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ALTERNATE PROOFS OF TWO CLASSICAL THEOREMS ON FINITE SOLVABLE GROUPS AND SOME RELATED RESULTS FOR p-GROUPS

Yakov Berkovich
University of Haifa, Israel

Abstract. We offer a new proof of the classical theorem asserting that if a positive integer n divides the order of a solvable group G and the set L_n of solutions of the equation $x^n = 1$ in G has cardinality n, then L_n is a subgroup of G. The second proof of that theorem is also presented. Next we offer an easy proof of Philip Hall's theorem on solvable groups independent of Schur-Zassenhaus' theorem. In conclusion, we consider some related questions for p-groups. For example, we study the irregular p-groups G satisfying $|L_{p^k}| \leq p^{k+p-1}$ for $k > 1$.

1. Introduction

We consider finite groups only. Section 1 of this note is a continuation of [B1, §1]. All groups considered in §1, are solvable or π-solvable. All groups of §2 are p-groups.

Suppose that a positive integer n divides the order $|G|$ of a group G and L_n the set of solutions of $x^n = 1$ in G. By fundamental Frobenius' theorem, $|L_n| = kn$ for some positive integer k. Frobenius posed the following problem: Is it true that if $k = 1$, then L_n is a subgroup of G? As I think, this problem aroused from his study of the following situation. Let $H < G$ be of index n and assume that $H \cap H^x = \{1\}$ for all $x \in G - H$. In that case, the set $L_n = G - \bigcup_{x \in G-H} H^x$ has cardinality n. Using character theory, Frobenius succeeded to show that L_n is a (characteristic) subgroup of G. In general, Frobenius problem was solved in the positive only after classification of finite simple groups (see [I] and listed there papers of the same authors).

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2. Solvable groups

This section does not contain new results. Our aim is to produce easy proofs of some known results about solvable groups.

Very important partial case of Frobenius conjecture many years ago was proved in elementary way in the case when G is solvable (see [H2, Theorem 9.4.1]; see also [BZ, §5.11]). Below we present two other proofs of this result. In our proofs we use only those facts that were known before 1900 (otherwise, Lemma 2.2 would be superfluous), namely, that sections of solvable groups are solvable. We also use Frattini argument, an easy consequence of Sylow's theorem. In this sense, our exposition in this section is self contained.

Theorem 2.1. *If a positive integer n is a divisor of the order of a solvable group G and the set L_n of solutions of $x^n = 1$ in G has cardinality n, then L_n is a subgroup of G.*

We need the following

Lemma 2.2. *If p is a prime divisor of the order of a solvable group G, then there is in G a proper subgroup whose index in G is a power of p.*

Proof. We proceed by induction on $|G|$. Let R be a minimal normal subgroup of G; then R is an elementary abelian q-group for some prime q.

If p divides $|G/R|$, then, by induction, there is in G/R a proper subgroup H/R, whose index is a power of p. Since $|G : H| = |(G/R) : (H/R)|$, we are done in this case. This is the case if $q \neq p$. Therefore, in what follows we assume that G has no nonidentity normal p'-subgroup.

Now let p does not divide $|G/R|$. Then R is a Sylow p-subgroup of G. Let S/R be a minimal normal subgroup of G/R; then S/R is an s-subgroup for some prime $s \neq p$. Let Q be a Sylow s-subgroup of $S = QR = RQ$. Then the Frattini argument yields $G = S\mathrm{N_G(Q)}$. But $S\mathrm{N_G(Q)} = \mathrm{RQN_G(Q)} = \mathrm{RN_G(Q)}$ and $\mathrm{N}_G(Q) < G$, by the previous paragraph. Clearly, $\mathrm{N}_G(Q)$ is maximal in G. Since $|G : \mathrm{N_G(Q)}|$ divides (even equals) $|R|$, a power of p, the proof is complete. □

