GLASNIK MATEMATIČKI
Vol. 46(66)(2011), 249 – 268

A 2-EQUIVALENT KELLEY CONTINUUM

Carlos Islas
Universidad Autónoma de la Ciudad de México, Mexico

Abstract. The main purpose of this paper is to construct a 2-equivalent compactification X of a ray whose remainder is homeomorphic to X and such that X is a Kelley Continuum. In order to construct this example, we prove a theorem which gives conditions for an inverse limit of arcs X to be the compactification of a ray and X is a Kelley continuum.

1. Introduction

We construct a 2-equivalent continuum which is a compactification X of a ray whose remainder is homeomorphic to X and such that X is a Kelley continuum. In order to construct this example, we prove a theorem which gives conditions for an inverse limit of arcs to be the compactification of a ray and such that it is a Kelley continuum.

W. T. Ingram in [8, Theorem 2.3., p. 193] gives different conditions to obtain a Kelley continuum which is a compactification of a ray. R. A. Beane and W. J. Charatonik proved in [1, Theorem 2.3., p. 105] that for every chainable Kelley continuum C, there exists a compactification D of a ray with remainder homeomorphic to C, and such that D is a Kelley continuum.

A continuum is a compact and connected metric space, a map is a continuous function. Let X and Y be continua, a map $f : X \to Y$ is said to be confluent provided that for any subcontinuum B of Y and any component A of $f^{-1}(B)$, $f(A) = B$. A monotone map $f : X \to Y$, is a map such that $f^{-1}(C)$ is a connected set, for every connected subset C of Y (see [9, Lema 2.1.12, p. 74]). An arc means a space homeomorphic to the closed interval $[0, 1]$. The set of positive integers is denoted by $\mathbb{N}$.

2010 *Mathematics Subject Classification.* 54F15, 54F50.
Key words and phrases. 2-equivalent continuum, Kelley Continuum, inverse limits.
The author thanks ICyTDF for the financial support provided.

For a positive integer n, a continuum X is said to be n-equivalent provided that X contains exactly n topologically distinct subcontinua. The arc and the pseudo-arc are the only known 1-equivalent continua (see [6, 12, 13]). In [16], Whyburn has shown that each planar 1-equivalent continuum is tree-like, and the planarity assumption has been deleted after 40 years by Cook ([5]) who proved the tree-likeness of any 1-equivalent continuum. But it is still not known whether or not the arc and the pseudo-arc are the only 1-equivalent continua.

The class of 2-equivalent continua was studied by Mahavier in [11]. He proved that, if a 2-equivalent continuum contains an arc, then it is a simple triod, a simple closed curve or it is an irreducible continuum, and that the only locally connected 2-equivalent continua are a simple triod and a simple closed curve. In [11] it is also shown that if X is a decomposable, not locally connected, 2-equivalent continuum containing an arc, then X is arc-like and it is the compactification of a ray R such that the remainder $K = cl(R) \setminus R$ is a subcontinuum of X (a ray is a space homeomorphic to the interval $[0, \infty)$). It is well known that the $\sin\frac{1}{x}$ continuum is an example of this kind of continua, such that the remainder K is an arc, these are called Elsa continua. By a suggestion of the referee, we define that a continuum X is n-equivalent compactification, if X is a compactification of a ray and its remainder is n-equivalent. With this definition, we obtain that the $\sin\frac{1}{x}$ curve is a 1-equivalent compactification and in this paper we will present an example of a 2-equivalent compactification.

Let us recall some definitions and facts on inverse limits.

Let $\{X_1, X_2, X_3, \ldots\}$ be a sequence of continua and let $\{f_1^2, f_2^3, f_3^4, \ldots\}$ be a sequence of maps, such that $f_i^{i+1} : X_{i+1} \to X_i$ for every $i \in \{1, 2, \ldots\}$. The sequence $\{X_i, f_i^{i+1}\}_{i=1}^{\infty}$ is called an inverse sequence and the inverse limit space is defined by

$$X_\infty = \varprojlim \{X_n, f_n^{n+1}\} = \{(x_1, x_2, \ldots) : \text{for every } n \in \mathbb{N},\ f_n^{n+1}(x_{n+1}) = x_n\},$$

as a subspace of the product $\prod_{n=1}^{\infty} X_n$.

Every space X_n is called a factor space and f_n^{n+1} a bonding map. We denote by $\pi_i : \varprojlim \{X_n, f_n^{n+1}\} \to X_i$ the i-th projection map, restricted to the inverse limit. If $n > m$, with $n, m \in \mathbb{N}$, f_m^n denotes the composition $f_m^{m+1} \circ \cdots \circ f_{n-1}^n$. Sometimes we use f_n instead of f_n^{n+1}. If K is a subcontinuum of X_∞, we denote $K_i = \pi_i(K)$. If $X_n = X$, for every $n \in \mathbb{N}$, we denote $X_\infty = \varprojlim \{X, f_n^{n+1}\}$ or $\varprojlim \{X, f\}$ if every $f_n^{n+1} = f$. We will use the sequences $\{I_1, I_2, I_3, \ldots\}$ of subintervals of $I = [0, 1]$, and sequences of maps $\{f_1^2, f_2^3, f_3^4, \ldots\}$, with $f_i^{i+1}(I_{i+1}) = I_i$. The Hilbert cube is a space homeomorphic to the product $\prod_{n=1}^{\infty} I_n$, where $I_n = [0, 1]$ and the distance between two points $(x_1, x_2, x_3, \ldots)$, $(y_1, y_2, y_3, \ldots)$ is defined by $d((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$.

Now, let us recall some facts about Kelley continua.

Definition 1.1. *A continuum X is a Kelley continuum at $p \in X$ if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if A is a subcontinuum of X, $p \in A$, $q \in X$ and $d(p,q) < \delta$, then there exists a subcontinuum B of X such that, $q \in B$ and $\mathcal{H}(A,B) < \varepsilon$ (d denotes the distance on X and $\mathcal{H}$ denotes the Hausdorff distance on the hyperspace of subcontinua of X, $C(X)$). X is Kelley continuum if, X is a Kelley continuum at every one of its points (see [3, p. 74]).*

It is necessary to mention that a Kelley continuum is well known as a continuum with the property of Kelley.

Theorem 1.2. *[15, 16.11, p. 413] If X is locally connected at p, then X is a Kelley continuum at p.*

Theorem 1.3. *[4, Theorem 2, p. 190] If $X_\infty = \varprojlim \left\{X_n, f_n^{n+1}\right\}$ and every factor space X_n is a Kelley continuum and every bonding map f_n^{n+1} is confluent, then X_∞ is a Kelley continuum.*

2. Theorem on Inverse Limits

2.1. *A Theorem on Inverse Limits.* The following theorem gives conditions under which an inverse limit of intervals is the compactification of a ray, another proof of Theorem 2.1 is in [2]. Nevertheless we include our proof for completeness and because the techniques are different.

Theorem 2.1. *Let $f : I \longrightarrow I$ be a map, where $I = [0,1]$, given by:*

$$f(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right]; \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]; \end{cases}$$

and $\operatorname{Im} f|_{\left[\frac{1}{2},1\right]} \subseteq \left[\frac{1}{2},1\right]$ (Im denote the image). Let $X = \varprojlim \{I, f\}$. Then X is the compactification of a ray R and $K = \varprojlim \left\{\left[\frac{1}{2},1\right], f|_{\left[\frac{1}{2},1\right]}\right\}$ is the remainder ($f|_{\left[\frac{1}{2},1\right]}$ means the restriction of the function f on the set $\left[\frac{1}{2},1\right]$).

Proof. Let X be as in the hypothesis. For every positive integer n, let

$$\alpha_n = \left\{(x_1, x_2, \ldots) \in X : x_n < \frac{1}{2}\right\}.$$

We note that:

1. $\alpha_n \subseteq \alpha_{n+1}$.
2. $\pi_n|_{\alpha_n}$ is a homeomorphism from α_n onto $\left[0, \frac{1}{2}\right)$.
3. $x \in \alpha_n \setminus \alpha_{n-1}$ if and only if $x_n \in \left[\frac{1}{8}, \frac{1}{2}\right)$.

We will show that $R = \cup_{n=1}^{\infty} \alpha_n$ is a ray. Observe that $R = \alpha_1 \cup \cup_{n=2}^{\infty} (\alpha_n \smallsetminus \alpha_{n-1})$, and define $\sigma : R \longrightarrow [0,\infty)$ by:

$$\sigma(x) = \begin{cases} x_1, & \text{if } x \in \alpha_1; \\ x_n + \frac{3(n-1)}{8}, & \text{if } x \in \alpha_n \smallsetminus \alpha_{n-1} = \pi_n^{-1}\left[\frac{1}{8}, \frac{1}{2}\right). \end{cases}$$

We will prove that σ is a homeomorphism. If $r \in [0,\infty)$, then either $r \in \left[0, \frac{1}{2}\right) = \sigma(\alpha_1)$ or there exists $n > 1$ such that $r \in \left[\frac{3n-2}{8}, \frac{3(n+1)-2}{8}\right) = \sigma(\alpha_n \smallsetminus \alpha_{n-1})$. This proves that σ is surjective.

We denote by B_n the set $\sigma(\alpha_n \smallsetminus \alpha_{n-1})$, if $n > 1$ and by $B_1 = \left[0, \frac{1}{2}\right) = \sigma(\alpha_1)$. We will see that σ is injective. Since $B_n \cap B_m = \emptyset$ if $n \neq m$, the equality $\sigma(x) = \sigma(y)$ implies that either $x, y \in \alpha_1$ or $x, y \in \alpha_n \setminus \alpha_{n-1}$, $n > 1$. Therefore, either $\sigma(x) = x_1 = y_1 = \sigma(y)$ in the first case or $\sigma(x) = x_n + \frac{3(n-1)}{8} = y_n + \frac{3(n-1)}{8} = \sigma(y)$ in the second case. In both cases $x_n = y_n$ and since $\pi_n|_{\alpha_n}$ is a homeomorphism, $x = y$.

