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ON FINITE p-GROUPS CONTAINING A MAXIMAL ELEMENTARY ABELIAN SUBGROUP OF ORDER p^2

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Abstract. We continue investigation of a p-group G containing a maximal elementary abelian subgroup R of order p^2, $p > 2$, initiated by Glauberman and Mazza [GM]; case $p = 2$ also considered. We study the structure of the centralizer of R in G. This reduces the investigation of the structure of G to results of Blackburn and Janko (see references). Minimal nonabelian subgroups play important role in proofs of Theorems 2 and 5.

Glauberman and Mazza ([GM]) have proved that if a p-group G, $p > 2$, possesses a maximal elementary abelian subgroup R of order p^2 (i.e., R is not contained in an elementary abelian subgroup of G of order p^3), then G has no elementary abelian subgroup of order p^{p+1}. The proof of this deep result is not elementary.

In this note we continue to study the structure of groups from [GM] clearing the structure of $\mathrm{C}_G(R)$ also in case $p = 2$. In the last case, there is G containing an elementary abelian subgroup of order 2^4, and all such G are classified in [BJ1, Theorem 127.1].

We use elementary prerequisites only and standard notation (see [BJ1, BJ2, B].) Only finite p-groups are considered, p is a prime. By C_{p^n}, E_{p^n}, D_{2^n} and Q_{2^n} we denote cyclic, elementary abelian, dihedral and generalized quaternion groups of orders p^n, p^n, 2^n and 2^n, respectively. Next, $\mathrm{Z}(G)$ is the center of G and $\Phi(G)$ its Frattini subgroup, $\mathrm{d}(G) = \log_p(|G : \Phi(G)|)$.

The N/C-theorem ([B, Introduction, Proposition 12]) asserts that if $H \leq G$, then the number $|\mathrm{N}_G(H)/\mathrm{C}_G(H)|$ divides $|\mathrm{Aut}(H)|$.

Let G be a minimal nonabelian p-group (see [B, Exercise 1.8a] and [BJ1, Lemma 65.1]). Then (i) $\mathrm{Z}(G) = \Phi(G)$ has index p^2 in G, (ii) $|\Omega_1(G)| \leq p^3$

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and, if $|\Omega_1(G)| \le p^2$, then G is metacyclic. If, in addition, G is metacyclic of order $> p^3$, then $G = B \cdot A$, a semidirect product with cyclic kernel A and cyclic complement B.

If all minimal nonabelian subgroups of a nonabelian 2-group G are $\cong \mathrm{Q}_8$, then $G = Q \times E$, where Q is generalized quaternion and $\exp(E)$ divides 2 ([BJ1, Corollary A.17.3]). This result is used essentially in the proof of Theorem 5.

If a 2-group G has a maximal elementary abelian subgroup R of order 4, then every subgroup of G is generated by four elements so G has no elementary abelian subgroup of order 2^5. Indeed, in view of MacWilliams' theorem (see [BJ1, Theorem 50.3]), it suffices to show that G has no normal elementary abelian subgroup of order 8. Assume that $E \cong \mathrm{E}_8$ is a normal subgroup of G. Then, since RE is not of maximal class ([B, Proposition 1.6]), we get $\mathrm{C}_{RE}(R) > R$ ([B, Proposition 1.8]), and $\mathrm{C}_{RE}(R)$ is elementary abelian, by the modular law, and this is a contradiction. Note that the wreath product $G = \mathrm{Q}_{2^n} \,\mathrm{wr}\, \mathrm{C}_2$ has a maximal elementary abelian subgroup of order 4 and a maximal subgroup B (the base of this wreath product) with $\mathrm{d}(B) = 4$.