Remark 2.3. Let G be a solvable group and $n \mid |G|$, $|L_n| = n$ and R a minimal normal subgroup of G. Let $|R| = p^\alpha$ and write $\bar{G} = G/R$. (i) If $|R| \mid n$, then the set $\bar{L}$ of solutions of $\bar{x}^{n/|R|} = \bar{1}$ in $\bar{G}$ has cardinality $n/|R|$ and $L = L_n$, where L is the inverse image of $\bar{L}$ in G. Indeed, let $\bar{L} = \{\bar{x}_1, \dots, \bar{x}_s\}$. Then $\bar{x}_i = x_iR$ is a coset of R so that $L = x_1R \cup \dots \cup x_sR$. We compute $x_i^n = (x_i^{n/|R|})^{|R||} = 1$ since $x_i^{n/|R|} \in R$. There one can take, instead of x_i, any element of the coset x_iR. Thus, $x_iR \in L_n$ so that $L \subseteq L_n$. By Frobenius' theorem, $n/|R|$ divides s so that $s \geq n/|R|$. One has $|L| = s|R| \geq (n/|R|)|R| = n = |L_n|$, and we conclude that $L = L_n$ whence $s = n/|R|$. (ii) Let $p \mid n$ but R does not divide n. Set $n = n_pn_{p'}$. Let $\bar{L}$ be the set of all solutions of $\bar{x}^{n_{p'}}$ in $\bar{G}$ and L the inverse image of $\bar{L}$ in

G. By Frobenius, $|\bar{L}| \geq n_{p'}$; then $|L| \geq |R|n_{p'} > n_p n_{p'} = n = |L_n|$ since $|R| > n_p$, by assumption. If $y \in L$, then $y^n = (y^{n_{p'}})^{n_p} = 1$ since $y^{n_{p'}} \in R$ and $\exp(R) = p \mid n_p$. Thus, $L \subseteq L_n$, contrary to what has just been proved. We see that if $p \mid n$, then L_n is a union of cosets of R, i.e., $|R| \mid n$. (iii) Now suppose that p does not divide n. We claim that then the set $\bar{L}$ of solutions of $\bar{x}^n = \bar{1}$ has cardinality n. Write $\bar{L} = \{\bar{x}_1, \dots, \bar{x}_s\}$. By Frobenius' theorem, $s \geq n$. Let $L = x_1 R \cup \dots \cup x_s R$ be the inverse image of $\bar{L}$ in G. We have $x_i^n \in R$ so that $x_i = y_i z_i = z_i y_i$, where $o(y_i) = o(\bar{x}_i)$ and $z_i \in R$ (note that $o(\bar{x}_i) \mid n$). Since $y_i R = y_i z_i R = x_i R$ for all i, we have $y_i \neq y_j$ for $i \neq j$. Therefore, the set $L^0 = \{y_1, \dots, y_s\} \subseteq L_n$, moreover, $L^0 = L_n$ since $|L^0| = s \geq n = |L_n|$, completing this case.

Proof of Theorem 2.1. We use induction on $|G|+n$. One may assume that $1 < n < |G|$. Write, as above, $L_n = \{x \in G \mid x^n = 1\}$.

Suppose that n divides the order of a proper subgroup M of G. Then the number of solutions of equation $x^n = 1$ in M is a nonzero multiple of n, and we conclude that $L_n \leq M$, by induction since $|L_n| = n$, and we are done.

Next we assume that n does not divide orders of all $M < G$. Let $p \mid |G|$. Then there is $H < G$ such that $|G : H|$ is a power of p (Lemma 2.2). Since n does not divide $|H|$, we conclude that p divides n. Thus, $\pi(n) = \pi(|G|)$, where $\pi(n)$ is the set of primes dividing n.

Let R be a minimal normal subgroup of G; then $|R| = p^\alpha$ for some prime p and positive integer α and $\exp(R) = p$. Since, by the previous paragraph, $p \mid n$, all elements of R satisfy $x^n = 1$ so that $R \subseteq L_n$. Write $\bar{G} = G/R$. By Remark 2.3(i,ii), $|R| = p^\alpha \mid n$ and, if $\bar{L}$ is the set of solutions of $\bar{x}^{n/p^\alpha} = \bar{1}$ in $\bar{G}$, then its inverse image coincides with L_n and $|\bar{L}| = n/p^\alpha$. Therefore, by induction, $\bar{L}$ is a subgroup of $\bar{G}$. In that case, L_n, the inverse image of $\bar{L}$ in G, is also a subgroup of G. ◻

