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# COVERINGS OF FINITE GROUPS BY FEW PROPER SUBGROUPS

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Abstract. A connection between maximal sets of pairwise non-commuting elements and coverings of a finite group by proper subgroups is established. This allows us to study coverings of groups by few proper subgroups. The $p$-groups without $p+2$ pairwise non-commuting elements are classified. We also prove that if a $p$-group admits an irredundant covering by $p+2$ subgroups, then $p=2$. Some related topics are also discussed.

## 1. Introduction

In what follows all groups are finite and $p$ is a prime.

We say that a group $G$ is covered by proper subgroups $A_1, \dots, A_n$ if

$$G = A_1 \cup \dots \cup A_n. \tag{1.1}$$

We have, in (1.1), $G > \{1\}$ and $n > 1$. A group is covered by its proper subgroups if and only if it is not cyclic. Every noncyclic group is covered by (proper) cyclic subgroups. A group is not covered by two proper subgroups. Covering (1.1) is said to be *irredundant* if every proper subset of the set $\{A_1, \dots, A_n\}$ does not cover $G$. In what follows, we assume that (1.1) is an irredundant covering of $G$ by proper subgroups.

Remark 1.1. If, in (1.1), $|A_1| \geq \dots \geq |A_n|$, then $|G| \leq |A_1|n - (n-1) < n|A_1|$, and hence $|G : A_1| < n$. A more general situation is considered in the following theorem of B. H. Neumann ([N]): If an arbitrary (finite or infinite) group $G$ is covered by $n$ cosets $H_1x_1, \dots, H_nx_n$ ($H_1, \dots, H_n \leq G$), then at least one subgroup $H_i$ has index $\leq n$ in $G$, and this estimate is best possible

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(Neumann's theorem is obvious for finite $G$, however, for infinite groups it is a fairly deep result).

Let $\mathcal{M}$ be a maximal subset (with respect to inclusion) of pairwise noncommuting elements of a nonabelian group $G$. We denote the set of all such subsets by $\Lambda(G)$. Write

$$\gamma(G) = \max\{|\mathcal{M}| \mid \mathcal{M} \in \Lambda(G)\}.$$

For an abelian group $G$, we set $\gamma(G) = 1$. If $H$ is a subgroup of $G$, then $\gamma(H) \leq \gamma(G)$. Recall that two groups $G$ and $G_1$ are lattice isomorphic if there is a bijective mapping $\phi$ of the set of subgroups of $G$ onto the set of subgroups of $G_1$ such that, provided $F, H \leq G$, then $(F \cap H)^\phi = F^\phi \cap H^\phi$ and $\langle F, H\rangle^\phi = \langle F^\phi, H^\phi\rangle$. If groups $G$ and $G_1$ are lattice isomorphic, then the inequality $\gamma(G) \neq \gamma(G_1)$ is possible owing to the fact that some nonabelian groups are lattice isomorphic to abelian groups (indeed, there exist nonabelian modular groups which are lattice isomorphic to abelian groups).

Let $\Gamma_1$ be the set of all maximal subgroups of $G$.

As Lemma 1.3(a) shows, if $\mathcal{M} \in \Lambda(G)$, then $G = \bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x)$ and this covering is irredundant.

Every nonabelian group contains three pairwise non-commuting elements (this follows from Lemma 1.3(a)). For $p$-groups one can prove a stronger result.

Lemma 1.2. *Let $G$ be a nonabelian $p$-group. Then*

(a) *If $G$ is minimal nonabelian, then $\gamma(G) = p + 1$.*

(b) *$\gamma(G) \geq p + 1$.*

Proof. (a) We have $\mathrm{d}(G) = 2$. Since all maximal subgroups of $G$ are abelian, any two non-commuting elements of $G$ are contained in distinct maximal subgroups of $G$. Therefore, $\gamma(G) \leq p + 1$. If $\Gamma_1 = \{M_1, \dots, M_{p+1}\}$ and $x_i \in M_i - \Phi(G)$, then, for $i \neq j$, $\langle x_i, x_j\rangle = G$ is nonabelian, so $x_i x_j \neq x_j x_i$. Thus, $x_1, \dots, x_{p+1}$ are pairwise non-commuting elements so $\gamma(G) \geq p + 1$, completing the proof.

(b) Let $H$ be a minimal nonabelian subgroup of $G$. Then $\gamma(H) = p + 1$, by (a), and so $\gamma(G) \geq \gamma(H) = p + 1$. □

The following lemma establishes a connection between members of the set $\Lambda(G)$ with some irredundant coverings of a nonabelian group $G$. Part (b) of this lemma also shows that members of $\Lambda(G)$ of cardinality $\gamma(G)$ have a special property.

Lemma 1.3. *Let $G$ be a nonabelian group and $\mathcal{M} \in \Lambda(G)$. Then*

(a) *We have*

$$\bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x) = G. \tag{1.2}$$

*If $\mathcal{N} \subseteq \mathcal{M}_1 \in \Lambda(G)$ and $\bigcup_{x\in\mathcal{N}} \mathrm{C}_G(x) = G$, then $\mathcal{N} = \mathcal{M}_1$; in particular, (1.2) is an irredundant covering.*

(b) *Suppose, in addition, that $|\mathcal{M}| = \gamma(G)$. If $x \in \mathcal{M}$, then the subgroup $\langle G - \bigcup_{y\in\mathcal{M}-\{x\}} \mathrm{C}_G(y)\rangle$ is abelian.*

Proof. (a) Assume that there is $g \in G - \bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x)$. Since $\mathcal{M} \subset \mathcal{M} \cup \{g\}$, it follows from maximality of $\mathcal{M}$ that $gx = xg$ for some $x \in \mathcal{M}$; then $g \in \mathrm{C}_G(x)$, contrary to the choice of $g$. Thus, $\bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x) = G$.