We begin with the following

Proposition 1. *Suppose that a p-group G contains a maximal elementary abelian subgroup R of order p^2 and R is not G-invariant. If $x \in R - \mathrm{Z}(G)$, then $C_G(x) = \mathrm{C}_G(R)$ has no metacyclic subgroup of order p^4 and exponent p^2.*

Proof. By hypothesis, G is nonabelian and $\Omega_1(\mathrm{Z}(G)) < R$. Write $C = \mathrm{C}_G(R)$ and $N = \mathrm{N}_G(R)$; then $|N : C| = p$, by the N/C-theorem. Obviously, $\Omega_1(C) = R$. Assume, by way of contradiction, that $L \le C$ is metacyclic of order p^4 and exponent p^2. Clearly, $\mathrm{Z}(G) \cap R = U$ has order p (indeed, $R\Omega_1(\mathrm{Z}(G)) = R$) so $\mathrm{Z}(G)$ is cyclic. If $y \in R - U$, then $\mathrm{C}_G(y) = C(= \mathrm{C}_G(R))$ since $R = \langle y \rangle \times U$ and $U \le \mathrm{Z}(G)$. Since $\exp(L) = p^2$ and L is metacyclic, we have $L = AB$, where A and B are cyclic of order p^2 such that $A \cap B = \{1\}$. At least one of subgroups $\Omega_1(A)$, $\Omega_1(B)$ is different from $U(= \Omega_1(\mathrm{Z}(G)))$; denote that subgroup by $\langle x \rangle$; then, as we have noticed, $\mathrm{C}_G(x) = C$. Let, for definiteness, $x \in A$. We have $\langle x \rangle \times U = R$.

If G has no normal abelian subgroup of type (p, p), it is a 2-group of maximal class ([B, Lemma 1.4]), and such G has no subgroup isomorphic to L, a contradiction. Let $E \lhd G$ be abelian of type (p, p); then $R \ne E$, by hypothesis, and $U < E$. Write $F = \langle x, E \rangle$, where x is chosen in the previous paragraph. Clearly, $x \notin E$ (otherwise, $E = \langle x \rangle \times U = R$). Since $R < F$ and $\Omega_1(F) = F$, it follows that F is nonabelian of order p^3. Recall that $A < L$ is the cyclic subgroup of order p^2 containing x. In this case, $W = A \cdot E$ is the natural semidirect product with kernel E; then $F < W$ and F/E is a unique subgroup of order p in the cyclic group W/E of order p^2. Since the centralizer $\mathrm{C}_W(E)$ has index $\le p$ in W, it contains F so F is abelian, a contradiction. Thus, L does not exist. □

THEOREM 2. *Suppose that a nonabelian p-group G, $p > 2$, possesses a maximal elementary abelian subgroup R of order p^2. Set $C = \mathrm{C}_G(R)$. Then one of the following holds:*

(a) *G is metacyclic; then $R = \Omega_1(G)$.*

(b) *G is a p-group of maximal class. If, in addition, $R \triangleleft G$, then $p = 3$.*[1]

(c) *C has a cyclic subgroup of index p so it is abelian of type (p^n, p), $n > 1$.*

PROOF. If $R \leq \mathrm{Z}(G)$, then $C = G$ has no elementary abelian subgroup of order p^3 so it is metacyclic ([Bla1]; see also [B, Theorem 13.7]). In what follows we assume that $R \not\leq \mathrm{Z}(G)$; then $\mathrm{Z}(G)$ is cyclic. It follows that, in any case, C is metacyclic.

Suppose that $R \triangleleft G$. If G contains an elementary abelian subgroup A of order p^3, then $\mathrm{C}_{RA}(R) > R$ is elementary abelian, contrary to the hypothesis. Then, by [B, Theorem 13.7], one of the following holds: (i) G is metacyclic. (ii) $G = \Omega_1(G)Z$, where $\Omega_1(G)$ is nonabelian of order p^3 and exponent p and Z is cyclic. (iii) G is a 3-group of maximal class. In case (i), there are no further restrictions on the structure of G. In case (ii), $\mathrm{C}_G(R)$ has a cyclic subgroup of index p. In case (iii), if $|G| > 3^4$, we have $R = \Omega_1(\Phi(G))$ and $\mathrm{C}_G(R) = C$ has no cyclic subgroups of index 3, by [B, Exercise 9.1(c)]; in this case, C is either abelian or minimal nonabelian. In what follows we assume that R is not G-invariant; then G is not metacyclic. We also assume that G is not of maximal class. Then $|C| > p^2$ ([B, Proposition 1.8]).