Remark 2.4. Suppose that G and L_n are such that L_n is a subgroup of G of order n (here we do not assume that G is solvable). Let $p \in \pi(|G : L_n|)$; then $P \in \mathrm{Syl_p(G)}$ is not contained in L_n (Lagrange). In that case, $L_n \cap P$ is the unique subgroup of order n_p in P since all subgroups of P of exponent $\leq n_p$ are contained in L_n and so $P \cap L_n \in \mathrm{Syl}_p(L_n)$. It follows that either P is cyclic or $p = 2 = n_2$ and P is a generalized quaternion group ([B2, Proposition 1.3]). In the first case, L_n is p-nilpotent, by [B1, Theorem A.28.9]. In the second case, 4 does not divide $|L_n|$, so L_n is 2-nilpotent again.

Combining some arguments from the above text, one can produce another proof of Theorem 2.1. We need two additional lemmas.

Lemma 2.5. *If a prime p divides $|G|/n$, then the set L_n of cardinality n does not contain a subset of cardinality pn_p which is a subgroup of G.*

Proof. (Independent of Theorem 2.1) One may assume that $p \mid n$ (otherwise, in view of Lagrange's theorem, it is nothing to prove).

We use induction on $|G|+n$. Let R be a minimal normal subgroup of G of order, say q^α. Write $\bar{G}=G/R$.

Assume that $M<G$ has order divisible by n. Then $L_n \subseteq M$, and, by induction, there is not a subgroup of order pn_p that is a subset of L_n. In what follows we assume that n does not divide orders of all proper subgroups of G.

Then, by Lemma 2.2, $q \mid n$. By Remark 2.3(i,ii), $q^\alpha \mid n$ and the set $\bar{L}$ of solutions of $\bar{x}^{n/q^\alpha}=\bar{1}$ in $\bar{G}$ has cardinality n/q^α.

If $q \neq p$, then $(n/q^\alpha)_p = n_p$ so, by induction, the set $\bar{L}$ has no subset of cardinality pn_p which is a subgroup of G. Assume that a p-subgroup $P<G$ of order pn_p is a subset of L_n. Then $PR/R \subseteq \bar{L}$ since all elements of PR/R satisfy $\bar{x}^n=\bar{1}$. In that case, the p-subgroup PR/R of order $pn_p = p(n/|R|)_p$ is a subset of $\bar{L}$, contrary to what has just been said.

Now let $q=p$. By Remark 2.3(i), the set $\bar{L}$ of solutions of $\bar{x}^{n/p^\alpha}=\bar{1}$ has cardinality n/p^α hence the inverse image L of $\bar{L}$ in G has cardinality n so coincides with L_n. If a p-subgroup $\bar{P}$ is a subset of $\bar{L}_n=\bar{L}$ (see Remark 2.3(i)), then $|\bar{P}| \leq (n/p^\alpha)_p$, by induction, and we obtain $|P|=|\bar{P}||R| \leq (n/p^\alpha)_p \cdot p^\alpha = n_p$. ∎

Of course, Lemma 2.5 follows from Theorem 2.1, but we prefer to give an independent proof since our aim now is to produce another proof of Theorem 2.1.

The following lemma is due essentially to Galois.

Lemma 2.6. *Every index of a maximal subgroup of a solvable group G is equal to some index of a principal series of G.*

Proof. We proceed by induction on $|G|$. Let $H<G$ be maximal. One may assume that H is not normal in G (otherwise, it is nothing to prove). Let R be a minimal normal subgroup of G; then $|R|$ is an index of a principal series of G containing R. If $R<H$, then $|G:H|=|(G/R):(H/R)|$ is equal to an index of a principal series of G/R, by induction, completing this case. If R is not contained in H, then $G=HR$, $H \cap R=\{1\}$ so that $|G:H|=|R|$. ∎

The second proof of Theorem 2.1. As in the first proof of Theorem 2.1, it suffices to show that n divides the order of some maximal subgroup of G. Assume that this is false. In that case, as we have noticed in the first proof of Theorem 2.1, $\pi(n)=\pi(|G|)$. Let p be a prime divisor of $|G|/n$ and P a Sylow p-subgroup of G; then P is not contained in the set L_n (Lemma 2.5). Let $P_1 \leq P$ be of order pn_p. Since, by Lemma 2.5, P_1 is not contained in L_n, we get $\exp(P_1) > n_p$ hence P_1 is cyclic. Since $p \mid n$, we get $|P_1|>p$. Thus, all subgroups of order pn_p in P are cyclic, whence P is either cyclic or a generalized quaternion group and $|P_1|=4$ ([B2, Proposition 1.3]). In the second case, clearly, $n_2=2$. Note that all factors of a principal series of G

are elementary abelian. It follows that all $\{p\}$-indices of a principal series of G are equal to p in the first case and 2 or 4 in the second case.