To see that σ is a continuous function we observe that the functions $\sigma_{|\alpha_1}$ and $\sigma_{|\alpha_n \smallsetminus \alpha_{n-1}}$ are continuous. Then if either $x \in \pi_1^{-1}\left[0, \frac{1}{2}\right) = \alpha_1$ or $x \in \pi_n^{-1}\left(\frac{1}{8}, \frac{1}{2}\right)$, then σ is continuous at x, since α_1 and $\pi_n^{-1}\left(\frac{1}{8}, \frac{1}{2}\right)$ are open sets of R. It is only necessary to prove that σ is continuous at every element of $\pi_n^{-1}\left(\frac{1}{8}\right)$. Let $x \in \pi_n^{-1}\left(\frac{1}{8}\right)$; i.e., $\pi_n(x) = x_n = \frac{1}{8}$. In this case

$$\sigma(x) = x_n + \frac{3(n-1)}{8} = \frac{1}{8} + \frac{3(n-1)}{8} = \frac{3n}{8} - \frac{1}{4}. \tag{2.1}$$

Let $\varepsilon > 0$ and choose positive numbers δ_0, δ_1 and δ_2 with the following properties:

a) Since $\sigma_{|\alpha_n \smallsetminus \alpha_{n-1}}$ is a map, if $y \in \alpha_n \smallsetminus \alpha_{n-1}$ and $d(x,y) < \delta_0$, then $|\sigma(x) - \sigma(y)| < \varepsilon$.
b) If $s, t \in [0,1]$ and $|s - t| < \delta_1$, then $|f(s) - f(t)| < \varepsilon$.
c) If $y \in R$ and $d(x,y) < \delta_2$, then $|\pi_n(x) - \pi_n(y)| < \min\left\{\delta_1, \frac{3}{32}\right\}$.

Let $\delta = \min\{\delta_0, \delta_2\}$ and $y \in R$. We consider two cases:

Case 1 $\pi_n(y) = y_n \geq \frac{1}{8}$.

In this case $y \in \alpha_n \smallsetminus \alpha_{n-1}$ and since $\delta \leq \delta_0$, it follows by a), that $|\sigma(x) - \sigma(y)| < \varepsilon$.

Case 2 $y_n < \frac{1}{8}$.

Since $d(x,y) < \delta_2$, by c), we have that $x_n - y_n < \frac{3}{32}$, $\frac{1}{8} > y_n > \frac{1}{32}$, $y_{n-1} > \frac{1}{8}$ and thus,

$$\sigma(y) = y_{n-1} + \frac{3(n-2)}{8}. \tag{2.2}$$

Now, since $x_n - y_n < \delta_1$, by b), we have that $|f(x_n) - f(y_n)| < \varepsilon$. It is sufficient to show that $|\sigma(x) - \sigma(y)| = |f(x_n) - f(y_n)|$. Since $f(x_n) = \frac{1}{2}$ and $f(y_n) = y_{n-1} < \frac{1}{8} < \frac{1}{2}$, then $|f(x_n) - f(y_n)| = \frac{1}{2} - y_{n-1}$.

By (1) and (2), $\sigma(x) - \sigma(y) = \frac{1}{2} - y_{n-1}$. Therefore, σ is continuous.

To show that the function $\sigma^{-1}:[0,\infty)\to R$ is continuous, we define $\lambda:[0,\infty)\to\left[0,\frac{1}{2}\right)$ by

$$\lambda(t)=\begin{cases} t, & \text{if} \quad t\in B_1; \\ t-\frac{3(n-1)}{8}, & \text{if} \quad t\in B_n \text{ and } n>1. \end{cases}$$

Recall that $B_1=[0,\frac{1}{2})=\sigma(\alpha_1)$ and $B_n=[\frac{3n-2}{8},\frac{3(n+1)-2}{8})=\sigma(\alpha_n\setminus\alpha_{n-1})$.

It is easy to verify the following:

1) $\lambda|_{B_n}: B_n\to[0,\frac{1}{2})$ is continuous, for every n.
2) If $n>1$ and $t\in B_n$, then $\frac{1}{8}\leq\lambda(t)<\frac{1}{2}$.
3) If $t\in B_n$, then $\pi_n^{-1}(\lambda(t))$ has exactly one point in $\bigcup_{m=1}^{\infty}\alpha_m$ and $\sigma^{-1}(t)=\pi_n^{-1}(\lambda(t))\in\alpha_n\setminus\alpha_{n-1}\subset R$.

By 3), $\sigma^{-1}|_{B_n}$ is continuous for every n. Since $[0,\infty)=\cup_{n=1}^{\infty}B_n$, we only have to prove that σ^{-1} is continuous at the points of the form $\frac{3n-2}{8}$. Given $x\in[0,\infty)$ we denote $x_n=\pi_n\left(\sigma^{-1}(x)\right)$. Let $t=\frac{3n-2}{8}$. Then $\frac{1}{8}=\lambda(t)$, let $t_n=\lambda(t)$, $t_1=t_2=t_3=\cdots=t_{n-1}=\frac{1}{2}$ and $t_{n+j}=f^{-j}\left(\frac{1}{8}\right)=\frac{1}{4j}\left(\frac{1}{8}\right)$ if $j\geq 1$. Let $\varepsilon>0$ and $\delta'>0$ such that, if $\left|u-\frac{1}{2}\right|<\delta'$ then $\left|f^k(u)-f^k\left(\frac{1}{2}\right)\right|<\varepsilon$ for every $k\in\{1,2,\ldots,n-2\}$. Let $\delta=\min\left\{\frac{1}{8},\varepsilon,\delta'\right\}$. If $s\in[0,\infty)$ and $|s-t|<\delta$, then, since $\delta\leq\frac{1}{8}$, $s\in B_{n-1}\cup B_n$. Since $\sigma^{-1}|_{B_n}$ is continuous, we only have to consider the case when $s\in B_{n-1}$. We will prove that $|s_j-t_j|<\varepsilon$ for every $j\in\mathbb{N}$. Since $s_{n-1}=\lambda(s)=s-\frac{3(n-1)-2}{8}$, $|s_{n-1}-t_{n-1}|=\left|\left(s-\frac{3(n-1)-2}{8}\right)-\frac{1}{2}\right|=\left|s-\frac{(3n-2)}{8}\right|=|s-t|<\delta\leq\varepsilon$. Now, since $t_{n-1}=\frac{1}{2}$, the choice of δ' implies that $\left|f^k(s_{n-1})-f^k\left(\frac{1}{2}\right)\right|=|s_{n-1-k}-t_{n-1-k}|<\varepsilon$, for every $k\in\{1,2,\ldots,n-2\}$; i.e., $|s_j-t_j|<\varepsilon$ if $j\in\{1,2,\ldots,n-2\}$. Since $s_{n-1},t_{n-1}\in[0,\frac{1}{2})$ and $f^{-1}(u)=\frac{u}{4}$ if $u\in[0,\frac{1}{2})$, we have that $\left|f^{-k}(s_{n-1})-f^{-k}(t_{n-1})\right|=\left|\frac{s_{n-1}-t_{n-1}}{4^k}\right|<\frac{\varepsilon}{4^k}<\varepsilon$, for every $k>n-1$. Thus, $|s_j-t_j|<\varepsilon$ if $j\geq n$. Then σ^{-1} is continuous.

Let us note that $f([\frac{1}{2},1])\subset[\frac{1}{2},1]$. Then we define:

$$K=\varprojlim\left\{\left[\frac{1}{2},1\right],f|_{[\frac{1}{2},1]}\right\}.$$

Then $X=R\cup K$ and $R\cap K=\emptyset$.

We will show that $\overline{R}\setminus R=K$. We only have to prove that $K\subseteq\overline{R}$. By the definition of the distance, if $u,v\in X=\varprojlim\{I,f\}$ and, for some $k\in\mathbb{N}$, $u_k=v_k$, then $d(u,v)<\frac{1}{2^k}$.

Let $x\in K$. Then $x_n\in\left[\frac{1}{2},1\right]$ for every $n\in\mathbb{N}$. Let $\varepsilon>0$ and $n\in\mathbb{N}$ such that $\frac{1}{2^n}<\varepsilon$. By definition of f, $f\left([0,\frac{1}{2})\right)=[0,1]$. Then there exists $y\in[0,\frac{1}{2})$ such that $f(y)=x_n$. Hence, there is a point p_n in α_{n+1} such that, its $(n+1)$-th coordinate is y, which implies that $d(x,p_n)<\frac{1}{2^n}<\varepsilon$. Thus, $K\subset\overline{R}$. Therefore, X is a compactification of a ray R such that $\overline{R}\setminus R=K$, which is a continuum. ∎

2.2. *Main Theorem on Inverse Limits.* To prove the main Theorem we need the following Lemma.

LEMMA 2.2. *[7, Lemma 2.2, p. 193] Let $M = \varprojlim \{X_i, f_i\}$. Assume that, for every $i \in \mathbb{N}$, X_i is a continuum and $\varepsilon > 0$. Then there exist a positive integer N and a positive number δ, such that, if H and K are subcontinua of M, such that $\mathcal{H}_N(H_N, K_N) < \delta$, then $\mathcal{H}(H, K) < \varepsilon$ ($\mathcal{H}_N$ and $\mathcal{H}$ denote the Hausdorff distance on $C(X_N)$ and $C(M)$, respectively).*

THEOREM 2.3. *Let $X = \varprojlim \{I, f\}$, where $I = [0,1]$ and $f : I \to I$ is a map such that:*

1. $f(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right]; \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$
2. $\operatorname{Im} f|_{\left[\frac{1}{2},1\right]} = \left[\frac{1}{2}, 1\right]$;
3. $C = \varprojlim \left\{ \left[\frac{1}{2}, 1\right], f|_{\left[\frac{1}{2},1\right]} \right\}$ *is a Kelley continuum.*

Then X is a compactification of a ray with remainder C and X is a Kelley continuum.

PROOF. By Theorem 2.1, X is a compactification of a ray and its remainder is C.

i) Note that, if p is a point of the ray, then, by Theorem 1.2, X is a Kelley continuum at p.

Let $\varepsilon > 0$.

ii) Since C is a Kelley continuum, there exists a positive number δ_1 such that, if A is a subcontinuum of C, r is a point of A and s is a point of C and $d(r, s) < \delta_1$, then there exists a subcontinuum B of C with s in B and $\mathcal{H}(A, B) < \frac{\varepsilon}{3}$.

