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CONVERGENCE THEOREMS OF ZEROS OF A FINITE FAMILY OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we continue to study convergence problems for a Ishikawa-like iterative process for a finite family of m-accretive mappings. Strong convergence theorems are established in uniformly smooth Banach spaces.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and dual space of E, respectively. Let C be a nonempty subset of E and $T : C \to C$ be a mapping. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. It is well known that, if E^* is strictly convex, then J is single-valued. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([1, 19]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. Browder ([1]) proved that if

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E is a Hilbert space, then x_t converges strongly to a fixed point of T that is nearest to u. Reich ([14]) extended Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Recall that T is said to be pseudo-contractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2, \quad \forall x, y \in C.$$

Clearly, the class of nonexpansive mappings is a subset of the class of pseudo-contractive mappings. Closely related to the class of pseudo-contractive mappings is the class of accretive mappings. Recall that a (possibly multi-valued) operator A with domain $D(A)$ and range $R(A)$ in E is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0.$$

An accretive operator A is m-accretive if $R(I + rA) = X$ for each $r > 0$. The set of zeros of A is denoted by $N(A)$. Hence,

$$N(A) = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from E to $C := \overline{D(A)}$ which will be assumed convex.

We observe that p is a zero of the accretive mapping A if and only if it is a fixed point of the pseudo-contractive mapping $T := I - A$. It is now well known (see [27]) that, if A is accretive, then the solutions of the equation $Ax = 0$ correspond with the equilibrium points of some evolution systems. Consequently, considerable research works, especially, for the past 15 years or more, have been devoted to the iterative methods for approximating the zeros of an accretive mapping A (see [1–3, 5, 6, 9, 11, 15, 26]).

Dominguez Benavides et al. in [10] studied the sequence $\{x_n\}$ generated by the following algorithm:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0 \tag{1.2}$$

and proved strongly convergence of iterative scheme (1.2) in uniformly smooth Banach spaces which have a weakly continuous duality mapping.

Kim and Xu in [12] also studied the iterative process (1.2) and improved the results of Dominguez Benavides et al. ([10]). To be more precisely, they obtained a strong convergence theorem just in uniformly smooth Banach spaces.

Recall that the normal Mann's iterative process was introduced by Mann in [13] in 1953. Since then, construction of fixed points for nonexpansive mappings via the normal Mann's iterative process has been extensively investigated by many authors.

The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.3}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $(0,1)$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.3) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [18]). Therefore, many authors try to modify normal Mann's iteration process to have strong convergence for nonexpansive mappings and other extensions; see, e.g., [7-9,14,16,17,22,25] and the references therein.

This paper, motivated by Cho and Qin ([6]), Ceng et al. ([9]), Dominguez Benavides et al. ([10]), Kim and Xu ([12]), Qin and Su ([17]) and Zegeye and Shahzed ([26]), introduces a Ishikawa-like iterative algorithm as follows.

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ y_n = \beta_n x_n + (1-\beta_n) S_r x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.4}$$

where $u \in C$ is a given point, $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0,1,2,\cdots,r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $(0,1)$. We prove strong convergence of a finite family of m-accretive mappings in a uniformly smooth Banach space. Our results improve the recent ones announced by many authors.

2. Preliminaries

The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for $(x,y) \in U \times U$.

We need the following definitions and lemmas for the proof of our main results.

A Banach space E is said to be strictly convex if, for $a_i \in (0,1)$, $i = 1,2,\cdots,r$, such that $\sum_{i=1}^{r} a_i = 1$,

$$||a_1x_1 + a_2x_2 + \cdots + a_rx_r|| < 1, \quad \forall x_i \in E, \, i = 1,2,\cdots,r,$$

with $||x_i|| = 1$, $i = 1,2,\cdots,r$, and $x_i \neq x_j$ for some $i \neq j$. In a strictly convex Banach space E, we have that, if

$$||x_1|| = ||x_2|| = \cdots = ||x_r|| = ||a_1x_1 + a_2x_2 + \cdots + a_rx_r||$$

for $x_i \in E$, $a_i \in (0,1)$, $i = 1,2,\cdots,r$, where $\sum_{i=1}^{r} a_i = 1$, then $x_1 = x_2 = \cdots = x_r$ (see [23]).

LEMMA 2.1. *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, the following holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

LEMMA 2.2 ([21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

LEMMA 2.3 ([26]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $A_i : C \to E$, $i = 1,2,\cdots,r$, be a family of m-accretive mappings such that $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. Let $a_0, a_1, a_2, \cdots, a_r$ be real numbers in $(0,1)$ such that $\sum_{i=0}^{r} a_i = 1$ and $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, where $J_{A_i} := (I + A_i)^{-1}$. Then S_r is nonexpansive and $F(S_r) = \bigcap_{i=1}^{r} N(A_i)$.*

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny provided $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $Q(x) + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows ([4,20]): if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in D.$$

Reich ([20]) showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows.