Suppose that $P \in \mathrm{Syl}_p(G)$ is cyclic. Then, by Lemma 2.6, G has a maximal subgroup H of index p. In that case, $n \mid |H|$, contrary to the assumption.

Now suppose that $p = 2$ and a Sylow 2-subgroup of G is generalized quaternion; then $n_2 = 2$ and all $\{2\}$-indices of a chief series of G divide 4. In that case, by Lemma 2.6, G has a maximal subgroup H of index dividing 4. Since $|P| \geq 8$ and $n_2 = 2$, we get $n \mid |H|$, contrary to the assumption. $\square$

Let π be a set of primes. Recall that a group G is π-solvable if all indices of its composition series that are not primes from π, are π'-numbers.

Remark 2.7. If π is a nonempty set of prime divisors of the order of a π-solvable group G and $p \in \pi$, then G has a proper subgroup whose index is a power of p and a proper subgroup whose index is a π'-number (of course, all this follows from an analog of Theorem 2.8, below, for π-solvable groups; that result is due to S. A. Chunikhin). However, the proof of this result depends on the Schur-Zassenhaus theorem. In view of Odd Order theorem, π-separable groups, in sense of Hall-Higman, are either π- or π'-solvable, so one can state an analog of Lemma 2.2 for such groups.

Supplement to Theorem 2.1 Suppose that G is a π-solvable group and a positive integer n is divisible by $|G|_{\pi'}$. If $|L_n| = n$, then L_n is a subgroup of G.

Proof. In view of Theorem 2.1, one may assume that G is nonsolvable; then $\pi \neq \pi(|G|)$. We also assume that $n < |G|$ and $\pi \neq \emptyset$ (otherwise, it is nothing to prove). Let $H \leq G$ be of order $|G_{\pi'}|$. If $n = |G|_{\pi'}$, then $L_n = H$. Next we assume that $|G_{\pi'}| < n$; then $\pi \cap \pi(|G|) \neq \emptyset$. Let R be a minimal normal subgroup of G. Write $\bar{G} = G/R$, $|R| = r$. Then r is either power of a prime from π or a π'-number.

As above, we use induction on $|G| + n$. By Remark 2.3, for every $p \in \pi \cap \pi(G)$, there is $M < G$ such that $|G : M|$ is a power of p. As in the proof of Theorem 2.1, if $n \mid |M|$, the result follows by induction. So one may assume that $\pi \subset \pi(n)$.

(a) Suppose that $(|R| =)r = p^\alpha$, where $p \in \pi$. By what has been said in the previous paragraph, $p \mid n$; then $R \subset L_n$. By Remark 2.3(i,ii) (these parts also hold if $|R|$ is a prime power), $p^\alpha \mid n$ and the set $\bar{L}$ of solutions of $\bar{x}^{n/p^\alpha} = \bar{1}$ has cardinality n/p^α so, by induction, $\bar{L}$ is a subgroup of $\bar{G}$; then its inverse image L in G is also subgroup in G. Since $|L| = |\bar{L}||R| = n$, we get $L = L_n$, completing this case.

(b) Now let r be a π'-number. By Remark 2.3(iii) (this part holds for arbitrary groups in the case under consideration and induction, $\bar{L} = \{\bar{x} \in$

$\bar{G} \mid \bar{x}^{n/r} = \bar{1}\}$ is a subgroup of G of order n/r. Then (the subgroup) L, the inverse image of $\bar{L}$ in G has order n so coincides with L_n. ☐

THEOREM 2.8 (P. Hall [H2]). *If m is a divisor of the order of a solvable group G such that $\mathrm{GCD}(m, |G|/m) = 1$, then all largest $\pi(m)$-subgroups of G have the same order m and are conjugate in G.*

PROOF. We use induction on $|G| + m$. Let R be a minimal normal subgroup of G with $|R| = p^\alpha$ for some prime p.