In that case, by Proposition 1, C has no metacyclic subgroup of order p^4 and exponent p^2. We claim that then C has a cyclic subgroup of index p. One may assume that $|C| > p^4$. Since C is regular ([B, Theorem 7.1(c)]) and metacyclic, we have $|\Omega_2(C)| \leq p^4$ and $\exp(\Omega_2(C)) = p^2$ so $|\Omega_2(C)| = p^3$ by what has just been said. It follows that C/R has only one subgroup, namely, $\Omega_2(C)/R$, of order p, and hence it is cyclic. If $Z < C$ is maximal such that $R \not\leq Z$ (Z exists since $R \not\leq \Phi(C)$), then Z is cyclic of index p in C. Since $R \leq \mathrm{Z}(C)$, the subgroup C is abelian of type (p^n, p) as in (c). □

Note that the p-groups G, $p > 2$, such that $\mathrm{C}_G(x)$ is abelian of type (p^n, p) for some $x \in G$ of order p, were studied in great detail in rarely cited important Blackburn's paper [Bla2]; that paper yields essential additional information on groups in part (c) of Theorem 2.

A subgroup A of a p-group G is said to be *soft* in G, if $\mathrm{C}_G(A) = A$ and $|\mathrm{N}_G(A) : \mathrm{C}_G(A)| = p$ ([H]). Thus, soft subgroups are abelian. A subgroup C of Theorem 2(c) is soft in G as we have noticed in the first paragraph of the proof of Proposition 1. Moreover, if a nonnormal $R < G$ is of order p^2, then $|\mathrm{N}_G(R) : \mathrm{C}_G(R)| = p$, and, in addition, $\mathrm{C}_G(R)$ is abelian, then it is soft in G.

[1] If a 3-group G of maximal class is not isomorphic to a Sylow 3-subgroup of the symmetric group of degree 3^2, then all maximal elementary abelian subgroups of G have order 3^2 ([B, Exercise 9.13]). If $p > 3$, then there is a p-group G of maximal class and order $> p^4$ that has no such a subgroup as R (this is a case, if $\Omega_1(G) \leq \Phi(G)$).

Soft subgroups have a number of remarkable properties (see [H] and further papers of L. Hethelyi listed in MathSciNet; see also [BJ2, §130]). One of such properties is proved in Remark 3 (note that this proof is distinct from the original one due to L. Hethelyi in [H]).

REMARK 3. (The result of this remark coincides with [BJ2, Lemma 130.2] and taken from [H]). Let A be a nonnormal maximal abelian subgroup of a group G and $|\mathrm{N}_G(A) : A| = p$. Let us prove that, if $A < H < G$, then $|\mathrm{N}_G(H) : H| = p$ (it follows from this that there is only one maximal chain connecting A with G). Set $N_0 = \mathrm{N}_G(A)$; then N_0 is nonabelian. Set $N_1 = \mathrm{N}_G(N_0)$. Then N_0 contains $|N_1 : N_0| > 1$ conjugates of A under N_1. Since N_0 is nonabelian, the number of abelian subgroups of index p in N_0 is equal to $p+1$ (see [B, Exercise 1.6(a)]), therefore, we get $|N_1 : N_0| = p$. The intersection of all abelian subgroups of index p in N_0 coincides with $\mathrm{Z}(N_0) = Z \triangleleft N_1$. The quotient group N_1/Z is nonabelian since its subgroup A/Z (of index p^2) is not normal. Since $\mathrm{C}_G(A) = A$, we get $\mathrm{Z}(G) \le Z < A$. Let $R \le \mathrm{Z}(G)$ be of order p. Then either A/R or N_0/R is a maximal abelian subgroup of G/R since N_1/R, having a nonabelian epimorphic image $N_1/\mathrm{Z}(G)$, is nonabelian. Clearly, the pair $K/R < G/R$, where K/R is the chosen above a maximal abelian subgroup of G/R containing A/R, satisfies $|\mathrm{N}_{G/R}(K/R) : (K/R)| = p$, since $K \in \{A, N_0\}$. Thus, K/R is soft in G/R. By induction, there is only one maximal chain connecting K/R and G/R so there is only one maximal chain connection A and G. Indeed, it is nothing to prove if $K = A$. If $K > A$, then $K = N_0$ so the result also holds since N_0 is a unique subgroup of G of order $p|A|$ containing A. Similarly, by induction, we obtain the second assertion on indices.

