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SOME REMARKS ON DERIVATIONS IN SEMIPRIME RINGS AND STANDARD OPERATOR ALGEBRAS

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Abstract. In this paper identities related to derivations on semiprime rings and standard operator algebras are investigated. We prove the following result which generalizes a classical result of Chernoff. Let X be a real or complex Banach space, let $L(X)$ be the algebra of all bounded linear operators of X into itself and let $A(X) \subseteq L(X)$ be a standard operator algebra. Suppose there exists a linear mapping $D : A(X) \to L(X)$ satisfying the relation $2D(A^3) = D(A^2)A + A^2D(A) + D(A)A^2 + AD(A^2)$ for all $A \in A(X)$. In this case D is of the form $D(A) = AB - BA$ for all $A \in A(X)$ and some fixed $B \in L(X)$, which means that D is a linear derivation.

This research has been motivated by the work of Brešar ([3]) and Chernoff ([4]) and it is a continuation of our recent work ([11–13]). Throughout, R will represent an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. In case we have a ring R an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [x, a]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in

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general not true. A classical result of Herstein ([6]) asserts that any Jordan derivation on a $2-$torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [1]. Cusack ([5]) generalized Herstein's result to $2-$torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation D on an arbitrary $2-$ torsion free ring R is a Jordan triple derivation (see, for example, [1]). Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subseteq L(X)$ is said to be standard in case $F(X) \subseteq A(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem.

Let us start with the following result proved by Brešar ([3]).

THEOREM 1. *Let R be a $2-$torsion free semiprime ring and let $D : R \to R$ be a Jordan triple derivation. In this case D is a derivation.*

Since, as we mentioned above, any Jordan derivation D on arbitrary $2-$torsion free ring is a Jordan triple derivation, one can conclude that Theorem 1 generalizes Cusack's generalization of Herstein's theorem. We proceed with the following result which is motivated by Theorem 1.

THEOREM 2. *Let R be a $2-$torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either*

$$D(xyx) = D(xy)x + xyD(x) \tag{1}$$

or

$$D(xyx) = D(x)yx + xD(yx) \tag{2}$$

holds for all pairs $x, y \in R$. In both cases D is a derivation.

The approach we use in the proof of Theorem 2 differs from those used by Brešar in his proof of Theorem 1. For the proof of Theorem 2 we need the lemma bellow (see [10, Lemma 3]).

LEMMA 3. *Let R be a semiprime ring and let $f : R \to R$ be an additive mapping. Suppose that either*

$$f(x)x = 0$$

or

$$xf(x) = 0$$

holds for all $x \in R$. In both cases $f = 0$.

PROOF OF THEOREM 2. Let us assume that the relation (1) is fulfilled. The linearization of the relation (1) gives

$$D(xyz + zyx) = D(xy)z + D(zy)x + xyD(z) + zyD(x)$$

for all $x, y, z \in R$. In particular, for $z = x^2$ the above relation gives

(3) $D(xyx^2 + x^2yx) = D(xy)x^2 + D(x^2y)x + xyD(x^2) + x^2yD(x), x, y \in R.$

The substitution $xy + yx$ for y in the relation (1) gives

$$D(xyx^2 + x^2yx) = D(x^2y)x + D(xyx)x + x^2yD(x) + xyxD(x), x, y \in R.$$

We have therefore using (1)

(4) $D(xyx^2+x^2yx) = D(x^2y)x+D(xy)x^2+xyD(x)x+x^2yD(x)+xyxD(x)$

for all pairs $x, y \in R$. By comparing (3) and (4) we obtain

(5) $$xyA(x) = 0$$

for all pairs $x, y \in R$, where $A(x)$ stands for $D(x^2) - D(x)x - xD(x)$. Right multiplication of (5) by x and left multiplication by $A(x)$ gives $A(x)xyA(x)x = 0$ for all pairs $x, y \in R$, whence it follows

(6) $$A(x)x = 0$$

for all $x \in R$ by semiprimeness of R. The substitution $A(x)yx$ for y in the relation (5) gives $xA(x)yxA(x) = 0$ for all pairs $x, y \in R$, which gives

(7) $$xA(x) = 0$$

for all $x \in R$. The linearization of the relation (6) gives

$$B(x, y)x + A(x)y + B(x, y)y + A(y)x = 0$$

for all pairs $x, y \in R$, where $B(x, y)$ denotes $D(xy + yx) - D(x)y - xD(y) - D(y)x - yD(x)$. Putting in the above relation $-x$ for x and comparing the relation so obtained with the above relation one obtains

$$B(x, y)x + A(x)y = 0$$

for all pairs $x, y \in R$. Right multiplication of the above relation by $A(x)$ gives, because of the relation (7), $A(x)yA(x) = 0$ for all pairs $x, y \in R$, whence it follows $A(x) = 0$ for all $x \in R$. In other words, D is a Jordan derivation. By Cusack's generalization of Herstein's theorem one can conclude that D is a derivation. In [3] Brešar has proved Theorem 1 without using Cusack's generalization of Herstein's theorem. It is our aim to show that Theorem 2 can be proved without using Cusack's generalization of Herstein's theorem as well. From the fact that D is a Jordan derivation it follows that D is a Jordan triple derivation. Now, comparing the relation $D(xyx) = D(x)yx + xD(y)x + xyD(x), x, y \in R$, with the relation (1) one obtains

$$(D(xy) - D(x)y - xD(y))x = 0, \ x, y \in R.$$

For any fixed $y \in R$ we have an additive mapping $x \mapsto D(xy) - D(x)y - xD(y)$ on R. Thus from the above relation and Lemma 3 it follows $D(xy) - D(x)y - xD(y) = 0$ for all pairs $x, y \in R$. In other words, D is a derivation. The proof that D is a derivation in case $D(xyx) = D(x)yx + xD(yx)$ holds for all pairs $x, y \in R$ goes through in a similar way and will therefore be omitted. ∎

Disadvantage of Theorem 2 is that in identities (1) and (2) there is no symmetry. Theorem 2 together with the desire for symmetry leads to the following conjecture.