(i) First we prove that G contains a subgroup of order m. Let q be a prime divisor of $|G|/m$; then there is $H < G$ such that $|G : H|$ is a power of q (Lemma 2.2). In that case, $m \mid |H|$ and $\mathrm{GCD}(m, |H|/m) = 1$ so, by induction, (the solvable subgroup) H contains a subgroup of order m.

(ii) We claim that all subgroups of order m are conjugate in G. Let $F, H < G$ be of order m. Then $\pi(m)$-Hall subgroups FR/R and HR/R are conjugate in G/R, by induction, so $(FR)^x = HR$ for some $x \in G$ whence $F^x \le HR$.

If $p \mid m$, then $R \le F \cap H$ so $FR = F$, $HR = H$ and hence $F^x = H$. Next assume that $\mathrm{O}_{\pi(m)}(G) = \{1\}$.

If $HR < G$, there is $y \in HR$ such that $(F^x)^y = H$, by induction. Now assume that $HR = G$ for any choice of R; then also $FR = G$ and $R \in \mathrm{Syl}_p(G)$ is the unique minimal normal subgroup of G so H and F are maximal in G. Let Q/R be a minimal normal subgroup of G/R; then Q/R is a q-subgroup for some prime $q \ne p$. In that case, $H \cap Q, F \cap Q \in \mathrm{Syl}_q(Q)$ are not normal in G, by assumption. Therefore, $\mathrm{N}_G(H \cap Q) = H$ and $\mathrm{N}_G(F \cap Q) = F$. By Sylow's theorem, $F \cap Q = (H \cap Q)^y$ for some $y \in Q$. Then

$$F = \mathrm{N}_G(F \cap Q) = \mathrm{N}_G((H \cap Q)^y) = \mathrm{N}_G(H \cap Q)^y = H^y.$$

(iii) It remains to show that if $K < G$ is the greatest $\pi(m)$-subgroup, then $|K| = m$. By induction, $KR/R \le H/R$, where H/R is a $\pi(m)$-Hall subgroup of G/R.

If p divides m, then $|H| = m$, $R \le K$ and $K = H$, by maximality of K.

Now let p does not divide m. We have $K < KR \le H$. If $H < G$, then, by induction, K is contained in a subgroup of order m in H so $|K| = m$, by maximality of K. Now let $H = G$. Then $G = FR$, where $F < G$ is of order m (F exists, by (i)). Set $K_1 = KR \cap F$; then $|K_1| = |K|$, by the product formula. By (ii), $K = K_1^z$ for some $z \in KR$ so $K \le F^z$ and, since $|F^z| = |F| = m$, we obtain $K = F^z$. ☐

3. p-GROUPS

In this section G is a p-group always.

Let k be a positive integer and $p^{k+1} < |G|$. If $|L_{p^k}| = p^k$, then G has exactly one subgroup of order p^k so G is either cyclic or $p = 2$, $k = 1$ and

G is generalized quaternion group ([B2, Proposition 1.3]). So Frobenius' case $|L_{p^k}| = p^k$ is trivial for p-groups.

Next we assume that $|L_{p^k}| > p^k$. In that case, G is noncyclic so such group G contains a noncyclic subgroup H of order p^{k+1}, unless G is generalized quaternion and $k = 1$. Since $H \subseteq L_{p^k}$, we get $|L_{p^k}| \geq p^{k+1}$.

A. Assume that $|L_{p^k}| = p^{k+1}(< |G|)$. Then $L_{p^k} = H$, where, as above, $H < G$ is noncyclic of order p^{k+1}. In that case, there is in G a subgroup F of order p^{k+1} that is $\neq H$ ([B2, Proposition 1.3]). Since all subgroups of order p^k are subsets of L_{p^k}, it follows that F is not generated by its maximal subgroups so it is cyclic. Thus, H is the unique noncyclic subgroup of order p^{k+1} in G. Then there is in G a normal abelian subgroup R of type (p,p) so $R \leq L_{p^k} = H$.

If $k = 1$, then $R = H = \Omega_1(G)$. In that case, G is either metacyclic or a 3-group of maximal class or $p = 2$ and G has exactly three involutions (see [B2, Theorem 13.7] and [BJ, §82]).