REMARK 4. If R and G are as in Theorem 2, then every subgroup $H \le G$ such that $R < H$ and $\exp(H) = p$, has order $\le p^p$. Indeed, $|\mathrm{C}_H(R)| = p^2$ so H is of maximal class ([B, Proposition 1.8]), and now the claim follows from Blackburn's theory of p-groups of maximal class (see [B, Theorems 9.5, 9.6]).

Case $p = 2$ is considered in the following theorem.

THEOREM 5. *Suppose that a nonabelian* 2*-group G contains a maximal elementary abelian subgroup R of order* 4 *and R is not normal in G.*[2] *Then one of the following holds:*

(a) *The subgroup $\mathrm{C}_G(R)$ has a cyclic subgroup of index* 2 *(so it is abelian).*
(b) *The subgroup $\mathrm{C}_G(R) = Q \times Z$, where Q is a generalized quaternion group and $|Z| = 2$.*[3]

[2]If $R \triangleleft G$, then G has no normal elementary abelian subgroup of order 2^3; the structure of such G is described in [BJ1, §50]. Note that a minimal nonmetacyclic group X of order 2^5 satisfies $|\Omega_1(X)| = 4$ and $\mathrm{d}(X) = 3$; the group X is special.

[3]In that case, $\mathrm{Z}(G) = \Omega_1(Q)$ has order 2 since this subgroup is characteristic in C. It follows that if $Z = \langle z \rangle$, then $\mathrm{C}_G(z) = C$. The 2-groups G containing an involution x

PROOF. Set $C = \mathrm{C}_G(R)$; then $\Omega_1(C) = R$. Since R is not normal in G, the subgroup C has no metacyclic subgroup of order 16 and exponent 4, by Proposition 1.

If C is abelian, we get case (a) since C has no abelian subgroup of type $(4,4)$ and so $|\Omega_2(C)| \le 2^3$ (see the proof of Theorem 2).

Now suppose that C is nonabelian. Then C contains a minimal nonabelian subgroup A. Since $\Omega_1(A) \le \Omega_1(C) = R$, it follows that A is metacyclic ([BJ1, Lemma 65.1]). Assume that $|A| > 8$. Then $R < A$ since $\Omega_1(A) \cong \mathrm{E}_4 \cong R = \Omega_1(C)$, so $R = \Omega_1(A) \le \mathrm{Z}(A)$, and we conclude that A has no cyclic subgroup of index 2 (otherwise, A will be abelian). Since, by Proposition 1, A has no metacyclic subgroup of order 16 and exponent 4, we get a contradiction. Therefore, $|A| = 8$. Since $A \not\cong \mathrm{D}_8$, it follows that $A \cong \mathrm{Q}_8$. Thus, all minimal nonabelian subgroups of C are isomorphic to Q_8. It follows that $C = Q \times Z$, where Q is a generalized quaternion group and $|Z| = 2$ ([BJ1, Corollary A.17.3]), and the proof is complete. ∎