Let ε_1 be a positive number, such that, $\varepsilon_1 < \min\left\{\frac{\varepsilon}{3}, \delta\right\}$.

iii) By Lemma 2.2, there exist a positive number $\delta_2 < \delta_1$ and a positive integer N such that, if H and K are subcontinua of X with $\mathcal{H}_N(H_N, K_N) < \delta_2$, we have that $\mathcal{H}(H, K) < \varepsilon_1$.

iv) Since f is uniformly continuous, there exists a positive number δ_3 such that if x and y are points of X_{N+1} with $|x - y| < \delta_3$, then $|f(x) - f(y)| < \frac{\delta_2}{2}$.

Let $\delta = \frac{\delta_3}{2^{N+1}}$ and suppose that H is a subcontinuum of X. Let $p \in H$ and $q \in X$ such that $d(p, q) < \delta$. Therefore, $d(p, q) < \frac{\delta_3}{2^{N+1}}$ and $|p_{N+1} - q_{N+1}| < \delta_3$.

Let J be the arc irreducible respect to $H_{N+1} \cup \{q_{N+1}\}$; i.e., J is the arc such that $H_{N+1} \cup \{q_{N+1}\} \subset J$ and, if A is a subarc such that $H_{N+1} \cup \{q_{N+1}\} \subset A$, then $J \subset A$.

We will see that $\mathcal{H}_N(H_N, f(J)) \leq \frac{\delta_2}{2} < \delta_2$. Let $x \in f(J)$, then there exists $y \in J$ such that $x = f(y)$. Now, since $y \in J$, there exists $z \in H_{N+1}$ such that $|y - z| < |p_{N+1} - q_{N+1}|$ (by the irreducibility of J). Then $|y - z| < \delta_3$ and $|f(y) - f(z)| < \frac{\delta_2}{2} < \delta_2$. In consequence, for every element x in $f(J)$, there exists $f(z)$ in H_N such that $|x - f(z)| < \frac{\delta_2}{2}$, and, since $H_N \subset f(J)$, $\mathcal{H}_N(H_N, f(J)) < \delta_2$.

We will consider two cases:

CASE 1. $q_{N+2} < \frac{1}{2}$.

In this case we have two possibilities

CASE 1.1. H_{N+1} contains 1.

By the definition of f, there exists a subinterval of I, $[x,y] \subset \left[0, \frac{1}{2}\right]$ such that $f([x,y]) = J$, moreover, we could choose $[x,y]$ in such a way that $q_{N+2} \in [x,y]$, because $q_{N+1} \in J$ and $q_{N+2} < \frac{1}{2}$. Now, define K as the subcontinuum of X such that $K_{N+2} = [x,y]$ and

$$K_i = \begin{cases} f^{N+2-i}([x,y]), & \text{if } \ i < N+2; \\ f^{-(i-(N+2))}([x,y]), & \text{if } \ i > N+2. \end{cases}$$

Thus, $f(K_{i+1}) = K_i$ and $K \subset X$. Moreover, $q \in K$, $K_N = f(J)$ and, since $\mathcal{H}(H_N, f(J)) < \delta_2$, $\mathcal{H}(H,K) < \frac{\varepsilon}{3}$.

CASE 1.2 $1 \notin H_{N+1}$.

Consider two possibilities.

a) $H_{N+1} \subset \left[\frac{1}{2}, 1\right]$, in this case, we choose: $[x,y] \subset \left[0, \frac{1}{4}\right]$ such that $f([x,y]) = J$ if $q_{N+2} \leq \frac{1}{4}$ or $[x,y] \subset \left[\frac{1}{4}, \frac{1}{2}\right]$ such that $f([x,y]) = J$ if $\frac{1}{4} < q_{N+2} < \frac{1}{2}$. Now, define K in the same way as in Case 1.1. We obtain that $\mathcal{H}(H,K) < \varepsilon$.

b) If $H_{N+1} \cap \left[0, \frac{1}{2}\right) \neq \emptyset$, since H_{N+2} is a continuum and $H_{N+2} \cap \left[0, \frac{1}{4}\right) \neq \emptyset$, $H_{N+2} \subset \left[0, \frac{1}{4}\right)$, so $H \subset \alpha_{N+2}$, with $\alpha_{N+2} = \left\{(x_1, \ldots) \in X \mid x_{N+2} < \frac{1}{2}\right\}$ (α_{N+2} is the subset of the ray defined in Theorem 2.1). In this case we obtain that $p \in H \subset \alpha_{N+2}$; hence, p is an element of the ray and, by i), X has the property of Kelley in p.

CASE 2 $q_{N+2} \geq \frac{1}{2}$.

In this case, we have two possibilities:

CASE 2.1 $J \subseteq \left[\frac{1}{2}, 1\right]$.

Since $q_{N+2} \geq \frac{1}{2}$ and $J \subseteq \left[\frac{1}{2}, 1\right]$, $H_{N+1} \subseteq \left[\frac{1}{2}, 1\right]$. We define A as follows: If $H_i \subseteq \left[\frac{1}{2}, 1\right]$ for every $i \in \mathbb{N}$, we define $A = H$. If $H_i \cap \left[0, \frac{1}{2}\right) \neq \emptyset$ for some i, we define A in the following way:

Let $j+1 = \min\left\{i \in \mathbb{N} : H_i \cap \left[0, \frac{1}{2}\right) \neq \emptyset\right\}$ (it is clear that $j > N+1$). By [14, 13.71, p. 310], $f|_{\left[\frac{1}{2},1\right]}$ is weakly confluent. Then we may choose a component A_{j+1} of $f^{-1}|_{\left[\frac{1}{2},1\right]}(H_j)$ such that $f(A_{j+1}) = H_j$. We define A inductively. In general, for every $i > j$, we choose A_{i+1} to be a component of $f^{-1}|_{\left[\frac{1}{2},1\right]}(A_i)$ such that $f(A_{i+1}) = A_i$. Hence, $A = \varprojlim\left\{A_i, f|_{A_{i+1}}\right\}$ is a subcontinuum of C, which coincides with H at least in the first $N+1$ coordinates. By the choice of δ, by iii) and ii), we have that $\mathcal{H}(A,H) < \frac{\varepsilon}{3}$.

We define r and s in the same way: r and s are elements of C, which coincide with p and q respectively in at least the first $N+1$ coordinates. In fact, s coincides with q at least in the first $N+2$ coordinates, because of our assumption; i.e., $J \subseteq \left[\frac{1}{2}, 1\right]$.

Since $|r_{N+1} - s_{N+1}| < \delta_3$ we have $|r_N - s_N| < \delta_2$. By the choice of δ_2, $d(r,s) < \delta_1$ and since C is a Kelley continuum, there exists a subcontinuum B of C such that $s \in B$ and $\mathcal{H}(A,B) < \frac{\varepsilon}{3}$. Thus, $\mathcal{H}(H,B) < \frac{2\varepsilon}{3}$. If $q \in B$, we choose $K = B$ to obtain the conclusion. Moreover, since C is a Kelley continuum, if $q \in C$, we also obtain the conclusion. Suppose that $q \notin B$ and that there exists $i > N+2$ such that $q_i < \frac{1}{2}$.

Let $j+1 = \min\left\{i \in \mathbb{N} : q_i < \frac{1}{2}\right\}$. We will construct K as in Case 1.1.

Since $f\left(\left[0, \frac{1}{2}\right]\right) = [0,1]$, there is a subinterval $[x,y]$ contained either in $\left[\frac{1}{8}, \frac{1}{4}\right]$ or in $\left[\frac{1}{4}, \frac{1}{2}\right]$ such that $q_{j+1} \in [x,y]$ and such that $f([x,y]) = B_j$.

Let $K_{j+1} = [x,y]$, $K_{j+2} = f^{-1}(K_{j+1})$, $K_{j+i+1} = f^{-1}(K_{j+i})$, for every i, and if $i < j+1$, let $K_i = f^{j+1-i}(K_{j+1})$. Thus, $K = \varprojlim\left\{K_i, f|_{K_{i+1}}\right\}$ is a continuum containing q and such that $K_N = B_N$. Since $K_j = B_j$, by iii), $\mathcal{H}(H,K) < \varepsilon$.

Case 2.2 $J \cap \left[0, \frac{1}{2}\right) \neq \emptyset$.

We have $q_{N+2} \geq \frac{1}{2}$ and $J \cap \left[0, \frac{1}{2}\right) \neq \emptyset$.

If $f(J) \subseteq \left[\frac{1}{2}, 1\right]$, then we proceed as in case 2.1, using $f(J)$ instead of J.

If $f(J) \cap \left[0, \frac{1}{2}\right) \neq \emptyset$, there exists an element $y < \frac{1}{2}$ in $f(J)$, and by the definition of f, $y = f(x)$, for some $x \in J$, where $x < \frac{1}{4}$, now, since $q_{N+2} \geq \frac{1}{2}$, $q_{N+1} \geq \frac{1}{2}$, so $f\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right) \subseteq J$, thus $\left[\frac{1}{2}, 1\right] \subseteq f(J)$.

Let $y_0 = \min\{y : y \in f(J)\}$. Then $f(J) = [y_0, 1]$. Let $K_N = f(J)$, $K_{N+1} = f^{-1}(K_N) = \left[\frac{y_0}{4}, 1\right]$, $K_{N+2} = f^{-1}(K_{N+1}) = \left[\frac{y_0}{16}, 1\right]$, ... If $i < N$, $K_i = f^{N-i}(K_N)$. Let $K = \varprojlim\left\{K_i, f|_{K_{i+1}}\right\}$. Then K is a continuum containing q.

Since $q \in K$ and $\mathcal{H}(H_N, f(J)) < \delta_2$ and $K_N = f(J)$, we obtain by iii), that $\mathcal{H}(H,K) < \varepsilon$. ∎

3. A 2-equivalent Kelley continuum.

In this section, we describe the factor spaces of our example, which is a 2-equivalent continuum with the following properties: It is the compactification of a ray and its remainder is homeomorphic to the whole space and it is a Kelley continuum. Then we give some properties of the factor spaces and the bonding maps. We show the continuity and the confluence of the bonding maps. Finally, we construct the example and prove its properties.

3.1. *Factor spaces.* The function g, that we define below, will help us to define the factor spaces:

Let $g : \left[\frac{1}{2}, 1\right] \longrightarrow [0,1]$ defined by $g(x) = 2x - 1$ and let g^{-1} its inverse.