Now assume that there is $u \in \mathcal{M}$ such that $\bigcup_{x\in\mathcal{M}-\{u\}} \mathrm{C}_G(x) = G$. Then there is $v \in \mathcal{M} - \{u\}$ such that $u \in \mathrm{C}_G(v)$, so that $u, v$ are distinct commuting members of the set $\mathcal{M}$, a contradiction. Thus, the covering $\bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x) = G$ is irredundant.

(b) Given $x \in \mathcal{M}$, set $D = G - \bigcup_{y\in\mathcal{M}-\{x\}} \mathrm{C}_G(y)$. Assume that there are noncommuting $u, v \in D$. By the choice, every element of the set $\mathcal{M} - \{x\}$ does not commute with $u$ and $v$. It follows that the set $(\mathcal{M} - \{x\}) \cup \{u, v\} \subseteq M_2 \in \Lambda(G)$, a contradiction since $|\mathcal{M}_2| > |\mathcal{M}| = \gamma(G)$. Thus, any two elements of the set $D$ commute so the subgroup $\langle D\rangle$ is abelian. ∎

It follows from Lemma 1.3(a) that if $H < G$ is such that $\gamma(H) = \gamma(G)$ and $\mathcal{M} = \{x_1, \dots, x_{\gamma(G)}\} \in \Lambda(H)$, then there are $i \neq j$ with $\mathrm{C}_G(x_i) \not\leq H$ and $\mathrm{C}_G(x_j) \not\leq H$. Indeed, there is $i$ such that $\mathrm{C}_G(x_i) \not\leq H$ since $\bigcup_{k=1}^{\gamma(G)} \mathrm{C}_G((x_k) = G > H$ (Lemma 1.3(a)). If for all $j \neq i$ we have $\mathrm{C}_G(x_j) \leq H$, then $H \cup \mathrm{C}_G(x_i) = G$, which is impossible since $H$ and $\mathrm{C}_G(x_j)$ are non-incident so cannot cover $G$.

There are dozens of papers devoted to irredundant coverings of groups (without finiteness assumption); see MathSciNet and [Bh]. I state some results from those papers that are mentioned in [Bh]. Let $\sigma(G)$ be a minimal number $n$ such that $G$ is covered by $n$ proper subgroups. As we have noticed, $\sigma(G) \geq 3$. As Scorza (see [Z]) has showed, $\sigma(G) = 3$ if and only if there is $N \lhd G$ such that $G/N$ is a four-group. The groups $G$ satisfying $\sigma(G) \in \{4, 5, 6\}$ are also described (see, for example, [C]). On the other hand, it is proved in [T] that $\sigma(G) \neq 7$. In contrast, in this note we consider irredundant coverings by $n$ subgroups such that inequality $n > \sigma(G)$ is possible. For noncyclic $p$-groups $G$, we have $\sigma(G) = p + 1$ always. In the same time, in investigation of irredundant coverings of $p$-groups we meet a number of deep problems, and our note is not more than an introduction in this fascinating topic.

In the following section we study the $p$-groups containing a maximal subset (with respect to inclusion) of pairwise non-commuting elements of cardinality $p+1$. Some related results are also established and discussed. Next, we study the $p$-groups which are covered by $\leq 2p$ proper subgroups. It is proved that if a $p$-group $G$ admits an irredundant covering by $p + 2$ subgroup, then

$p = 2$. We also consider coverings of nonnilpotent groups by few proper subgroups. Minimal nonabelian and minimal nonnilpotent groups play a crucial role in what follows.

## 2. $p$-groups

A noncyclic $p$-group $G$ admits an irredundant covering by $p+1$ maximal subgroups (indeed, if $T \triangleleft G$ is such that $G/T$ is abelian of type $(p,p)$, then $p+1$ maximal subgroups of $G$ containing $T$, cover $G$). Moreover, Lemma 2.1 shows that if a $p$-group $G$ is covered by $p+1$ proper subgroups $A_1, \dots, A_{p+1}$, then $|G : \bigcap_{i=1}^{p+1} A_i| = p^2$, i.e., all $A_i$ are maximal in $G$.

Lemma 2.1 is known; it is proved to make our exposition self contained.

For $X \subseteq G$, we write $X^\# = X - \{1\}$

Lemma 2.1. *Suppose that a noncyclic $p$-group $G$ of order $p^m$ is covered by $n$ proper subgroups $A_1, \dots, A_n$ as in (1.1). Then*

(a) $n \geq p+1$.

(b) *If $n = p+1$, then covering (1.1) is irredundant and $|G : \bigcap_{i=1}^{p+1} A_i| = p^2$. In particular, all the $A_i$'s are maximal in $G$.*

Proof. (a) If $n \leq p$, then

$$|\sum_{i=1}^{n} A_i^\#| \leq p(p^{m-1} - 1) = p^m - p < |G^\#|,$$

which is a contradiction.