PROPOSITION 6 ([GM, Lemma 2.5] for $p > 2$). *Suppose that a p-group G, that is not a 2-group of maximal class, contains a non-G-invariant maximal elementary abelian subgroup R of order p^2. Then G has only one normal elementary abelian subgroup of order p^2, unless $p = 2$ and G contains a proper subgroup of order 2^4 that is isomorphic to the group $K \cong \mathrm{D}_8 * \mathrm{C}_4$ of order 16.*[4]

PROOF. Assume that E and F are distinct G-invariant abelian subgroups of type (p,p) in G. Since $\mathrm{Z}(G)$ is cyclic, we get $E \cap F = U$, where $U = \Omega_1(\mathrm{Z}(G))$ so $|E \cap F| = p$ and the subgroup $H = EF$ has order p^3, by the product formula. The subgroups $E/U, F/U \le \mathrm{Z}(G/U)$. If H is abelian, it is elementary, and so $R \not\le H$. If H is nonabelian, it is either of exponent $p > 2$ or isomorphic to D_8 (this follows from the description of groups of order p^3). In that case, all noncyclic subgroups of index p in H are normal in G since $H/U \le \mathrm{Z}(G)$ and $U = \Phi(H)$. It follows that $R \not\le H$. Write $D = HR$; then $\Omega_1(D) = D$ and, since $U = H \cap R$ has order p, we get $|D| = p^4$, by the product formula. Since E/U and F/U are distinct central subgroups of G/U, it follows that $D/U \cong \mathrm{E}_{p^3}$ so that $\mathrm{d}(D) = 3$ and $\mathrm{cl}(D) = 2$.

Suppose that H is abelian. In that case, $\mathrm{C}_D(R)$ is of exponent p so it coincides with R, by hypothesis, and it follows from [B, Proposition 1.8] that $\mathrm{cl}(D) = 3 > 2$, contrary to the last sentence of the previous paragraph.

Now let H be nonabelian. By [B, Proposition 10.17], $\mathrm{C}_D(H) \not\le H$ since D is not of maximal class, and so $\mathrm{Z}(D)$ has order p^2. It follows that $\mathrm{Z}(D)$ is cyclic (otherwise, $R < R\mathrm{Z}(D) \cong \mathrm{E}_{p^3}$, contrary to the hypothesis). In that case, we

such that $\mathrm{C}_G(x) = Q \times \langle x \rangle$, where Q is either cyclic or a generalized quaternion group, are described in Janko's papers [Jan1] and [Jan2], respectively (see also [BJ1, §§48, 49]), and these sources contain essential additional information on this case.

[4] Note that all abelian subgroups of type $(2,2)$ are normal in K.

have $p = 2$ (if $p > 2$, then $D = \Omega_1(D)$ is of exponent p, a contradiction). It follows that $D \cong \mathrm{D}_8 * \mathrm{C}_4$ has order 2^4 (note that $\mathrm{D}_8 * \mathrm{C}_4 \cong \mathrm{Q}_8 * \mathrm{C}_4$). □

In particular, if, in Proposition 6, $p > 2$, then G has only one normal abelian subgroup of type (p, p), as asserted in [GM, Lemma 2.5].

Definition 7. *A proper subgroup A of a p-group G is said to be generalized soft if, whenever $A \leq H < G$, then $|\mathrm{N}_G(H) : H| = p$ (in that case, there is only one maximal chain connecting A and G but the converse is not true).*

In the following proposition we consider the p-groups containing a subgroup of order p that is, as a rule, generalized soft.