Let $X_1 = \varprojlim \{I, f_1\}$, where $I = [0,1]$ and $f_1 : I \longrightarrow I$ is defined by:

$$f_1(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right] ; \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ \frac{1}{2}, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then, by Theorem 2.1, X_1 is homeomorphic to the interval $[0,1]$, since X_1 is the compactification of a ray R_1 with remainder $K_1 = \varprojlim \left\{ \left[\frac{1}{2}, 1\right], f_1 |_{\left[\frac{1}{2},1\right]} \right\} = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \ldots\right) \right\}$. Recall that

$$X_1 = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \ldots \right) \right\} \cup \left(\cup_{n=1}^{\infty} \alpha_n^1 \right) \text{ where } \alpha_n^1 = \left\{ x \in X_1 : x_n < \frac{1}{2} \right\}.$$

It is easy to see that X_1 is a Kelley continuum.

Let $X_2 = \varprojlim \{I, f_2\}$, with $f_2 : I \longrightarrow I$ defined by

$$f_2(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right]; \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ g^{-1}\left(f_1\left(g(x)\right)\right), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

By Theorem 2.1, $X_2 = K_2 \cup R_2$ where $K_2 = \varprojlim \left\{ \left[\frac{1}{2}, 1\right], f_2 |_{\left[\frac{1}{2},1\right]} \right\}$ and $R_2 = \sum_{n=1}^{\infty} \alpha_n^2$ where $\alpha_n^2 = \left\{ x \in X_2 : x_n < \frac{1}{2} \right\}$; i.e., X_2 is the compactification of a ray with remainder K_2. Note that K_2 is homeomorphic to X_1; hence, X_2 satisfies the hypotheses of the Theorem 2.3, then X_2 is a Kelley continuum.

In general, we define, for every positive integer n, $X_{n+1} = \varprojlim \{I, f_{n+1}\}$, where $I = [0,1]$ and $f_{n+1} : I \longrightarrow I$, given by

$$f_{n+1}(x) = \begin{cases} 4x, & \text{if } x \in \left[0, \frac{1}{4}\right]; \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ g^{-1}\left(f_n\left(g(x)\right)\right), & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then X_{n+1} is a compactification of a ray R_{n+1} such that its remainder, $K_{n+1} = \varprojlim \left\{ \left[\frac{1}{2}, 1\right], f_{n+1} |_{\left[\frac{1}{2},1\right]} \right\}$ is homeomorphic to X_n and, by Theorem 2.3, X_{n+1} is a Kelley continuum.

3.2. *Properties of the factor spaces.* We analyze the space X_i. As in the proof of Theorem 2.1, let

$$\alpha_n^i = \left\{ (x_1, x_2, \ldots) \in X_i : x_n < \frac{1}{2} \right\}.$$

Then:

1. $\alpha_n^i \subset \alpha_{n+1}^i$,
2. $\pi_n|_{\alpha_n^i}$ is a homeomorphism from α_n^i onto $\left[0, \frac{1}{2}\right)$,

3. $x \in \alpha_n^i \setminus \alpha_{n-1}^i$ if and only if $x_n \in \left[\frac{1}{8}, \frac{1}{2}\right)$,
4. $R_i = \cup_{n=1}^{\infty} \alpha_n^i$ is a ray and $R_i = \alpha_1^i \cup \cup_{n=2}^{\infty} \left(\alpha_n^i \setminus \alpha_{n-1}^i\right)$, and
5. $X_i = R_i \cup K_i$, where $K_i = \overline{R_i} \setminus R_i$, $\overline{R_i} = X_i$ and $K_i = \varprojlim \left\{ \left[\frac{1}{2}, 1\right], f_i \,|_{\left[\frac{1}{2},1\right]} \right\}$.

If $i > 1$, observe that K_i is homeomorphic to X_{i-1} with the homeomorphism $\overline{g}|_{K_i}$, where $\overline{g} : \left[\frac{1}{2}, 1\right]^{\infty} \longrightarrow I^{\infty}$ is defined by $\overline{g}\left((x_1, x_2, \ldots)\right) = \left(g\left(x_1\right), g\left(x_2\right), \ldots\right)$. Let us note that the function $\overline{g} : \left[\frac{1}{2}, 1\right]^{\infty} \to I^{\infty}$ is continuous, since it is continuous at each coordinate. Moreover, in the same way, we may define the map $\overline{g}^{-1}$ and thus, this map is continuous too.

We observe that $\alpha_1^i = \pi_1^{-1}\left(\left[0, \frac{1}{2}\right)\right)$ and $\alpha_n^i \setminus \alpha_{n-1}^i = \pi_n^{-1}\left(\left[\frac{1}{8}, \frac{1}{2}\right)\right)$.

3.3. *The bonding maps.*

3.3.1. *The map g_1.* We define the function $g_1 : X_2 \to X_1$ by:

$$g_1(x) = \begin{cases} x, & \text{if } x \in \alpha_2^2; \\ \left(y_1, y_2, \ldots, y_{n-2}, x_{n-1}, x_n, x_{n+1}, \ldots\right), & \text{if } x \in \alpha_n^2 \setminus \alpha_{n-1}^2, n > 2; \\ \left(\frac{1}{2}, \frac{1}{2}, \ldots\right), & \text{if } x \in K_2; \end{cases}$$

where $y_i = f_1^{n-1-i}\left(x_{n-1}\right)$, and $x = (x_1, x_2, x_3, \ldots) \in X_2$. We observe that g_1 is well defined, because $f_1|_{\left[0,\frac{1}{2}\right]} = f_2|_{\left[0,\frac{1}{2}\right]}$.

3.3.2. *Continuity of g_1 on the ray.* We will see that g_1 is continuous.

We will prove the continuity of g_1 on the ray R_2. Since $g_1|_{\alpha_2^2}$ is the identity, g_1 is continuous at $x \in \alpha_2^2$, because $x \in \pi_2^{-1}\left(\left[0, \frac{1}{2}\right)\right)$, which is an open set of X_2. In the same way, $g_1|_{\pi_n^{-1}\left(\frac{1}{8}, \frac{1}{2}\right)}$ is continuous, since it is continuous at each coordinate and the set $\pi_n^{-1}\left(\frac{1}{8}, \frac{1}{2}\right)$ is open in X_2. Therefore, if $x \in \pi_n^{-1}\left(\frac{1}{8}, \frac{1}{2}\right)$, g_1 is continuous at x. Now, we only have to prove the continuity of g_1 at every $x \in \pi_n^{-1}\left(\frac{1}{8}\right) \subset R^2$.

Let $x = (x_1, x_2, \ldots) \in \pi_n^{-1}\left(\frac{1}{8}\right)$; i.e., $\pi_n(x) = x_n = \frac{1}{8}$. By the definition of X_2, $x_{n-1} = f_2(x_n) = 4\left(\frac{1}{8}\right) = \frac{1}{2}$. Then $x \in \alpha_n^2 \setminus \alpha_{n-1}^2$ and thus $g_1(x) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \ldots\right)$; i.e., $g_1(x) = x$.

Let $\varepsilon > 0$. Then there exist positive numbers δ_0, δ_1 and δ_2 less than ε such that:

a) Since $g_1|_{\alpha_n^2 \setminus \alpha_{n-1}^2}$ is continuous, if $y \in \alpha_n^2 \setminus \alpha_{n-1}^2$ and $d(x, y) < \delta_0$, then $d\left(g_1(x), g_1(y)\right) < \varepsilon$.
b) By the uniform continuity of f_1. If $s, t \in [0, 1]$ and $|s - t| < \delta_1$, then $\sum_{i=1}^{n-2} \frac{\left|f_1^{n-i}(s) - f_1^{n-i}(t)\right|}{2^{i-1}} < \frac{\varepsilon}{4}$.
c) If $y \in R_2$ and $d(x, y) < \delta_2$, then $|\pi_n(x) - \pi_n(y)| < \min\left\{\delta_1, \frac{1}{16}\right\}$.

Let $\delta = \min\{\delta_0, \delta_2\}$ and $y \in X_2$ such that $d(x, y) < \delta$. We consider two cases:

Case 1 $\pi_n(y) = y_n \geq \frac{1}{8}$.

In this case $y \in \alpha_n^2 \setminus \alpha_{n-1}^2$ and since $\delta \leq \delta_0$, by a), $d\left(g_1\left(x\right), g_1\left(y\right)\right) < \varepsilon$.

Case 2 $\pi_n\left(y\right) = y_n < \frac{1}{8}$.

Since $d\left(x, y\right) < \delta_2$, c) implies that $x_n - y_n < \frac{1}{16}$; i.e., either $\frac{1}{8} > y_n > x_n - \frac{1}{16}$ or $\frac{1}{8} > y_n > \frac{1}{16}$. Then $\frac{1}{2} > y_{n-1} > \frac{4}{16} = \frac{1}{4} > \frac{1}{8}$, since $y_{n-1} = f_2\left(y_n\right)$ and $f_2\left(x\right) = 4x$ if $x \in \left[0, \frac{1}{4}\right]$. This implies that $y \in \alpha_{n-1}^2 \setminus \alpha_{n-2}^2$, hence, $g_1\left(y\right) = \left(f_1^{n-3}\left(y_{n-2}\right), \ldots, f_1\left(y_{n-2}\right), y_{n-2}, y_{n-1}, \ldots\right) = \left(f_1^{n-1}\left(y_n\right), \ldots, f_1\left(y_n\right), y_n, \ldots\right)$.

Now, since $f_1 = f_2$ on $\left[0, \frac{1}{2}\right)$,

$$d\left(g_1\left(x\right), g_1\left(y\right)\right) = d\left(\left(f_1^{n-1}\left(x_n\right), \ldots, f_1\left(x_n\right), \frac{1}{8}, \ldots\right), \left(f_1^{n-3}\left(y_{n-2}\right), \ldots, f_1\left(y_{n-2}\right), y_{n-2}, y_{n-1}, \ldots\right)\right).$$

Since $y_{n-2} = f_1^2\left(y_n\right)$, we have that

$$\begin{aligned} d\left(g_1\left(x\right), g_1\left(y\right)\right) &= \sum_{i=1}^{n-1} \frac{\left|f_1^{n-i}\left(x_n\right) - f_1^{n-i}\left(y_n\right)\right|}{2^i} + \sum_{i=1}^{\infty} \frac{\left|x_i - y_i\right|}{2^i} \\ &= \sum_{i=1}^{n-1} \frac{\left|f_2^{n-i}\left(x_n\right) - f_2^{n-i}\left(y_n\right)\right|}{2^i} + \sum_{i=n}^{\infty} \frac{\left|x_i - y_i\right|}{2^i} \\ &< \frac{\varepsilon}{4} + \left|x_n - y_n\right|. \end{aligned}$$

Because, for every $i > n$, $x_{i+1} = \frac{x_i}{4}$ and $y_{i+1} = \frac{y_i}{4}$. Thus, we have that $\left|x_{i+1} - y_{i+1}\right| < \left|x_i - y_i\right|$ and, by c), $d\left(g_1\left(x\right), g_1\left(y\right)\right) < \varepsilon$.