(b) Now let $n = p+1$. Then the covering (1.1) is irredundant, by (a). First assume that $A_1, \dots, A_{p+1}$ are maximal in $G$; then $|A_i \cap A_j| = p^{m-2}$ for $i \neq j$. We have

$$G = A_{p+1} \cup \left( \bigcup_{i=1}^{p} (A_i - A_{p+1}) \right). \tag{2.1}$$

Since $A_i - A_j = A_i - (A_i \cap A_j)$ for $i \neq j$, the right-hand side of (2.1) contains at most

$$p^{m-1} + p(p^{m-1} - p^{m-2}) = p^m = |G|$$

elements so (2.1) is a partition of $G$. It follows that $A_i \cap A_{p+1} = A_j \cap A_{p+1}$ and $(A_i - A_{p+1}) \cap (A_j - A_{p+1}) = \emptyset$ for all distinct $i, j < p+1$ (indeed, one can take in (2.1), $A_j$, $j \neq i$, instead of $A_{p+1}$). We conclude that $\bigcap_{i=1}^{p+1} A_i = A_1 \cap A_{p+1}$ has index $p^2$ in $G$.

It follows from the above computation (see the displayed formula after (2.1)) that, in fact, all subgroups $A_1, \dots, A_{p+1}$ must be maximal in $G$ (otherwise, we obtain $|\bigcup_{i=1}^{p+1} A_i| < |G|$).[1] ∎

[1]For another, longer proof, due to M. Roitman, see [B2, Remark 3.5].

It follows from Lemmas 1.3 and 2.1 that if $G$ is a nonabelian $p$-group, then $\gamma(G) \geq p+1$. In Theorem 2.3(b), the $p$-groups $G$ with $\gamma(G) = p+1$ are classified.

LEMMA 2.2. *Let $H$ be a minimal nonabelian subgroup of a $p$-group $G$. Then the intersection $\Lambda(H) \cap \Lambda(G)$ is not empty if and only if $G = H * \mathrm{C}_G(H)$; in that case, $\Lambda(H) \subseteq \Lambda(G)$.*

PROOF. (i) Let $\mathcal{M} \in \Lambda(H)$ and suppose that $\mathcal{M} \in \Lambda(G)$; then $|\mathcal{M}| = p+1$ (Lemma 1.2(a)). By hypothesis and Lemma 1.3(a), $G = \bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x)$ so, by Lemma 2.1(b), $|G : \bigcap_{x\in\mathcal{M}} \mathrm{C}_G(x)| = p^2$. Since $\bigcap_{x\in\mathcal{M}} \mathrm{C}_G(x) = \mathrm{C}_G(\mathcal{M})$ and $\langle\mathcal{M}\rangle = H$, we get $\mathrm{C}_G(H) = \mathrm{C}_G(\mathcal{M})$. Since $\mathrm{C}_G(H) \cap H = \mathrm{Z}(H)$ has index $p^2 = |G : \mathrm{C}_G(\mathcal{M})|$ in $H$, we get $G = H * \mathrm{C}_G(H)$, by the product formula. In particular, $H$ is $G$-invariant.

(ii) Now suppose that an (arbitrary) $p$-group $G = H * \mathrm{C}_G(H)$, where $H$ is minimal nonabelian, and let $\mathcal{M} = \{x_1, \dots, x_{p+1}\} \in \Lambda(H)$. Then $G = H * \mathrm{C}_G(H) \subseteq \bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x)$, so $\mathcal{M} \in \Lambda(G)$, by Lemmas 1.3(a) and 2.1(a). Thus, $\Lambda(H) \subseteq \Lambda(G)$.[2] □

If $M$ is a subset of a group $G$, then $\mathrm{C}_G(M) = \bigcap_{x\in M} \mathrm{C}_G(x)$.

THEOREM 2.3. *Let $G$ be a nonabelian $p$-group.*

(a) *If $\mathcal{M} \in \Lambda(G)$ has cardinality $p+1$, then $|G : \mathrm{C}_G(x)| = p$ for all $x \in \mathcal{M}$ and $|G : \mathrm{C}_G(\mathcal{M})| = p^2$.*
(b) *$\gamma(G) = p + 1$ if and only if $G = H\mathrm{Z}(G)$, where $H$ is an arbitrary minimal nonabelian subgroup of $G$, $H \cap \mathrm{Z}(G) = \mathrm{Z}(H)$. If, in addition, $G$ is of exponent $p$, then $G = H \times E$, where $H$ is nonabelian of order $p^3$ and $E$ is abelian.*

PROOF. Given $\mathcal{M} \in \Lambda(G)$, we have $G = \bigcup_{x\in\mathcal{M}} \mathrm{C}_G(x)$, and this covering is irredundant (Lemma 1.3(a)).

(a) follows from Lemma 2.1(b).

(b) Suppose that $\gamma(G) = p+1$. Let $H \leq G$ be minimal nonabelian. Then $\mathcal{M} \in \Lambda(H)$ has cardinality $p+1$ (Lemma 1.2(a)) so that $\mathcal{M} \in \Lambda(G)$. By Lemma 2.2, $G = H * \mathrm{C}_G(H)$ (central product).

We claim that $\mathrm{C}_G(H) = \mathrm{Z}(G)$. It suffices to show that $\mathrm{C}_G(H)$ is abelian. Assume that this is false. Then $\mathrm{C}_G(H)$ contains two non-commuting elements $b, b_1$. Let $\mathcal{M} = \{a_1, \dots, a_{p+1}\} \in \Lambda(H)$. Take $a \in L - \{\mathrm{Z}(H) \cup \{a_1\}\}$, where $L$ is an (abelian) maximal subgroup of $H$ containing $a_1$ (such $a$ exists since $|L - \mathrm{Z}(H)| > 1$). Then, since $[ab, a_i] = [a, a_i] \neq 1$ for $i > 1$ (indeed, for $i > 1$, the subgroup $\langle a, a_i\rangle = H$ is nonabelian), we obtain $\{ab, a_2, \dots, a_{p+1}\} \in \Lambda(G)$. Note, that $[ab, ab_1] = [b, b_1] \neq 1$ and, for $i > 1$, we have $[ab_1, a_i] = [a, a_i] \neq 1$.