Now let $k > 1$. In that case, H/R is the unique subgroup of order p^{k-1} in G/R so that either G/R is cyclic or $p = 2$, $k = 2$ and G/R is a generalized quaternion group. In both cases $R = \Omega_1(G)$. In the first case, G possesses a cyclic subgroup of index p so, by [B2, Theorem 1.2], either $G \cong \mathrm{M}_{p^n}$ or G is noncyclic abelian group of type (p^{n-1}, p). In the second case, $k = 2$, $p = 2$ and G is a metacyclic group as in [B2, Lemma 42.1(c)].

B. Now suppose that $k = 1$ and $p^2 < |L_p| < p^3$. In that case, G has no subgroups of order p^3 and exponent p. It follows that then $p = 2$ (see [B2, Theorems 12.1, 13.7 and §§9,7]). In the case under consideration, if G is of maximal class, it is easy to check that then $G \in \{\mathrm{D}_8, \mathrm{SD}_{16}\}$. Now assume that G is not of maximal class. Then it has a normal abelian subgroup of type $(2,2)$. By hypothesis, there is an involution $x \in G - R$; then $H = \langle x, R\rangle \cong \mathrm{D}_8$. Since the number of involutions in G is $\equiv 3 \pmod 4$ ([B2, Theorem 1.17(a)]), there are exactly 7 involutions in G. In that case, $|L_2| = 7 + |\{1\}| = 8$, contrary to the hypothesis. Thus, only groups D_8 and SD_{16} satisfy $4 < |L_2| < 8$.

C. The case of a p-group G satisfying $|L_p| = p^3$ is not tractable for $p = 2$ in this time. Now let $p > 2$; then $L_p = \Omega_1(G)$ is of exponent p (here we use the fact that a p-group of order $\leq p^p$ generated by elements of order p has exponent p). We suggest to the reader supply the case where G is a p-group, $k > 1$ and $|L_{p^k}| = p^{k+2}$ (as the following part D shows, this is more or less difficult for $p = 2$ only).

D. Now let G be an irregular p-group of order $p^m \geq p^{2p}$, $\exp(G) > p^k > p$ and $|L_{p^k}| \leq p^{k+p-1} < p^m (= |G|)$.

D1. Suppose that G has no normal subgroup of order p^p and exponent p. Then G is of maximal class ([B2, Theorem 12.1(a)]). In that case, $\exp(G) = p^e$, where $e = [\frac{m-1}{p-1}] + \epsilon$ with $\epsilon = 0$ if $p - 1 \mid m - 1$ and 1 otherwise (here $[x]$ is the integer part of the real number x). Our G has a maximal subgroup

G_1 satisfying $|G_1/\mho_1(G)| = p^{p-1}$ (see [B2, Theorem 9.6] where such G_1 are called absolutely regular p-groups) and such that all elements of the set $G - G_1$ have orders $\leq p^2$ ([B2, Theorem 13.19(b)]) so that $G - G_1 \subset L_{p^k}$. One has $|G - G_1| = p^{m-1}(p-1)$. The subgroup $\Omega_k(G_1)$ of order $p^{k(p-1)}$ is contained in L_{p^k}. Thus, $p^{k+p-1} \geq |L_{p^k}| = (p-1)p^{m-1} + p^{k(p-1)}$. It follows that $k + p - 1 > k(p-1)$ so that $p = 2$. In that case, $2^{k+1} \geq |L_{2^k}| = 2^{m-1} + 2^k$. It follows that $k \geq m - 1$, contrary to the hypothesis. Thus, G is not a group of maximal class.

D2. Then, by [B2, Theorem 12.1(a)], G possesses a normal subgroup R of order p^p and exponent p. Let $H/R < G/R$ be of order p^{k-1}, then $|H| = p^{p+k-1}$ and $\exp(H) \leq p^k$ so that $H = L_{p^k}$. It follows that all elements of the set $G - H$ have orders $> p^k$ so that $L_{p^k} = \Omega_k(G)$ and L_{p^k}/R is the unique subgroup of order p^{k-1} in G/R, and we conclude that G/R is either cyclic or generalized quaternion group (in the second case, $k = 2$). By hypothesis, $|G/R| \geq p^p$.