Therefore, g_1 is continuous at every point of R_2.

3.3.3. *Continuity of g_1 on K_2.* Let $\varepsilon > 0$ and $x = \left(x_1, x_2, \ldots\right) \in K_2$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$.

Since $x \in K_2$, $x_n \geq \frac{1}{2}$ for every $n \in \mathbb{N}$ and there exist δ_0, δ_1 and $\delta_2 > 0$ such that:

a) If $y \in K_2$ and $d\left(x, y\right) < \delta_0$, then $d\left(g_1\left(x\right), g_1\left(y\right)\right) < \varepsilon$; in fact, $d\left(g_1\left(x\right), g_1\left(y\right)\right) = 0$.
b) If $s, t \in [0, 1]$ and $\left|s - t\right| < \delta_1$, then $\left|f_2\left(s\right) - f_2\left(t\right)\right| < \frac{\varepsilon}{4}$.
c) If $y \in R_2$ and $d\left(x, y\right) < \delta_2$ then $\left|\pi_N\left(x\right) - \pi_N\left(y\right)\right| < \min\left\{\delta_1, \frac{1}{16}\right\}$.

Since $x_n \geq \frac{1}{2}$ for every $n \in \mathbb{N}$: If $z = \left(z_1, z_2, \ldots, z_{k-1}, z_k, \ldots\right) \in \alpha_k^2 \setminus \alpha_{k-1}^2$, then $z_k < \frac{1}{2}$, $z_i < \frac{1}{8}$ if $i \geq k+1$ and $z_{k-1} \geq \frac{1}{2}$; hence, $d\left(x, z\right) = \sum_{i=1}^{\infty} \frac{\left|x_i - z_i\right|}{2^i} = \sum_{i=1}^{k} \frac{\left|x_i - z_i\right|}{2^i} + \sum_{i=k+1}^{\infty} \frac{\left|x_i - z_i\right|}{2^i} + \frac{x_{k+1} - z_{k+1}}{2^{k+1}} \geq \frac{x_{k+1} - z_{k+1}}{2^{k+1}} > \frac{3/8}{2^{k+1}}$; i.e.:

$$(*) \qquad \text{if } z \in \alpha_k^2 \setminus \alpha_{k-1}^2,\ d\left(x, z\right) > \frac{3/8}{2^{k+1}}.$$

Let $y \in X_2$ such that $d(x,y) < \delta = \min\left\{\delta_0, \delta_2, \frac{\frac{3}{8}}{2^{N+2}}\right\}$. By the choice of δ and by $(*)$, $y \notin \cup_{k=1}^{N+1} \alpha_k^2 \smallsetminus \alpha_{k-1}^2$.

We consider two cases:

CASE 1. If $y \in K_2$.

This case is clear, since g_1 is constant on K_2.

CASE 2. If $y \notin K_2$.

Then $y \in \alpha_j^2 \smallsetminus \alpha_{j-1}^2$ for some $j > N+1$. Thus, $y_j < \frac{1}{2}$ and $y_{j-1} \geq \frac{1}{2}$. By c) and b), $|f_2(x_N) - f_2(y_N)| < \frac{\varepsilon}{2}$. Now, since $f_1(y_{j-1}) = \frac{1}{2}$,

$$g_1(y) = \left(f_1^{j-2}(y_{j-1}), \ldots, f_1(y_{j-1}), y_{j-1}, \ldots\right) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, y_{j-1}, \ldots\right).$$

This implies that

$$\begin{aligned} d(g_1(x), g_1(y)) &= d\left(\left(\frac{1}{2}, \frac{1}{2}, \ldots\right), \left(\frac{1}{2}, \ldots, \frac{1}{2}, y_{j-1}, \ldots\right)\right) \\ &= \sum_{i=1}^{j-2} \frac{\left|\frac{1}{2} - \frac{1}{2}\right|}{2^i} + \sum_{i=j-1}^{\infty} \frac{\left|\frac{1}{2} - y_i\right|}{2^i} < \frac{1}{2^j} < \frac{1}{2^N} < \varepsilon, \end{aligned}$$

by the choice of N. Thus, $d(g_1(x), g_1(y)) < \varepsilon$, and we obtain the continuity of g_1 at every point of X_2.

3.3.4. *The map g_r.* We define the function $g_2 : X_3 \to X_2$, for $x = (x_1, x_2, x_3, \ldots) \in X_3$, by

$$g_2(x) = \begin{cases} x, & \text{if } x \in \alpha_2^3; \\ (y_1, y_2, \ldots, y_{n-2}, x_{n-1}, x_n, x_{n+1}, \ldots), & \text{if } x \in \alpha_n^3 \smallsetminus \alpha_{n-1}^3, n > 2; \\ \overline{g}^{-1}(g_1(\overline{g}(x))), & \text{if } x \in K_3; \end{cases}$$

where $y_i = f_1^{n-1-i}(x_{n-1})$.

In general, we define $g_r : X_{r+1} \to X_r$, for $x = (x_1, x_2, x_3, \ldots) \in X_{r+1}$, by

$$g_r(x) = \begin{cases} x, & \text{if } x \in \alpha_2^{r+1}; \\ (y_1, y_2, \ldots, y_{n-2}, x_{n-1}, x_n, x_{n+1}, \ldots), & \text{if } x \in \alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}, n > 2; \\ \overline{g}^{-1}(g_{r-1}(\overline{g}(x))), & \text{if } x \in K_{r+1}; \end{cases}$$

where $y_i = f_1^{n-1-i}(x_{n-1})$.

3.3.5. *The continuity of g_r.* We only prove the continuity of g_2 since the continuity of g_r is similar, except for the complexity of the cases and the indexes.

We omit the proof of the continuity of g_2 on the ray R_3, because it is similar to the continuity of g_1 on the ray R_2. We will prove de continuity of g_2 on K_3.

Let $x = (x_1, x_2, \ldots) \in K_3$. Then $x_i \geq \frac{1}{2}$ for every $i \in \mathbb{N}$. Since K_3 is homeomorphic to X_2, we have that $K_3 = R_1^3 \cup K_{3,2}$; i.e., K_3 is a compactification of a ray R_1^3 with remainder $K_{3,2}$, also $R_1^3 = \cup_{n=1}^{\infty} \alpha_n^{3,2}$, where $\alpha_n^{3,2} = \left\{y \in K_3 : y_n < \frac{3}{4}\right\}$.

Hence, we have that either $x \in \alpha_s^{3,2} \setminus \alpha_{s-1}^{3,2}$ for some $s \in \mathbb{N}$ or $x \in K_{3,2}$.

Case 1. Suppose that $x \in \alpha_s^{3,2} \setminus \alpha_{s-1}^{3,2}$ for some $s \in \mathbb{N}$.

We note that

$$\begin{aligned} g_2(x) &= g_2(x_1, x_2, \ldots) = \overline{g}^{-1}\left[g_1\left(\overline{g}(x_1, x_2, \ldots)\right)\right] \\ &= \overline{g}^{-1}\left[g_1\left(g(x_1), g(x_2), \ldots, g(x_{s-2}), g(x_{s-1}), g(x_s), \ldots\right)\right] \\ &= \overline{g}^{-1}\left[f_1^{s-2}\left(g(x_{s-1})\right), \ldots, f_1\left(g(x_{s-1})\right), g(x_{s-1}), g(x_s), \ldots\right] \\ &= \left(g^{-1}\left(f_1^{s-2}\left(g(x_{s-1})\right)\right), \ldots, g^{-1}\left(f_1\left(g(x_{s-1})\right)\right), g^{-1}\left(g(x_{s-1})\right),\right. \\ &\qquad \left. g^{-1}\left(g(x_s)\right), \ldots\right). \end{aligned}$$

Also, we note that, since $x \in \alpha_s^{3,2} \setminus \alpha_{s-1}^{3,2}$, then $x_i \geq \frac{3}{4}$ for every $i \leq s-1$; and thus, $g(x_i) \geq \frac{1}{2}$ for every $i \leq s-1$, $f_1(g(x_i)) = \frac{1}{2}$ and $g^{-1}(f_1(g(x_i))) = \frac{3}{4}$. $g_2(x) = \left(\frac{3}{4}, \ldots, \frac{3}{4}, x_{s-1}, x_s, \ldots\right) = \left(f_2^{s-1}(x_s), \ldots, f_2^2(x_s), f_2(x_s), x_s, \ldots\right)$. This follows from the definition of f_2.

Let $\varepsilon > 0$. Then there exist positive numbers δ_0, δ_1 and δ_2, every one less than $\frac{\varepsilon}{2}$, such that:

a) If $y \in K_3$ and $d(x, y) < \delta_0$, then $d(g_2(x), g_2(y)) < \varepsilon$.

b) If $r, t \in [0,1]$ and $|r - t| < \delta_1$, then $\sum_{i=1}^{s-1} \frac{\left|f_2^i(r) - f_2^i(t)\right|}{2^i} < \frac{\varepsilon}{2}$.

c) If $y \in R_3$ and $d(x, y) < \delta_2$, then $|\pi_s(x) - \pi_s(y)| < \min\left\{\delta_1, \frac{1}{64}\right\}$.

Let us note that, as in $(*)$ of the Section 3.3.3,

$$(**) \qquad \text{if } z \in \alpha_k^3,\ d(x, z) > \frac{\frac{3}{32}}{2^{k-1}}.$$

Let $\delta = \min\left\{\delta_0, \delta_2, \frac{\frac{3}{32}}{2^{N+2}}\right\}$. Let $y \notin K_3$ such that $d(x, y) < \delta$. Then, by the choice of δ, $y \in \alpha_j^3 \backslash \alpha_{j-1}^3$ for some $j > s$.