Conjecture 4. *Let R be a $2-$torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that*

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx)$$

holds for all pairs $x, y \in R$. In this case D is a derivation.

Our next result is in the spirit of the conjecture above.

Theorem 5. *Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Suppose there exists a linear mapping $D : A(X) \to L(X)$ satisfying the relation*

$$2D(A^3) = D(A^2)A + A^2D(A) + D(A)A^2 + AD(A^2)$$

for all $A \in A(X)$. In this case D is of the form $D(A) = [A, B]$ for all $A \in A(X)$ and some fixed $B \in L(X)$, which means that D is a linear derivation.

Theorem 5 generalizes the result below first proved by Chernoff ([4]) (see also [8, 9]).

Theorem 6. *Let X be a real or complex Banach space, let $A(X)$ be a standard operator algebra on X and let $D : A(X) \to L(X)$ be a linear derivation. In this case D is of the form $D(A) = [A, B]$ for all $A \in A(X)$ and some fixed $B \in L(X)$.*

In the proof of Theorem 5 we use Herstein's theorem, Theorem 6 and methods which are similar to those used in [11–13].

Proof of Theorem 5. We have therefore the relation

$$2D(A^3) = D(A^2)A + A^2D(A) + D(A)A^2 + AD(A^2) \tag{8}$$

for all $A \in A(X)$. Let us first consider the restriction of D on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$ be a projection with $AP = PA = A$. Putting $A + P$ for A in the relation (8) we obtain after some calculations

$$\begin{aligned} 6D(A^2) + 6D(A) = {} & 4D(A)A + 4AD(A) + D(A^2)P + PD(A^2) \\ & + D(P)A^2 + A^2D(P) + 3D(A)P + 3PD(A) \\ & + 3AD(P) + 3D(P)A. \end{aligned}$$

Putting in the above relation $-A$ for A and comparing the relation so obtained with the above relation we obtain

$$6D(A^2) = 4D(A)A + 4AD(A) + D(A^2)P + PD(A^2) + D(P)A^2 + A^2D(P) \tag{9}$$

and

$$2D(A) = D(A)P + PD(A) + AD(P) + D(P)A. \tag{10}$$

Putting A^2 for A in the relation (10) we obtain

$$2D(A^2) = D(A^2)P + PD(A^2) + A^2D(P) + D(P)A^2$$

which reduces the relation (9) to

$$D(A^2) = D(A)A + AD(A). \tag{11}$$

The relation (11) is fulfilled for any $A \in F(X)$. From the relation (10) one can conclude that D maps $F(X)$ into itself. We have therefore a linear mapping which maps $F(X)$ into itself satisfying the relation (11) for all $A \in F(X)$, which means that D is a Jordan derivation on $F(X)$. Since $F(X)$ is prime it follows that D is a derivation by Herstein's theorem. Applying Theorem 6 one can conclude that D is of the form

$$D(A) = [A, B] \tag{12}$$

for all $A \in F(X)$ and some fixed $B \in L(X)$. It remains to prove that the relation (12) holds for all $A \in A(X)$ as well. For this purpose we introduce $D_1 : A(X) \to L(X)$ by $D_1(A) = [A, B]$ and consider $D_0 = D - D_1$. The mapping D_0 is, obviously, linear and satisfies the relation (8). Besides, D_0 vanishes on $F(X)$. It is our aim to prove that D_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be an one-dimensional projection and let us introduce $S \in A(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to see that $D_0(S) = D_0(A)$ and $D_0(S^2) = D_0(A^2)$. Now we have

$$\begin{aligned}
&D_0(S^2)S + S^2D_0(S) + D_0(S)S^2 + SD_0(S^2)\\
&=2D_0(S^3) = 2D_0(S^3 + P) = 2D_0((S + P)^3)\\
&=D_0(S^2)(S + P) + (S^2 + P)D_0(S) + D_0(S)(S^2 + P) + (S + P)D_0(S^2).
\end{aligned}$$

We have therefore $D_0(S^2)P + PD_0(S) + D_0(S)P + PD_0(S^2) = 0$ and since $D_0(S) = D_0(A)$ and $D_0(S^2) = D_0(A^2)$ we arrive at

$$D_0(A^2)P + PD_0(A) + D_0(A)P + PD_0(A^2) = 0. \tag{13}$$

Putting in the above relation $-A$ for A and comparing the relation so obtained with the relation (13) we obtain

$$PD_0(A) + D_0(A)P = 0. \tag{14}$$

Multiplying the above relation from both sides by P we obtain

$$PD_0(A)P = 0. \tag{15}$$

Right multiplication of the relation (14) by P gives, because of (15),

$$D_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection we have $D_0(A) = 0$ for all $A \in A(X)$, which was our intension to prove. The proof of the theorem is complete. $\square$

In the proof of Theorem 5 we used some ideas similar to those used by Molnár in [7].

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