[2] We do not assert that here, in the case under consideration, $\gamma(G) = \gamma(H)$ (however, this equality holds, by Theorem 2.3(b)).

It follows that $p+2(>\gamma(G))$ elements $ab, ab_1, a_2, \dots, a_{p+1}$ are pairwise non-commuting, a contradiction. Thus, $\mathrm{C}_G(H)$ is abelian so coincides with $\mathrm{Z}(G)$.

Let us show that for our group $G = H\mathrm{Z}(G)$ we have $\gamma(G) < p+2$ (by Lemma 2.2(b), $\Lambda(H) \subseteq \Lambda(G)$, but our assertion is stronger). Indeed, assume that $g_1, \dots, g_{p+2} \in G$ are pairwise non-commuting. Then $g_i = h_i z_i$, where $h_i \in H$, $z_i \in \mathrm{Z}(G)$ $(i = 1, 2, \dots, p+2)$. Let $i \neq j$. Then $[h_i, h_j] = [h_i z_i, h_j z_j] = [g_i, g_j] \neq 1$ so the minimal nonabelian $p$-group $H$ contains $p+2$ pairwise non-commuting elements $h_1, \dots, h_{p+2}$, contrary to Lemma 1.2(a).

Now suppose that $G = H\mathrm{Z}(G)$ is of exponent $p$ (here $H$ is of order $p^3$ as minimal nonabelian group of exponent $p$, and $\mathrm{Z}(G)$ is elementary abelian). In that case, $H \cap \mathrm{Z}(G) = \mathrm{Z}(H)$ is of order $p$ so $\mathrm{Z}(G) = \mathrm{Z}(H) \times E$, where $E$ is elementary abelian. Then $G = H \times E$, and this completes the proof of (b). □

Theorem 2.3(b), in particular, classifies the nonabelian $p$-groups possessing exactly $p+1$ distinct centralizers of noncentral elements (note that paper [P] yields an estimate of $|G : \mathrm{Z}(G)|$ is terms of $\gamma(G)$).

Proposition 2.4. *The following assertions for a nonabelian $p$-group $G$ are equivalent:*

(a) *If $H \leq G$ is minimal nonabelian, then $\Lambda(H) \subseteq \Lambda(G)$.*
(b) *$G = (B_1 * \cdots * B_k)\mathrm{Z}(G)$, where $B_1, \dots, B_k$ are minimal nonabelian.*

Proof. (a) $\Rightarrow$ (b): We proceed by induction on $|G|$. Let $B_1 \leq G$ be minimal nonabelian. Then $G = B_1 * \mathrm{C}_G(B_1)$, by Lemma 2.2. If $\mathrm{C}_G(B_1)$ is abelian, we are done. If $\mathrm{C}_G(B_1)$ is nonabelian, the result follows by induction applied to $\mathrm{C}_G(B_1)$ since $\mathrm{Z}(\mathrm{C}_G(B_1)) = \mathrm{Z}(G)$.

(b) $\Rightarrow$ (a): Let $G$ be as in (b) and $H \leq G$ minimal nonabelian. Since $|G'| = p$, then, by [B1, Lemma 4.3(a)], we obtain $G = H * \mathrm{C}_G(H)$ so $\Lambda(H) \subseteq \Lambda(G)$, by Lemma 2.2. □

Remark 2.5. The argument in part (ii) of the proof of Lemma 2.2 shows that if $H$ is a nonabelian subgroup of an arbitrary group $G = H * \mathrm{C}_G(H)$, then $\Lambda(H) \subseteq \Lambda(G)$.

## 3. Nonnilpotent groups

In this section $G$ is a nonnilpotent group.

Let $p$ be a prime divisor of $|G|$ such that $G$ has no normal $p$-complement. Then there is in $G$ a minimal nonnilpotent subgroup $H = Q \cdot P$, where $P = H' \in \mathrm{Syl}_p(H)$ and $Q \in \mathrm{Syl}_q(H)$ is cyclic (this follows from Frobenius' normal $p$-complement theorem; see, for example, [I, Theorem 9.18]). We have $|P| = p^{b+c}$, where $b$ is the order of $p \pmod q$ and $p^c = |P \cap \mathrm{Z}(H)|$ (see [BZ, Lemma 11.2]). In that case, there are in $H$ exactly $p^b$ Sylow $q$-subgroups, say

$$Q_1 = \langle x_1 \rangle, \dots, Q_{p^b} = \langle x_{p^b} \rangle.$$

Then $x_1, \dots, x_{p^b}$ are pairwise non-commuting elements (indeed, if $i \neq j$, then $\langle x_i, x_j \rangle$ is nonnilpotent so coincides with $H$: it has two distinct Sylow $q$-subgroups $Q_i$ and $Q_j$). If $\{y_1, \dots, y_s\}$ is a maximal subset of pairwise non-commuting elements of $P$, then $y_1, \dots, y_s, x_1, \dots, x_{p^b}$ is a maximal subset (with respect to inclusion) of pairwise non-commuting elements of $H$ of cardinality $p^b + s \geq p + 1$ (note that $s = 1$ if and only if $P$ is abelian). Thus, $\gamma(G) \geq \gamma(H) = p^b + s \geq p^b + 1$.

