $X_3' = \langle c_3 \rangle$ and so we must have $g^2 c_1 \in \langle c_3 \rangle$. But we know that $g^2 = c_3$ and so $g^2 c_1 = c_3 c_1 = c_2 \in \langle c_3 \rangle$, a contradiction. We have proved that such 2-groups do not exist!

(ii) Finally, assume that $d(G) = 3$. For any $x, y \in G$ we have $\langle x, y \rangle < G$ and so Proposition 1 implies that $o([x, y]) \leq p$ and $[x, y] \in Z(G)$. But then G' is elementary abelian (of order p^2 or p^3) and $G' \leq Z(G)$ and so we have obtained the groups from part (d) of our theorem. For any $a, b \in G$, $[a^p, b] = [a, b]^p = 1$ and so $\Phi(G) \leq Z(G)$. If $Z(G) \not\leq \Phi(G)$, then there is a maximal subgroup M of G such that $G = \langle M, x \rangle$, where $x \in Z(G)$. But then $G' = M'$ and so $|G'| = 2$, a contradiction. Hence $\Phi(G) = Z(G)$. Theorem 7 is completely proved. □

References

[1] Y. Berkovich, Groups of prime power order, Vol. 1, Walter de Gruyter, Berlin, 2008.

[2] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 2, Walter de Gruyter, Berlin, 2008.

[3] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 3, Walter de Gruyter, Berlin, 2011.

[4] V. Ćepulić and O. Pylyavska, *Determination of p-groups all of whose proper subgroups have a commutator subgroup of order equal or less than p ($p \geq 3$)*, Naukovi zapysky, Kyyevo **39** (2005), 28–34.

[5] J. Q. Zhang and X. H. Li, *Finite p-groups all of whose proper subgroups have small derived subgroups*, Sci. China Math. **53** (2010), 1357–1362.

Z. Janko
Mathematical Institute
University of Heidelberg
69120 Heidelberg
Germany
E-mail: janko@mathi.uni-heidelberg.de

Received: 16.11.2010.
Revised: 1.12.2010.