Now, since $x_{s-1} \geq \frac{3}{4}$ and $\frac{9}{16} < x_s < \frac{3}{4}$, we obtain the following: $f_2|_{\left[\frac{1}{2}, \frac{3}{4}\right]} = f_3|_{\left[\frac{1}{2}, \frac{3}{4}\right]}$, $\frac{1}{2} + \frac{1}{64} < x_{s+1} < \frac{9}{16}$ and $\frac{1}{2} + \frac{1}{4^{i+2}} < x_i < \frac{1}{2} + \frac{1}{4^{i+1}}$ where $i > s$.

By c) $|\pi_s(x) - \pi_s(y)| < \delta_1$; thus, $\frac{1}{2} + \frac{1}{64} < y_s < \frac{3}{4}$ and, by definition of f_3, $\frac{1}{2} + \frac{1}{4^{j+2}} < y_{j-1} < \frac{1}{2} + \frac{1}{4^j}$. So $y_i \in \left(\frac{1}{2}, \frac{5}{8}\right]$ where $i \in \{s, s+1, \ldots, j-1\}$ and $f_2(y_i) \in \left(\frac{1}{2}, \frac{3}{4}\right]$. Therefore, $y_{i+1} = f_2(y_i)$, $i \in \{s, s+1, \ldots, j-1\}$. Now, if we calculate $g_2(y) = \left(f_2^{j-2}(y_{j-1}), \ldots, f_2^{j-1-s}(y_{j-1}), \ldots, f_2(y_{j-1}), y_{j-1}, \ldots\right)$, where the s-th coordinate is $f_2^{j-1-s}(y_{j-1})$ then

$$\begin{aligned} g_2(y) &= \left(f_2^{j-2}(y_{j-1}), \ldots, f_2^{j-2-s}(y_{j-1}), y_s, y_{s+1}, \ldots, y_{j-1}, y_j, \ldots\right) \\ &= \left(f_2^{s-1}(y_s), \ldots, f_2(y_s), y_s, \ldots\right). \end{aligned}$$

Thus,

$$\begin{aligned} &d\left(g_2\left(x\right), g_2\left(y\right)\right) \\ &\qquad = d\Big(\left(f_2^{s-1}\left(x_s\right), \ldots, f_2^2\left(x_s\right), f_2\left(x_s\right), x_s, \ldots\right), \\ &\qquad\qquad\qquad\qquad \left(f_2^{s-1}\left(y_s\right), \ldots, f_2\left(y_s\right), y_s, \ldots\right)\Big) \\ &\qquad = \sum_{i=1}^{s-1} \frac{\left|f_2^i\left(x_s\right) - f_2^i\left(y_s\right)\right|}{2^i} + \sum_{i=s}^{\infty} \frac{\left|x_s - y_s\right|}{2^i} < \frac{\varepsilon}{2} + \delta_1 < \varepsilon. \end{aligned}$$

By b), the first sum is less than $\frac{\varepsilon}{2}$. Regarding second sum, we note that $f_3|_{\left[\frac{1}{2},\frac{5}{8}\right]}$ is the monotone map $f_3\left(t\right) = 4t - \frac{3}{2}$; and thus, if $i > s$, $\left|x_{i+1} - y_{i+1}\right| = \left|\frac{x_i}{4} + \frac{3}{8} - \frac{y_i}{4} - \frac{3}{8}\right| = \left|\frac{x_i}{4} - \frac{y_i}{4}\right| < \left|x_i - y_i\right|$. Then $\left|x_s - y_s\right| > \left|x_i - y_i\right|$ for every $i > s$, thus, the second sum is less than $\left|x_s - y_s\right|$ and, by c) it is less than δ_1. Therefore, g_2 is continuous at x.

Case 2. If $x \in K_3$.

In this case

$$\begin{aligned} g_2\left(x\right) &= g_2\left(x_1, x_2, \ldots\right) = \overline{g}^{-1}\left[g_1\left(\overline{g}\left(x_1, x_2, \ldots\right)\right)\right] \\ &= \overline{g}^{-1}\left[g_1\left(g\left(x_1\right), g\left(x_2\right), \ldots, g\left(x_{s-2}\right), g\left(x_{s-1}\right), g\left(x_s\right), \ldots\right)\right] \end{aligned}$$

and, since $x_i \geq \frac{3}{4}$ for every i, $g\left(x_i\right) \geq \frac{1}{2}$. Then $g_2\left(x\right) = \overline{g}^{-1}\left(\frac{1}{2}, \frac{1}{2}, \ldots\right) = \left(g^{-1}\left(\frac{1}{2}\right), g^{-1}\left(\frac{1}{2}\right), \ldots\right) = \left(\frac{3}{4}, \frac{3}{4}, \ldots\right)$.

Let $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$. By (**), if $z \in \alpha_k^3$ then $d\left(x, z\right) > \frac{\frac{3}{32}}{2^{k-1}}$. Then, let $\delta < \frac{\frac{3}{32}}{2^{N-1}}$, thus, if $y \in R_3$ and $d\left(x, y\right) < \delta$, then $y \in \alpha_j^3 \backslash \alpha_{j-1}^3$, where $j > N + 1$; and thus, $g_2\left(y\right) = \left(f_2^{j-2}\left(y_{j-1}\right), \ldots, f_2\left(y_{j-1}\right), y_{j-1}, y_j, \ldots\right)$. We note that $y_{j-1} \geq \frac{1}{2}$ and $y_i \geq \frac{1}{2}$ for every $i \leq j-1$. By c) $\left|y_N - x_N\right| < \delta_1 < \frac{1}{32}$, but $x_N = \frac{3}{4}$. Thus, $y_N > \frac{5}{8}$. Hence, $y_{N-1} \geq \frac{3}{4}$ and $y_i \geq \frac{3}{4}$ for every $i < N$.

Since $y_i \geq \frac{1}{2}$ for every $i \leq j - 1$, we have two subcases:

Case 2.1. If for some $i \in \{N, \ldots, j-1\}$, $y_i \geq \frac{3}{4}$.

In this case $g_2\left(y\right) = \left(\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, y_i, y_{i+1}, \ldots, y_{j-1}, y_j, \ldots\right)$.

Case 2.2. The first coordinate i where $y_i \geq \frac{3}{4}$ is $i = N - 1$.

In both cases, the image of y is $\frac{3}{4}$ in the first $(N-1)$ coordinates. Therefore,

$$\begin{aligned} d\left(g_2\left(x\right), g_2\left(y\right)\right) &= \left(\left(\frac{3}{4}, \frac{3}{4}, \ldots\right)\left(\frac{3}{4}, \ldots, \frac{3}{4}, y_i, y_{i+1}, \ldots\right)\right) \\ &= \sum_{i=1}^{N-1} \frac{\left|\frac{3}{4} - \frac{3}{4}\right|}{2^i} + \sum_{i=N}^{\infty} \frac{\left|\frac{3}{4} - \pi_i\left(g_2\left(y\right)\right)\right|}{2^i} < \varepsilon. \end{aligned}$$

Since the first sum is zero and the other is less than $\frac{1}{2^N}$, then $d\left(g_2\left(x\right), g_2\left(y\right)\right) < \varepsilon$ and we obtain the continuity of g_2 at every point of X_3.

3.3.6. *The function g_r is monotone.* We start proving that $g_r|_{R_{r+1}}$ is a homeomorphism. Then we show that $g_r|_{R_s^{r+1}}$ is a homeomorphism. Finally we will prove that g_r is monotone. Recall

$$g_r(x) = \begin{cases} x, & \text{if } x \in \alpha_2^{r+1}; \\ (y_1, y_2, \ldots, y_{n-2}, x_{n-1}, x_n, x_{n+1}, \ldots), & \text{if } x \in \alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}, n > 2; \\ \overline{g}^{-1}(g_{r-1}(\overline{g}(x))), & \text{if } x \in K_{r+1}; \end{cases}$$

where $y_i = f_1^{n-1-i}(x_{n-1})$.

Proposition 3.1. *The image of R_{r+1} under the function g_r is R_r.*

Proof. Let $x = (x_1, x_2, \ldots) \in R_{r+1}$. Then either $x \in \alpha_2^{r+1}$ or $x \in \alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}$ for some $n > 2$. If $x \in \alpha_2^{r+1}$, then $g_r(x) = x$ and, since $x_2 < \frac{1}{2}$, $g_2(x) \in \alpha_2^r \subset R_r$. On the other hand, if $x \in \alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}$, we obtain that $g_r(x) = \left(f_r^{n-2}(x_{n-1}), f_r^{n-1}(x_{n-1}), \ldots, f_r(x_{n-1}), x_{n-1}, x_n, \ldots\right)$, which is a point of X_r such that its n-th coordinate, x_n, is less than $\frac{1}{2}$, and every coordinate $i < n$, satisfies that $x_i \geq \frac{1}{2}$. Thus, $g_r(x) \in \alpha_n^r \smallsetminus \alpha_{n-1}^r$; i.e., the image of x under g_r is in R_r.

If $y = (y_1, y_2, \ldots) \in R_r$, then either $y \in \alpha_2^r$ or $y \in \alpha_n^r \smallsetminus \alpha_{n-1}^r$ for some $n > 2$. If $y \in \alpha_2^r$ then, if we take $x = y$, we obtain that $g_r(x) = y$, if $y \in \alpha_n^r \smallsetminus \alpha_{n-1}^r$, then, if we take the point $x = \left(f_{r+1}^{n-2}(y_{n-1}), f_{r+1}^{n-1}(y_{n-1}), \ldots, f_{r+1}(y_{n-1}), y_{n-1}, y_n, \ldots\right)$, we obtain that x is a point in $\alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}$, which satisfies the following

$$g_r(x) = \left(f_r^{n-2}(y_{n-1}), f_r^{n-1}(y_{n-1}), \ldots, f_r(y_{n-1}), y_{n-1}, y_n, \ldots\right) = y.$$

This shows that every point of R_r is the image of a point of R_{r+1}. □

Proposition 3.2. *$g_r|_{R_{r+1}}$ is injective.*

Proof. Let $x, y \in R_{r+1}$ and suppose that $g_r(x) = g_r(y)$. By definition of the map g_r, it is clear that, either $x, y \in \alpha_2^{r+1}$ or there is an $n > 2$, such that, $x, y \in \alpha_n^{r+1} \smallsetminus \alpha_{n-1}^{r+1}$.