THEOREM 3.1. *Let $G$ be a nonabelian group and $p$ a prime divisor of $|G|$.*

(a) *If $G$ has no normal $p$-complement, then $\gamma(G) \geq p + 1$. If, in addition, $p$ is the minimal prime divisor of $|G|$, then $\gamma(G) \geq p^2 + 1$.*
(b) *Suppose that $G$ has a normal $p$-complement however a Sylow $p$-subgroup is not a direct factor of $G$. Then $\gamma(G) \geq p+2$. If, in addition, $\gamma(G) = p + 2$, then either $p = 2$ and $q = 3$ or $p$ is a Mersenne prime.*
(c) *If $G = P \times A$, where $P$ is nonabelian, $A$ is abelian and $\gamma(G) < p + 2$, then $P$ is such as in Theorem 2.3(b).*

PROOF. (a) was proved in the paragraph, preceding the theorem.

(b) Now assume that $G$ has a normal $p$-complement $H$ but $P \in \mathrm{Syl}_p(G)$ is not a direct factor of $G$. It follows that the $p$-solvable group $G$ contains a nonnilpotent subgroup $PQ$, where $Q \in \mathrm{Syl}_q(H)$; then $Q = PQ \cap H \lhd PQ$. In that case, $PQ$ contains a minimal nonnilpotent subgroup $F = P_1 Q_1$, where $P_1 \in \mathrm{Syl}_p(F)$ is cyclic and $Q_1 = F' \in \mathrm{Syl}_q(F)$. Then $|Q_1| = q^{\beta+c}$, where $\beta$ is the order of $q \pmod p$ and $q^c = |Q_1 \cap \mathrm{Z}(F)|$. As above, there is $\mathcal{M} \in \Lambda(F)$ of cardinality $\geq q^\beta + 1$. Since $q^\beta \geq p + 1$, we get $|\mathcal{M}| \geq p + 2$. Now assume that $|\mathcal{M}| = p + 2$; then $q^\beta = p + 1$ so either $p = 2$ and $q = 3$ or $q = 2$ and $p$ is a Mersenne prime.

(c) now follows from Remark 2.5 and Theorem 2.3(b). □

PROPOSITION 3.2. *Let $p$ be a minimal prime divisor of the order of a group $G$ and let $G = \bigcup_{i=1}^{p+1} A_i$ be an irredundant covering. Then $|G : \bigcap_{i=1}^{p+1} A_i| = p^2$. In particular, $|G : A_i| = p$ for $i = 1, \dots, p + 1$.*

PROOF. It follows from Remark 1.1 that, if $p$ is the minimal prime divisor of a group $G$, then it is not covered by $p$ proper subgroups. One may assume that $|A_1| \geq \cdots \geq |A_n|$. Then, by Remark 1.1, we have $|G : A_1| < p + 1$ so that $|G : A_1| = p$ and $A_1 \lhd G$.

First assume that all $A_i$ are maximal in $G$. Set $|G| = g$, $|G : A_i| = k_i$, $i = 2, \dots, p + 1$. Note, that $k_i \geq p$ for all $i$. We have

$$G = A_1 \cup \left( \bigcup_{i=2}^{p+1} (A_i - A_1) \right). \tag{3.1}$$

Since $A_i - A_1 = A_i - (A_i \cap A_1)$ and $|A_i : (A_i \cap A_1)| = p$ so $|G : (A_i \cap A_1)| = pk_i$ for $i > 1$, so obtain

$$|A_i - A_1| = \frac{g}{k_i} - \frac{g}{pk_i} = \frac{g}{k_i}\left(1 - \frac{1}{p}\right).$$

The right-hand side of (3.1) contains $v$ elements, where

$$v \leq \frac{g}{p} + \left(1 - \frac{1}{p}\right)\sum_{i=2}^{p+1}\frac{g}{k_i} \leq \frac{g}{p} + \left(1 - \frac{1}{p}\right)\sum_{i=2}^{p+1}\frac{g}{p} = \frac{g}{p} + \frac{g}{p}\left(1 - \frac{1}{p}\right)p = g.$$

Since $v = g$, it follows that (3.1) is a partition of $G$ and $k_i = p$ for all $i$; in that case, $|G : \bigcap_{i=1}^{p+1} A_i| = p^2$.

Now let $A_i \leq B_i < G$, where $B_i$ are maximal in $G$ for all $i$. Then $G = \bigcup_{i=1}^{p+1} B_i$ is an irredundant covering of $G$, by the first sentence of the proof, and so $|G : B_i| = p$ for all $i$, by the previous paragraph. If some $A_i < B_i$, then, taking in (3.1), $A_j = B_j$ for $j \neq i$, we get a contradiction. Thus, $B_i = A_i$ for all $i$ and so $|G : \bigcap_{i=1}^{p+1} A_i| = p^2$, by the previous paragraph. □

Lemma 2.1(b) is a partial case of Proposition 3.2.

Let $G$ be a non-$p$-nilpotent group. Then, using Theorem 3.1, one can show the following results:

(a) If $p = 2$, then $\gamma(G) \geq 5$.
(b) If $p > 2$, then $\gamma(G) \geq p + 1$.
(c) If $p > 2$ is a minimal prime divisor of $|G|$, then $\gamma(G) \geq p^3 + 1$.

## 4. On the number of maximal subgroups appearing in some coverings of $p$-groups

In this section we consider irredundant coverings of a $p$-group by $k$ proper subgroups, where $p + 1 < k \leq 2p$.

It is impossible to avoid some repetitions in computations (otherwise, the proofs will be unreadable).