Proposition 8. *Suppose that a p-group G contains a subgroup L of order p such that there is only one maximal chain connecting L and G. Then one of the following holds:*

(a) *G is abelian with cyclic subgroup of index p.*
(b) *$G = \langle a, b \mid a^{p^n} = b^p = 1, b^a = a^{1+p^{n-1}} \rangle \cong \mathrm{M}_{p^{n+1}}$ (see [B, Theorem 1.2]).*
(c) *G is a p-group of maximal class.*[5]

Proof. Write $N = \mathrm{N}_G(L)$; then N/L is cyclic. If $N = G$, we have case (a). Next we assume that $N < G$. If $|N/L| = p$, then G is of maximal class, by [B, Proposition 1.8]. Now assume that $|N/L| > p$. Since L is not G-invariant, it is not characteristic in N so N is not cyclic. Let $R = \Omega_1(N)$ and $\mathrm{N}_1 = \mathrm{N}_G(R)$. Since R is characteristic in N, we get $N < N_1$. By hypothesis, N_1/R is cyclic. Since $R < N < N_1$, it follows that $R = \Omega_1(N_1)$ is characteristic in N_1, and we conclude that $N_1 = G$. In that case, G possesses a cyclic subgroup of index p so $G \cong \mathrm{M}_{p^{n+1}}$, by [B, Theorem 1.2]. □

Remark 9. Below we describe the pairs $L < G$ of 2-groups such that $L \cong \mathrm{E}_4$, L is not G-invariant and there is only one maximal chain connecting L with G. Write $C = \mathrm{C}_G(L)$; then $C < G$. One may assume that $L < C$ (otherwise, G is of maximal class, by [B, Proposition 1.8]). In that case, $C/L > \{1\}$ is cyclic so C is a maximal abelian subgroup of G of rank 2 or 3. If $|C/L| = 2$, then $C \in \{\mathrm{E}_8, \mathrm{C}_4 \times \mathrm{C}_2\}$. Such G are described in [BJ1, §§50,77]. Next assume that $|C/L| > 2$. Let $\mathrm{d}(C) = 3$. Then $T = \Omega_1(C) \cong \mathrm{E}_8$ is a proper characteristic subgroup in C. In that case, $\mathrm{N}_G(T)/T > C/T > \{1\}$ is cyclic, by hypothesis, and so $T = \Omega_1(\mathrm{N}_G(T))$ is characteristic in $\mathrm{N}_G(T)$, and we conclude that $\mathrm{N}_G(T) = G$ hence $\Omega_1(G) = T$. Thus, $G/\Omega_1(G)$ is cyclic and $\Omega_1(G) \cong \mathrm{E}_8$. Then G has a cyclic subgroup of index 4 (such G are described in [BJ1, §74]). Now let C be abelian of rank 2; then $L = \Omega_1(C)$ so C has a cyclic

[5] Not all p-groups of maximal class contain such a subgroup as L (for example, an irregular p-group G of maximal class, $p > 3$, such that $\Omega_1(G)$ is abelian of order p^{p-1}, has no such subgroup).

subgroup of index 2, by Proposition 1. In that case, $\mathrm{N}_G(L)/L$ is cyclic, by hypothesis. Therefore, it follows from $L < C < \mathrm{N}_G(L)$ that $L = \Omega_1(\mathrm{N}_G(L))$ is characteristic in $\mathrm{N}_G(L)$ so $\mathrm{N}_G(L) = G$, i.e., $L \lhd G$, contrary to the hypothesis.

PROBLEMS

1. Suppose that a p-group G, $p > 2$, possesses a maximal elementary abelian subgroup of order p^2 and $H \leq G$. (i) Is it true that $\mathrm{d}(H) \leq p$? (ii) Is it true that $|H| < p^{p+1}$ provided $\exp(H) = p$?
2. Suppose that a p-group G, $p > 2$, possesses a maximal elementary abelian subgroup of order p^n. Is it true that G has no elementary abelian subgroup of order $p^{1+p^{n-1}}$?
3. Study the p-groups all of whose minimal nonabelian (so all nonabelian) subgroups are generalized soft.
4. Study the p-groups containing a cyclic generalized soft subgroup of order p^n (the problem is nontrivial even for $n = 2$).

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