LEMMA 2.4 ([20]). *Let X be a uniformly smooth Banach space and let $T: C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu+(1-t)tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q: C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \leq 0, \quad u \in C, z \in F(T).$$

LEMMA 2.5 ([24]). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

3. Main results

THEOREM 3.1. *Let E be a strictly convex and uniformly smooth Banach space. Let C be a nonempty closed convex subset of E. Let $A_i : C \to E$ for $i = 1, 2, \cdots, r$ be a family of m-accretive mappings with $\bigcap_{i=1}^{r} N(A_i) \neq \emptyset$. Let $J_{A_i} := (I + A_i)^{-1}$ for $i = 1, 2, \cdots, r$. Let $\{x_n\}$ be generated by the algorithm (1.4), where $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0, 1, 2, \cdots, r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $(0,1)$ which satisfy the following conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a common solution of the equations $A_i x = 0$ for $i = 1, 2, \cdots, r$.

PROOF. Noticing that Lemma 2.3, we have that S_r is a nonexpansive mapping and

$$F(S_r) = \bigcap_{i=1}^{r} N(A_i) \neq \emptyset.$$

We observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a fixed point p of S_r and notice that

$$\begin{aligned}\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)S_r x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|S_r x_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(u-p)+(1-\alpha_n)(y_n-p)\| \\ &\le \alpha_n\|u-p\|+(1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|.\end{aligned}$$

By simple inductions, we have

$$\|x_n-p\| \le \max\{\|x_0-p\|,\|u-p\|\},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{S_r x_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.1}$$

Put $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$. Now, we compute $l_{n+1}-l_n$. Observing that

$$\begin{aligned}l_{n+1}-l_n =& \frac{\alpha_{n+1}u+(1-\alpha_{n+1})y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ &- \frac{\alpha_n u+(1-\alpha_n)y_n-\beta_n x_n}{1-\beta_n} \\ =& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-y_{n+1}) + \frac{\alpha_n}{1-\beta_n}(y_n-u) \\ &+ S_r x_{n+1} - S_r x_n,\end{aligned}$$

we have

$$\begin{aligned}\|l_{n+1}-l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n-u\| + \|S_r x_{n+1}-S_r x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n-u\| + \|x_{n+1}-x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}&\|l_{n+1}-l_n\| - \|x_n-x_{n-1}\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n-u\|.\end{aligned}$$

Observe conditions (a) and (b) and take the limits as $n\to\infty$ to obtain

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Therefore, we can obtain $\lim_{n\to\infty}\|l_n-x_n\| = 0$ easily by Lemma 2.2. Since

$$x_{n+1}-x_n = (1-\beta_n)(l_n-x_n),$$

we have that (3.1) holds. Observing that

$$x_{n+1}-y_n = \alpha_n(u-y_n),$$

we can easily get

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0. \tag{3.2}$$

Notice that

$$\|x_n - y_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\|.$$

It follows from (3.1) and (3.2) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.3}$$

On the other hand, we have

$$\|S_r x_n - x_n\| \le \|x_n - y_n\| + \|y_n - S_r x_n\| \le \|x_n - y_n\| + \beta_n \|x_n - S_r x_n\|,$$

from which it follows that

$$(1-\beta_n)\|S_r x_n - x_n\| \le \|x_n - y_n\|.$$

Thanks to condition (b), we have

$$\lim_{n\to\infty} \|S_r x_n - x_n\| = 0. \tag{3.4}$$

Next we claim

$$\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \le 0, \tag{3.5}$$

where $q = Q(u) = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x_t \mapsto tu + (1-t)S_r x_t$ by Lemma 2.4. From x_t solves the fixed point equation

$$x_t = tu + (1-t)S_r x_t,$$

we have

$$\begin{aligned} \|x_t - x_n\|^2 &= \|(1-t)(S_r x_t - x_n) + t(u - x_n)\|^2 \\ &\le (1-t)^2 \|S_r x_t - x_n\|^2 + 2t\langle u - x_n, J(x_t - x_n)\rangle \\ &\le (1 - 2t + t)^2)\|x_t - x_n\|^2 + f_n(t) \\ &+ 2t\langle u - x_t, J(x_t - x_n)\rangle + 2t\langle x_t - x_n, J(x_t - x_n)\rangle, \end{aligned} \tag{3.6}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - S_r x_n\|)\|x_n - S_r x_n\| \to 0, \quad \text{as } n \to 0. \tag{3.7}$$