Remark 4.1. We claim that, if a $p$-group $G$ is neither cyclic nor $Q_8$, it admits an irredundant covering by $2p$ subgroups. Indeed, let $T \lhd G$ be such that $G/T$ is abelian of type $(p, p)$. Let $A_1/T, \ldots, A_{p+1}/T$ be all subgroups of order $p$ in $G/T$. Then $G = \bigcup_{i=1}^{p+1} A_i$ is an irredundant covering. Since $G$ is neither cyclic nor isomorphic to $Q_8$, one may assume that $A_1$ is noncyclic (here we use [B1, Theorem 1.2] which implies that if a $p$-group contains $> p$ cyclic subgroups of index $p$, it is $\cong Q_8$). In that case, there is in $T$ an $A_1$-invariant subgroup $T_0$ such that $A_1/T_0$ is abelian of type $(p, p)$. Let $T = T_1, T_2, \ldots, T_{p+1}$ be all maximal subgroups of $A_1$ containing $T_0$. Then $G$ is covered by $2p$ subgroups $A_2, \ldots, A_{p+1}, T_2, \ldots, T_{p+1}$, and this covering is irredundant.

THEOREM 4.2. *If a $p$-group $G$ admits an irredundant covering by $p+2$ subgroups $A_1, \dots, A_{p+2}$, then*

(a) *If $p > 2$, then at least $p+1$ of the $A_i$'s are maximal in $G$.*
(b) *If $p = 2$, then at least two of the $A_i$'s are maximal in $G$.*

PROOF. Let $|A_1| \geq \cdots \geq |A_{p+2}|$ and $|G| = p^n$. By Remark 1.1, $|G : A_1| = p$. Assume that $|G : A_{p+1}| > p$. Set $|G| = p^n$. Then

$$\begin{aligned} p^n &= |\bigcup_{i=1}^{p+2} A_i| \leq |A_1| + \sum_{i=2}^{p} |A_i - A_1| + \sum_{i=p+1}^{p+2} |A_i - A_1| \\ &= p^{n-1} + (p-1)(p^{n-1} - p^{n-2}) + 2(p^{n-2} - p^{n-3}) \\ &= p^n - p^{n-3}(p^2 - 3p + 2) = p^n - p^{n-3}(p-1)(p-2). \end{aligned}$$

If $p > 2$, then $p^n \leq p^n - p^{n-3}(p-1)(p-2) < p^n$, which is a contradiction. Thus, if $p > 2$, then at least $p+1$ subgroups $A_i$'s are maximal in $G$, completing this case.

Now let $p = 2$ and assume that $|A_2| < 2^{n-1}$. Then

$$2^n = |\sum_{i=1}^{4} A_i| \leq |A_1| + \sum_{i=2}^{4} |A_i - A_1| = 2^{n-1} + 3(2^{n-2} - 2^{n-3}) = 7 \cdot 2^{n-3} < 2^n,$$

a contradiction. Thus, if $p = 2$, then at least two $A_i$'s are maximal in $G$. ☐

Let $G = \bigcup_{i=1}^{4} A_i$ be an irredundant covering of a 2-group that is not two-generator, $|A_1| \geq |A_2| \geq |A_3| \geq |A_4|$; then $A_1, A_2 \in \Gamma_1$ (Theorem 4.2(b)). We claim that if $|G : A_3| = 2$, then $|G : A_4| = 2$ so all $A_i$ are maximal in $G$. We have

$$G = A_1 \cup (A_2 - A_1) \cup (A_3 - A_1 - A_2) \cup (A_4 - A_1). \tag{4.1}$$

Assume that $|G : A_4| > 2$. We have $|G : (A_1 \cap A_2 \cap A_3)| = 2^3$ (Lemma 2.1). Therefore,

$$|A_3 - A_1 - A_2| = 2^{n-1} - 2 \cdot 2^{n-2} + 2^{n-3} = 2^{n-3}.$$

It follows that the right-hand side of formula (4.1) contains $v$ elements, where

$$v \leq 2^{n-1} + (2^{n-1} - 2^{n-2}) + 2^{n-3} + (2^{n-3} - 2^{n-4}) < 2^n,$$

which is a contradiction. Thus, either exactly two or four of the $A_i$'s are maximal in $G$.

Let $G$ be a 2-group that is not generated by two elements. We claim that then $G$ admits an irredundant covering $G = \bigcup_{i=1}^{4} A_i$, where $A_i \in \Gamma_1$ for all $i$. Without loss of generality, one may assume that $\Phi(G) = \{1\}$. Let $A_1, A_2 \in \Gamma_1$ be distinct, and set $T = A_1 \cap A_2$. Let $T < A_3 \in \Gamma_1 - \{A_1, A_2\}$ and let $S < T$ be of index 2; then $A_3/S \cong \mathrm{E}_4$. Let $T/S, T_1/S, T_2/S < A_3/S$ be of index 2. Since $G/T_i \cong \mathrm{E}_4$, there is $B_1, B_2 \in \Gamma_1 - \{A_3\}$ such that $B_i \cap A_3 = T_i$, $i = 1, 2$. Since $A_3$ is a subset of the set $A_1 \cup B_1 \cup B_2$ (indeed, $A_3 = T \cup T_1 \cup T_2$

is a subset of $A_1 \cup B_1 \cup B_2$), it follows that $G = A_1 \cup A_2 \cup B_1 \cup B_2$ is a covering (indeed, $G = A_1 \cup A_2 \cup A_3$ is a subset of $A_1 \cup A_2 \cup B_1 \cup B_2$). Since the intersection of any three distinct elements of the set $\{A_1, A_2, B_1, B_2\}$ has index $2^3$ in $G$, our covering is irredundant (Lemma 2.1(b)).