Suppose that G/R is cyclic. Since $|G/R| > p$, G is not of maximal class. Assume that $R < \Omega_1(G)$. Then $|\Omega_1(G) = p^{p+1}|$ and, by [B2, Exercise 13.10(a)], $\exp(\Omega_1(G)) = p$. Let $M/\Omega_1(G) < G/\Omega_1(G)$ be of order p^{k-1}; then $|M| = p^{p+k}$ and $\exp(M) = p^k$. It follows that $M \subseteq L_{p^k}$, a contradiction since $|M| = p^{p+k} > p^{p+k-1} \geq |L_{p^k}|$. Thus, $\Omega_1(G) = R$ so that G is an L_p-group (see [B2, §17,18]).

Now suppose that G/R is a generalized quaternion group; then $p = 2$ and, as we have noticed, $k = 2$. If $\Omega_1(G) < R$, then $\Omega_1(G)$ is elementary abelian of order 8 ([B2, Exercise 13.10(a)]). If $H/\Omega_1(G) \leq G/\Omega_1(G)$ is noncyclic of order 2^2, then $\exp(H) = 2^2$ hence $H \subseteq L_{2^k}$, a contradiction since $|H| = 2^{3+2} > 2^{2+1} = |L_{2^2}|$. Thus, $\Omega_1(G) = R$ hence $|\Omega_2(G)| = 8$. It follows that G is as in [B2, Lemma 42.1(c)].

We state the results obtained in this section, in the following two propositions.

PROPOSITION 3.1. *Let G be a p-group. If $p < p^k < |L_{p^k}| \leq p^{k+1} < |G|$, then one of the following holds:*

(a) *G is either abelian with cyclic subgroup of index p or isomorphic to* M_{p^n}.

(b) $k = 2$, $p = 2$, $G = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = a^{2^{m-3}}, a^b = a^{-1}, m > 4\rangle$. *Here* $\mathrm{Z}(G) = \langle b^2\rangle$, $G' = \langle a^2\rangle$, $\Phi(G) = \langle a^2, b^2\rangle$, $\Omega_2(G) = \langle a^{2^{m-4}}, b^4\rangle$, G/G' and $\Omega_2(G)$ are abelian of type $(4, 2)$.

PROPOSITION 3.2. *Suppose that G is an irregular p-group of order $p^m \geq p^{2p}$ and exponent p^e, $1 < k < e$. If $|L_{p^k}| \leq p^{p+k-1} < p^m$, then one of the following holds:*

(a) *G is an L_p-group, i.e., $|\Omega_1(G)| = p^p$ and $G/\Omega_1(G)$ is cyclic (of order $\geq p^p$).*

(b) *$k = 2$, $p = 2$ and G is as in Proposition 3.1(b).*

PROBLEM 3.1. *Let n be a proper divisor of the order of a solvable group G. Study the embedding and the structure of L_n provided $|L_n| = 2n$ (the minimal nonabelian group G of order $3^2 \cdot 2^2$ is such that $|L_6| = 2 \cdot 6$).*

PROBLEM 3.2. *Let p be a minimal prime divisor of the order of a group G and n is a proper divisor of $|G|$. Study the embedding in G and the structure of the set L_n provided $|L_n| \leq pn$.*

PROBLEM 3.3. *Let G be an irregular p-group and $k > 1$. Study the structure of G provided $|L_{p^k}| \leq p^{k+p}$ (see §3).*

PROBLEM 3.4. *Suppose that G is a metacyclic p-group and H is a p-group such that $|L_{p^2}(H)| = |L_{p^2}(G)|$. Study the structure of H.*[1]

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Y. Berkovich
Department of Mathematics
University of Haifa
Mount Carmel, Haifa 31905
Israel

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[1]Mann has solved a partial case of this problem for $p > 2$ when $\mathrm{s}_1(H) = \mathrm{s}_1(G)$ and $\mathrm{s}_2(H) = \mathrm{s}_2(G)$, where $\mathrm{s}_k(G)$ is the number of subgroups of order p^k is G (it appears that the similar problem for $p = 2$ is surprisingly difficult); a small modification of Mann's argument allows to solve the general problem for $p > 2$ since, in the case under consideration, $|\Omega_2(G)| \leq p^4$.