It follows from (3.6) that

$$\langle x_t - u, J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\| + \frac{1}{2t} f_n(t). \tag{3.8}$$

Let $n \to \infty$ in (3.8) and note that (3.7) yields

$$\limsup_{n\to\infty} \langle x_t - u, J(x_t - x_n)\rangle \le \frac{t}{2} M, \tag{3.9}$$

where M is an appropriate constant. Taking $t \to 0$ in (3.9), we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle x_t - u, J(x_t - x_n)\rangle \le 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1 such that, for $t \in (0, \delta_1)$, we get

$$\limsup_{n\to\infty}\langle x_t - u, J(x_t - x_n)\rangle \le \frac{\epsilon}{2}. \tag{3.10}$$

On the other hand, since $x_t \to q$ as $t \to 0$, there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$ we have

$$\begin{aligned}
&|\langle u - q, J(x_n - q)\rangle - \langle x_t - u, J(x_t - x_n)\rangle| \\
&\le |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - x_t)\rangle| \\
&\quad + |\langle u - q, J(x_n - x_t)\rangle - \langle x_t - u, J(x_t - x_n)\rangle| \\
&\le |\langle u - q, J(x_n - q) - J(x_n - x_t)\rangle| + |\langle x_t - q, J(x_n - x_t)\rangle| \\
&\le \|u - q\|\|J(x_n - q) - J(x_n - x_t)\| + \|x_t - q\|\|x_n - x_t\| < \frac{\epsilon}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we have

$$\langle u - q, J(x_n - q)\rangle \le \langle x_t - u, J(x_t - x_n)\rangle + \frac{\epsilon}{2}.$$

That is,

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \limsup_{n\to\infty}\langle x_t - u, J(x_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (3.10) that

$$\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we have (3.5) holds. Now from Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(y_n - x^*) + \alpha_n(u - x^*)\|^2 \\
&\le \|(1 - \alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle \\
&\le (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, J(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{3.11}$$

Apply Lemma 2.4 to (3.11) to conclude $x_n \to q$ as $n \to \infty$. This completes the proof. ∎

REMARK 3.2. If we take $r = 1$, then we may take $S_1 := J_A = (I + A)^{-1}$, the strict convexity of E and real constants $a_i,\ i = 0, 1$ may not be needed.

REMARK 3.3. If $f : C \to C$ is a contraction map and we replace u by $f(x_n)$ in the recursion formula (1.4), we obtain what some authors now call viscosity iteration method. We note that our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, and using the fact that f is a contraction map, one can repeat the argument of this paper.

REMARK 3.4. Theorem 3.1 mainly improves Theorem 2.1 of Qin and Su ([15]) in the following sense:

(i) from a single mapping to a finite family of mappings;
(ii) relaxing the restrictions on parameters.

4. Applications

Closely related to the class of accretive mappings is the class of pseudo-contractive mappings. A mapping $A : D(A) \subset E \to E$ is said to be pseudo-contractive if $T := (I - A)$ is accretive . We observe that p is a zero of the accretive mapping A if and only if it is a fixed point of the pseudo-contractive mapping $T := I - A$. As applications of Theorem 3.1, we give a strong convergence theorem for pseudo-contractive mappings.

Theorem 4.1. *Let E be a strictly convex and uniformly smooth Banach space. Let C be a nonempty closed convex subset of E. Let $T_i : C \to E$, $i = 1, 2 \dots, r$, be a finite family of pseudo-contractive mappings such that, for each $i \in \{1, 2, \cdots, r\}$, $(I - T_i)$ is m-accretive on C with $\cap_{i=1}^r F(T_i) \neq \emptyset$. Let $J_{T_i} := (I + (I - T_i))^{-1} = (2I - T_i)^{-1}$ for $i = 1, 2, \dots, r$. For any given $x_0 \in C$, let $\{x_n\}$ be a sequence in C generated by the algorithm:*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_r x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where $S_r = a_0 I + a_1 J_{T_1} + a_2 J_{T_2} + \cdots, + a_r J_{T_r}$, for $0 < a_i < 1$, $i = 0, 1, \dots, r$, $\sum_{i=0}^r a_i = 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$ which satisfy the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_r\}$.

Proof. Let $A_i := (I - T_i)$ for each $i = 1, 2, \dots, r$. It follows clearly that $F(T_i) = N(A_i)$ and hence $\cap_{i=1}^r N(A_i) = \cap_{i=1}^r F(T_i) \neq \emptyset$. Furthermore, each A_i for $i = 1, 2, \cdots, r$ is m-accretive. We can conclude the desired conclusion easily from Theorem 3.1. □

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