In the first case: $x = g_r(x) = g_r(y) = y$. In the second case:

$$g_r(x) = \left(f_r^{n-2}(x_{n-1}), f_r^{n-1}(x_{n-1}), \ldots, f_r(x_{n-1}), x_{n-1}, x_n, \ldots\right)$$

and

$$g_r(y) = \left(f_r^{n-2}(y_{n-1}), f_r^{n-1}(y_{n-1}), \ldots, f_r(y_{n-1}), y_{n-1}, y_n, \ldots\right),$$

the second equality is true only when $x = y$. □

Theorem 3.3. *$g_r|_{R_{r+1}}$ is a homeomorphism.*

Proof. By the Propositions 3.1 and 3.2, $g_r|_{R_{r+1}}$ is a bijective function. The proof that $g_r|_{R_{r+1}}$ is continuous is similar to the continuity of $g_1|_{R_2}$ (which is in the Section 3.3.2).

Moreover, the inverse function of $g_r|_{R_{r+1}}$ is continuous, which is defined by

$$g_r^{-1}(x) = \begin{cases} x, & \text{if } x \in \alpha_2^r; \\ \left(f_{r+1}^{n-2}(x_{n-1}), \ldots, f_{r+1}(x_{n-1}), x_{n-1}, x_n, \ldots\right), & \text{if } x \in \alpha_n^r \setminus \alpha_{n-1}^r; \end{cases}$$

where $x = (x_1, x_2, \ldots) \in R_r$. The proof of the continuity is similar to proof of the continuity of g_r. ◻

Now, we will show that $g_2|_{R_1^3}$ is a homeomorphism from R_1^3 onto R_1^2.

First, we note that $X_r = \overline{R_r} = R_r \cup K_r$ and K_r is homeomorphic to X_{r-1}. Since $X_1 = R_1 \cup K_1$, $X_2 = R_2 \cup R_1^2 \cup K_1^2$, $X_3 = R_3 \cup R_1^3 \cup R_2^3 \cup K_1^3$ and, in general $X_r = R_r \cup R_1^r \cup R_2^r \cup \cdots \cup R_{r-1}^r \cup K_1^r$. We note that $R_{r-1}^r \cup K_1^r$ is homeomorphic to X_1 and, in general, that $R_{r-s}^r \cup \cdots \cup R_{r-1}^r \cup K_1^r$ is homeomorphic to X_s.

Since K_3 is homeomorphic to X_2, then $K_3 = R_1^3 \cup K_{3,2}$ and $K_{3,2}$ is homeomorphic to X_1.

Recall the definition of g_2, if $x = (x_1, x_2, \ldots) \in X_3$, then

$$g_2(x) = \begin{cases} x, & \text{if } x \in \alpha_2^3; \\ (y_1, y_2, \ldots, y_{n-2}, x_{n-1}, x_n, x_{n+1}, \ldots), & \text{if } x \in \alpha_n^3 \setminus \alpha_{n-1}^3, n > 2; \\ \overline{g}^{-1}\left(g_1\left(\overline{g}(x)\right)\right), & \text{if } x \in K_3; \end{cases}$$

where $y_i = f_1^{n-1-i}(x_{n-1})$.

Let us see the image of one element in R_1^3 under g_2.

If $x \in K_3$, then either $x \in \alpha_s^{3,2} \setminus \alpha_{s-1}^{3,2}$ for some $s \in \mathbb{N}$ or $x \in K_{3,2}$.

If $x \in \alpha_s^{3,2} \setminus \alpha_{s-1}^{3,2}$ for some $s \in \mathbb{N}$, then

$$\begin{aligned} g_2(x) &= \overline{g}^{-1}\left(g_1\left(\overline{g}(x)\right)\right) \\ &= \left(g^{-1}\left(f_1^{s-2}\left(g(x_{s-1})\right)\right), \ldots, g^{-1}\left(f_1\left(g(x_{s-1})\right)\right), x_{s-1}, x_s, \ldots\right). \end{aligned}$$

But $\frac{1}{2} \leq x_s \leq \frac{3}{4}$ and $x_{s-1} \geq \frac{3}{4}$. In fact, $x_i \geq \frac{3}{4}$ for every $i < s$, and $\frac{1}{2} \leq x_i \leq \frac{3}{4}$ for every $i \geq s$. Then $g_2(x) = \overline{g}^{-1}\left(g_1\left(\overline{g}(x)\right)\right) = \overline{g}^{-1}\left(g_1\left(g(x_1), g(x_2), \ldots, g(x_{s-1}), g(x_s), \ldots\right)\right)$.

Since $0 \leq g(x_s) \leq \frac{1}{2}$, $g(x_{s-1}) \geq \frac{1}{2}$; in fact, $0 \leq g(x_i) < \frac{1}{2}$ if $i \geq s$ and $g(x_i) \geq \frac{1}{2}$ if $i < s$. Then $\overline{g}(x) \in \alpha_s^2 \setminus \alpha_{s-1}^2$. Hence,

$$\begin{aligned} g_2(x) &= \overline{g}^{-1}\left(f_1^{s-2}\left(g(x_{s-1})\right), \ldots, f_1\left(g(x_{s-1})\right), g(x_{s-1}), g(x_s), \ldots\right) \\ &= \left(g^{-1}\left(f_1^{s-2}\left(g(x_{s-1})\right)\right), \ldots, g^{-1}\left(f_1\left(g(x_{s-1})\right)\right), x_{s-1}, x_s, \ldots\right). \end{aligned}$$

Since $x_{s-1} \geq \frac{1}{2}$ and $x_i \geq \frac{1}{2}$ for every i, and, since $x_{s-1} \geq \frac{3}{4}$, then $f_2(x_{s-1}) = \frac{3}{4}$. We note that $g^{-1}\left(f_1^2\left(g(x_{s-1})\right)\right) = g^{-1}\left(f_1\left(g\left(g^{-1}\left(f_1\left(g(x_{s-1})\right)\right)\right)\right)\right) = g^{-1}\left(f_1\left(g\left(f_2(x_{s-1})\right)\right)\right)$.

Moreover, $g^{-1}\left(f_1^n\left(g(x_{s-1})\right)\right) = f_2^n(x_{s-1})$. Thus,

$$g_2(x) = \left(f_2^{s-1}(x_{s-1}), \ldots, f_2(x_{s-1}), x_{s-1}, x_s, \ldots\right).$$

Since $x_{s-1} \geq \frac{3}{4}$, it follows, from the definition of f_2, that

$$(\#) \qquad g_2(x) = \left(\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, x_{s-1}, x_s, \ldots\right).$$

This implies $g_2(x) \in R_1^2$.

Moreover, if $y = (y_1, y_2, \ldots) \in R_1^2$, then $y_i \geq \frac{1}{2}$ for every $i \in \mathbb{N}$ and $y_s \leq \frac{3}{4}$ for some $s \in \mathbb{N}$; i.e., $y = \left(f_2^{s-1}(y_{s-1}), \ldots, f_3(y_{s-1}), y_{s-1}, \ldots\right)$; hence, if we define $x = \left(f_3^{s-3}(y_{s-1}), \ldots, f_3(y_{s-1}), y_{s-1}, y_s, \ldots\right)$, then $\frac{1}{2} \leq y_s < \frac{3}{4}$ and $y_{s-1} \geq \frac{3}{4}$; thus, $g_2(x) = y$.

Now we will prove that $g_2|_{R_1^3}$ is injective.

Let $x, y \in R_1^3$ and let $g_2(x) = g_2(y)$. By (#), we obtain that $\left(\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, x_{s-1}, x_s, \ldots\right) = \left(\frac{3}{4}, \frac{3}{4}, \ldots, \frac{3}{4}, y_{s-1}, y_s, \ldots\right)$, where $x_i = y_i$ for every $i \geq s-1$, but $x_i = f_3^{s-1-i}(x_{s-1})$ and $y_i = f_3^{s-1-i}(y_{s-1})$ for every $i < s-1$. Hence, $x = y$.

Thus, $g_2|_{R_1^3}^{-1}$ defined for $x = (x_1, x_2, \ldots) \in R_1^2$ by

$$g_2|_{R_2^3}^{-1}(x) = \left(f_3^{s-1}(x_{s-1}), \ldots, f_3(x_{s-1}), x_{s-1}, x_s, \ldots\right),$$

is continuous. The proof of the continuity is similar to the proof of the continuity of $g_2|_{R_2}$.

The following are Corollaries to Theorem 3.3.

Corollary 3.4. *$g_2|_{R_1^3}$ is a homeomorphism.*

Corollary 3.5. *$g_r|_{R_s^r}$, $s < r$ is a homeomorphism.*

Now, we will show, by induction, that g_r is monotone.

Proposition 3.6. *g_r is monotone.*

Proof. We will prove that g_1 is monotone. Let $x = (x_1, x_2, \ldots) \in X_1$. If $x \in R_1$, by Proposition 3.1, $g_1^{-1}(x) \in R_2$ and, by Proposition 3.2, $g_1^{-1}(x)$ is a point; and thus, connected. Now, if $x \in K_1$; i.e., $x = \left(\frac{1}{2}, \frac{1}{2}, \ldots\right)$, $g_1^{-1}(x) = K_2$ (it is a consequence of Proposition 3.1), which is connected. Thus, g_1 is monotone.

Now, if $r > 1$, let $x = (x_1, x_2, \ldots) \in X_r$. If $x \in R_r$, by Proposition 3.1, $g_r^{-1}(x) \in R_{r+1}$ and, by Proposition 3.2, $g_r^{-1}(x)$ is a point; and thus, it is connected. If $x \in K_r$, then, as a consequence of Proposition 3.1, $g_r^{-1}(x) \subseteq K_{r+1}$, from where we obtain that $g_r^{-1}(x) = \left(g_r|_{K_{r+1}}\right)^{-1}(x)$. But $g_r|_{K_{r+1}} = \overline{g}^{-1} \circ g_{r-1} \circ \overline{g}$, which is a composition of monotone maps, so $g_r^{-1}(x)$ is connected. Thus, g_r is monotone. ∎

3.4. *The continuum.* Let $X = \varprojlim \{X_n, g_n\}$, where X_n is the n-th factor space that we defined before and g_n is the bonding map between the spaces X_{n+1} and X_n.

First, we will prove that X is a compactification of a ray.