Similarly, if $p > 2$ and a $p$-group $G$ is not generated by two elements, then it admits an irredundant covering by $2p$ maximal subgroups.

LEMMA 4.3. *Suppose that $A, B, C, D$ are pairwise distinct maximal subgroups of a $p$-group $G$ of order $p^n$ such that $|G : (A \cap B \cap C)| = p^3$. Then*

$$|A \cup B \cup C| = 3p^{n-1} - 3p^{n-2} + p^{n-3}, \tag{4.2}$$

$$|D - (A \cup B \cup C)| \le p^{n-1} - 3p^{n-2} + 3p^{n-3} - p^{n-4}. \tag{4.3}$$

PROOF. Note that if distinct $U, V < G$ are maximal, then $|G : (U \cap V)| = p^2$. By the inclusion-exclusion identity [H, formula (2.2.1)],

$$\begin{aligned}|A \cup B \cup C| &= (|A| + |B| + C|) - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C| \\ &= 3p^{n-1} - 3p^{n-2} + p^{n-3}.\end{aligned}$$

By hypothesis, $A \cap B \neq B \cap C \neq C \cap A$ (if, for example, $A \cap B = A \cap C$, then $A \cap B = (A \cap B) \cap (A \cap C) = A \cap B \cap C$, a contradiction since $|A \cap B| = p^{n-2} > p^{n-3} = |A \cap B \cap C|$). We have

$$D - (A \cup B \cup C) = D - (D \cap (A \cup B \cup C)), \tag{4.4}$$

and so

$$D - (A \cup B \cup C) = D - ((D \cap A) \cup (D \cap B) \cup (D \cap C)).$$

By inclusion-exclusion identity,

$$\begin{aligned}|(D \cap A) \cup (D \cap B) \cup (D \cap C)| &= (|D \cap A| + |D \cap B| + |D \cap C|) \\ -(|D \cap A \cap B| + |D \cap A \cap C| &+ |D \cap B \cap C|) + |D \cap A \cap B \cap C|.\end{aligned}$$

If $A \cap B \subset D$, then $D \cap A \cap B \cap C = D \cap C$ has order $p^{n-2}$, a contradiction since $A \cap B \cap C \supseteq D \cap A \cap B \cap C$ has order $p^{n-3}$, by hypothesis. Thus, $A \cap B \not\subseteq D$, and the same is true for $A \cap C$ and $B \cap C$. Then, by the product formula and the previous displayed formula, we have

$$|D \cap A \cap B| = |D \cap A \cap C| = |D \cap B \cap C| = p^{n-3},$$

$$\begin{aligned}|D \cap (A \cup B \cup C)| &= |(D \cap A) \cup (D \cap B) \cup (D \cap C)| \\ &= 3p^{n-2} - 3p^{n-3} + |D \cap A \cap B \cap C|.\end{aligned}$$

Since $|D \cap A \cap B \cap C| \in \{p^{n-3}, p^{n-4}\}$, we obtain

$$|D \cap (A \cup B \cup C)| \ge 3p^{n-2} - 3p^{n-3} + p^{n-4}.$$

Now (4.3) follows from (4.4). ∎

THEOREM 4.4. *If a $p$-group admits an irredundant covering by $p+2$ proper subgroups, then $p = 2$.*

PROOF. Assume that a group $G$ of order $p^n$ admits an irredundant covering by $p+2$ proper subgroups $A_1, \dots, A_{p+2}$ and $p > 2$. By Theorem 4.2(a), one may assume that $A_1, \dots, A_{p+1}$ are maximal in $G$. Since $G \neq \bigcup_{i=1}^{p+1} A_i$, we have $|G : \bigcap_{i=1}^{p+1} A_i| \geq p^3$. One may assume, without loss of generality, that $|G : (A_1 \cap A_2 \cap A_3)| = p^3$. We may also assume that $|G : A_{p+2}| = p$. Then, by (4.2), we have

$$|A_1 \cup A_2 \cup A_3| = 3p^{n-1} - 3p^{n-2} + p^{n-3} \tag{4.5}$$

and, for $i > 3$,

$$|A_i - (A_1 \cup A_2 \cup A_3)| < p^{n-1} - 3p^{n-2} + 3p^{n-3} = p^{n-3}(p^2 - 3p + 3), \tag{4.6}$$

by (4.3).

Set $A_1 \cup A_2 \cup A_3 = U$. We have

$$G = U \cup (\bigcup_{i=4}^{p+2} (A_i - U)). \tag{4.7}$$

Therefore, taking in account (4.5) and (4.6), we obtain

$$\begin{aligned} |G| = p^n &\leq (3p^{n-1} - 3p^{n-2} + p^{n-3}) + (p-1)p^{n-3}(p^2 - 3p + 3) \\ &= p^n - p^{n-3}(p^2 - 3p + 2) = p^n - p^{n-3}(p-1)(p-2) < p^n, \end{aligned}$$

since $p > 2$, a final contradiction. Thus, we must have $p = 2$. ☐

PROPOSITION 4.5. *If a $p$-group $G$ is covered by at most $k \leq 2p$ proper subgroups $A_1, \dots, A_k$ (we do not assume that this covering is irredundant), then at least $p$ of these subgroups are maximal in $G$. If $p > 3$ and $p+2 < k < 2p$, then at least $p+1$ summands in our covering are maximal in $G$.*

PROOF. (i) One may assume that $G$ is not isomorphic to $Q_8$. Let $|G| = p^n$.