Let $R = \varprojlim \left\{R_n, g_n|_{R_{n+1}}\right\}$. We note that R is well defined, because $g_n|_{R_{n+1}}$ is a homeomorphism from R_{n+1} onto R_n. On the other hand, if $x = (x_1, x_2, \ldots) \in R$, then $x_i \in R_i$ where $x_i = \left(x_1^i, x_2^i, \ldots, x_{s-1}^i, x_s^i, \ldots\right)$. Recall that $R_i = \alpha_1^i \cup \cup_{n=2}^{\infty} \left(\alpha_n^i \setminus \alpha_{n-1}^i\right)$, which implies that $x_i \in \alpha_s^i \setminus \alpha_{s-1}^i$ for some $s \in \mathbb{N}$.

Hence, $x_i = \left(f_i^{s-2}\left(x_{s-1}^i\right), \ldots, f_i\left(x_{s-1}^i\right), x_{s-1}^i, x_s^i, \ldots\right)$ and

$$\begin{aligned}&\left(f_i^{s-2}\left(x_{s-1}^i\right), \ldots, f_i\left(x_{s-1}^i\right), x_{s-1}^i, x_s^i, \ldots\right)\\ &\qquad = \left(f_1^{s-2}\left(x_{s-1}^1\right), \ldots, f_1\left(x_{s-1}^1\right), x_{s-1}^1, x_s^1, \ldots\right).\end{aligned}$$

We obtain the last equality because $g_n|_{R_{n+1}}$ is a homeomorphism for every $n \in \{1, 2, \ldots, i-1\}$. Then there is a homeomorphism from R onto R_1, defined by the first projection map restricted to R, $\pi_1(x) = x_1$, because $\pi_1|_R$ is bijective, continuous and its inverse, defined by $\pi_1^{-1}(x_1) = \left(x_1, g_1^{-1}(x_1), g_2^{-1}\left(g_1^{-1}(x_1)\right), \ldots\right)$ is continuous, since it is continuous at each coordinate. Then we obtain that R is a ray.

Now, if $x = (x_1, x_2, \ldots) \in X \setminus R$, then $x_i \notin R_i$ for some i, but, by Proposition 3.1, we obtain that $x_j \notin R_j$ for every $j \in \mathbb{N}$.

We will show that, given a positive number ε, there exists a point $y \in R$ such that $d(x, y) < \varepsilon$.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$ and there exists $\delta > 0$ such that, if $d(s, t) < \delta$ then

$$\sum_{i=1}^{N-1} \frac{d\left(g_{N-i}\left(\ldots\left(g_{N-1}(s)\right)\right), g_{N-1}\left(\ldots,\left(g_{N-1}(t)\right)\right)\right)}{2^i} < \frac{\varepsilon}{2}.$$

Let $y_N \in R_N$ such that $d(x_N, y_N) < \delta$ (it is possible, because $\overline{R_N} = X_N$).

Since $g_n|_{R_{n+1}}$ is a homeomorphism from R_{n+1} onto R_n, we define $y \in R$, where $y = (y_1, y_2, \ldots, y_N, \ldots)$ such that $y_i = g_{N-i}\left(\ldots\left(g_{N-1}(y_N)\right)\right)$ if $i < N$ and $y_i = g_i^{-1}\left(\ldots\left(g_N^{-1}(y_N)\right)\right)$ if $i > N$.

Then, we obtain a point $y \in R$ and

$$\begin{aligned} d(x, y) &= d\left((x_1, x_2, \ldots), (y_1, y_2, \ldots)\right)\\ &= \sum_{i=1}^{N-1} \frac{d\left(g_{N-i}\left(\ldots\left(g_{N-1}(y_N)\right)\right), g_{N-1}\left(\ldots\left(g_{N-1}(x_N)\right)\right)\right)}{2^i}\\ &\quad + \sum_{i=N}^{\infty} \frac{d(x_i, y_i)}{2^i} < \varepsilon. \end{aligned}$$

Theorem 3.7. *X is 2-equivalent.*

Proof. Let $q_i = \left(1 - \frac{1}{2^i}, 1 - \frac{1}{2^i}, \ldots\right)$ for every $i \in \{1, 2, \ldots\}$ and $q = (q_1, q_2, \ldots)$. By the definition of X_i and X, $q_i \in X_i$ and $q \in X$.

If A is a subcontinuum of X, we will prove that, if $q \notin A$, then A is an arc and if $q \in A$, then A is homeomorphic to X. Let A a nondegenerate subcontinuum of X.

Case 1. $q \notin A$.

Then $q_i \notin A_i = \pi_i(A)$, for some i, which implies that A_i is a subcontinuum of X_i, such that X_i does not contain K_i and since $X_i = R_i \cup R_{i-1}^i \cup \ldots \cup R_1^i \cup \{q_i\}$, we obtain that either $A_i \subseteq R_i$ or $A_i \subseteq R_s^i$ for some $s \in \{1, 2, \ldots, i-1\}$. From here we obtain that A_i is an arc contained in a ray; thus, $A_{i+1} = \pi_{i+1}(A)$ is contained in either R_{i+1} or $A_{i+1} \subseteq R_s^{i+1}$ for some $s \in \{1, 2, \ldots, i-1\}$ such that $A_i \subseteq R_s^i$ (because, with the bonding map the image of R_{i+1} is R_i and the image of every R_s^{i+1} is R_s).

In general if $j > i$, either $\pi_j(A) = A_j \subseteq R_j$ or $A_j \subseteq R_s^j$ for the $s \in \{1, 2, \ldots, i-1\}$ which satisfies that $A_i \subseteq R_s^i$. Therefore, A is an arc.

Case 2. $q \in A$.

Then $q_i \in A_i = \pi_i(A)$ for every $i \in \mathbb{N}$.

If A_1 is nondegenerate, A_1 is an arc, such that, one of its end points is q_1, then A_1 is homeomorphic to X_1.

Let a_1 be the other end point of A_1. $a_1 \in R_1$ because $A_1 \neq \left\{\left(\frac{1}{2}, \frac{1}{2}, \ldots\right)\right\} = \{q_1\}$. Let $a_2 = g_1^{-1}(a_1)$. Recall that $g_r|_{R_{r+1}}$ is a homeomorphism, $a_2 \in A_2$ and, since $q_2 \in A_2$, we obtain that $A_2 \cap R_2 \neq \emptyset$ and $A_2 \cap K_2 \neq \emptyset$. Moreover, since K_2 is the remainder, $K_2 \subseteq A$. Then A_2 is homeomorphic to X_2.

In general, $a_i = g_{i-1}^{-1}\left(\ldots\left(g_1^{-1}(a_1)\right)\right) \in A_i$ and, since $q_i \in A_i$, we obtain that A_i is a compactification of a subray of R_i, i.e., A_i is homeomorphic to X_i. Thus, A is homeomorphic to X.

Now, let us suppose that A_1 is degenerate. Since A is nondegenerate, let $t > 1$ be the minimum of the numbers, such that A_t is nondegenerate. Then A_t is an arc contained in K_t (recall that $A_{t-1} = \pi_{t-1}(A) = \{q_{t-1}\}$, where a_t and q_t are its end points). Then $a_t \in R_{t-1}^t$.

On the other hand, in X_{t+1}, $a_{t+1} \in R_{t-1}^{t+1}$, and A_{t+1} does not intersect any ray neither R_{t+1} nor R_j^{t+1} for every $j < t-1$.

Since A_{t+1} intersect to K_{t+1}, it is a compactification of a subray of R_{t-1}^{t+1}, thus A_{t+1} is homeomorphic to X_1. If we continue with this process, we obtain that A_j is homeomorphic to X_{j-t} for every $j > t$, and thus A is homeomorphic to X. ∎

Theorem 3.8. *X is a Kelley continuum.*

Proof. Note that $X = \varprojlim\{X_n, g_n\}$, each X_n is a Kelley continuum and every bonding map g_n is confluent. By the Theorem 1.3, X is a Kelley continuum. ∎

Acknowledgements.

The author thanks the referee for his/her suggestions. Thanks, too, to

Rocío Leonel and Sergio Macías for their very useful help in the preparation of this paper.

References

[1] R. A. Beane and W. J. Charatonik, *Kelley remainders of* $[0, \infty)$, Topology Proc. **32** (2008), 101–114.

[2] R. Bennett, On inverse limit sequences, Master's Thesis, University of Tennesse, 1962.

[3] J. J. Charatonik and W. J. Charatonik, *Fans with the property of Kelley*, Topology Appl. **29** (1988), 73–78.

[4] W. J. Charatonik, *Inverse limits of smooth continua*, Comment. Math. Univ. Carolin. **23** (1982), 183–191.

[5] H. Cook, *Tree-likeness of hereditarily equivalent continua*, Fund. Math. **68** (1970), 203–205.

[6] G. W. Henderson, *Proof that every compact decomposable continuum which is topologically equivalent to each of its nondegenerate subcontinua is an arc*, Ann. of Math. (2) **72** (1960), 421–428.

[7] W. T. Ingram, *Periodicity and indecomposability*, Proc. Amer. Math. Soc. **123** (1995), 1907–1916.

[8] W. T. Ingram, *Families of inverse limits on* $[0, 1]$, Topology Proc. **27** (2003), 189–201.

[9] S. Macías, Topics on continua, Chapman & Hall/CRC, Boca Raton, 2005.

[10] T. Mackowiak, Singular arc-like continua, Dissertationes Math. **257** (1986), 40 pp.

[11] W. S. Mahavier, *Continua with only two topologically different subcontinua*, Topology Appl. **94** (1999), 243–252.

[12] S. Mazurkiewicz, *Problem 14*, Fund. Math. **2** (1921), p. 286.

[13] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. **63** (1948), 581–594.

[14] S. B. Nadler Jr., Continuum theory. An introduction, Marcel Deker, Inc., New York, 1992.

[15] S. B. Nadler Jr., Hyperspaces of sets. A text with research questions. Aportaciones Matemáticas, Serie Textos **33**, Sociedad Matemática Mexicana, Mexico, 2006.

[16] G. T. Whyburn, *A continuum every subcontinuum of which separates the plane*, Amer. J. Math. **52** (1930), 319–330.

C. Islas
Department of Mathematics
Faculty of Sciences
Universidad Autónoma de la Ciudad de México
04510 México DF
México
E-mail: islas@matem.unam.mx & carlos.islas@uacm.edu.mx

Received: 5.5.2009.
Revised: 2.2.2010. & 7.4.2010.