In view of Theorem 4.2(b), one may assume that $p > 2$. Let $|A_1| \geq \cdots \geq |A_k|$. Then $|G : A_1| = p$ (Remark 1.1). Since we do not assume that our covering is irredundant, one can add new summands of order $p^{n-2}$ to obtain $k = 2p$. We also may assume, by way of contradiction, that $A_1, \dots, A_{p-1}$ are maximal in $G$ and $A_p, \dots, A_{2p}$ have index $p^2$ in $G$. Indeed, if, for example, $|G : A_i| > p^2$, $(i > p-1)$, one can replace $A_i$ by subgroup that contains $A_i$ and has index $p^2$ in $G$. If, for example, $|G : A_i| > p$ $(i < p)$, one can replace $A_i$ by maximal subgroup of $G$ that contains $A_i$. We have

$$G = A_{p-1} \cup \left( \bigcup_{i=1}^{p-2} (A_i - A_{p-1}) \right) \cup \left( \bigcup_{i=p}^{2p} (A_i - A_{p-1}) \right). \tag{4.8}$$

The right-hand side of formula (4.8) contains $v$ elements, where

$$\begin{aligned} v &\leq p^{n-1} + (p-2)(p^{n-1} - p^{n-2}) + (p+1)(p^{n-2} - p^{n-3}) \\ &= p^n - p^{n-3}(p-1)^2 < p^n = |G|, \end{aligned}$$

a contradiction. Thus, at least $p$ subgroups $A_i$ ($i \leq 2p$) are maximal in $G$.

(ii) To prove the last assertion, one may assume, by way of contradiction, that $A_1, \dots, A_p$ are maximal in $G$ and $|G : A_i| = p^2$ for $i > p$ (see (i)). (Here $p > 3$ since $3+2 = 2\cdot 3-1$.) We also may assume that $k = 2p-1$ (if $k < 2p-1$, one can add to our union $2p-1-k$ new summands of order $p^{n-2}$). Then, as above, we obtain

$$\begin{aligned}|G| = p^n &\leq p^{n-1} + (p-1)(p^{n-1} - p^{n-2}) + (p-1)(p^{n-2} - p^{n-3}) \\ &= p^{n-1} + (p-1)(p^{n-1} - p^{n-3}) = p^n - p^{n-3}(p-1) < p^n,\end{aligned}$$

a contradiction. □

Proposition 4.6. *Suppose that a $p$-group $G$ of order $p^n \geq p^4$, $p > 2$, is covered by $k$ proper subgroups, say $A_1, \dots, A_k$, where $p + 2 < k \leq 2p$. Let, in addition, any $p + 2$ subgroups $A_i$ do not cover $G$, $|G : A_i| = p$ for $i \leq p$ and $|G : A_i| > p$ for $i > p$. Then*

(a) *$k = 2p$ and our covering is irredundant.*
(b) *$|\bigcap_{i=1}^{p} A_i| = p^{n-2}$.*
(c) *$|A_i| = p^{n-2}$ for $i > p$.*
(d) *$|\bigcap_{i=p+1}^{2p} A_i| = p^{n-3}$.*

Proof. In view of Proposition 4.5, $k = 2p$ so (a) is true since our covering must be irredundant.

We have

$$G = A_p \cup \left( \bigcup_{i=1}^{p-1} (A_i - A_p) \right) \cup \left( \bigcup_{i=p+1}^{2p} (A_i - A_p) \right). \tag{4.9}$$

(c) Assume that $|A_{p+1}| \geq \cdots \geq |A_{2p}|$ and $|A_{2p}| < p^{n-2}$. Then the right-hand side of (4.9) contains $v$ elements, where

$$\begin{aligned}v &\leq p^{n-1} + (p-1)(p^{n-1} - p^{n-2}) + (p-1)(p^{n-2} - p^{n-3}) + (p^{n-3} - p^{n-4}) \\ &= p^{n-1} + (p-1)(p^{n-1} - p^{n-3}) + (p^{n-3} - p^{n-4}) = p^n - p^{n-4}(p-1)^2 < p^n = |G|,\end{aligned}$$

which is a contradiction. This proves (c).

(b, d) We have $|G : A_i| = p^2$ for $i > p$, by (c). In that case, the right-hand side of (4.9) contains $v$ elements, where

$$v \leq p^{n-1} + (p-1)(p^{n-1} - p^{n-2}) + p(p^{n-2} - p^{n-3}) = p^n = |G|,$$

so (4.9) is a partition. This implies (b) and (d). □

A similar argument shows that if a $p$-group $G$ of order $p^n$ is covered by $p^2$ proper subgroups, then at least two of these subgroups are maximal in $G$. Indeed, if only one summand of our covering is maximal in $G$ (see Remark 1.1), we obtain

$$p^n \leq p^{n-1} + (p^2 - 1)(p^{n-2} - p^{n-3}) = p^n - p^{n-3}(p-1) < p^n,$$

a contradiction.

Corollary 4.7. *If a nonabelian $p$-group $G$ has at most $2p$ pairwise non-commuting elements, then centralizers of at least $p$ of these elements are maximal in $G$.*

Remark 4.8. Let $G$ be a group of maximal class and order $p^{n+2}$, $n \geq 2$, with abelian subgroup $A$ of index $p$. In that case, every $x \in G - A$ satisfies $|\mathrm{C}_G(x)| = p^2$ (indeed, $\mathrm{C}_A(x) = \mathrm{Z}(G)$ is of order $p$) and the number of maximal abelian subgroups of order $p^2$ not contained in $A$, is equal to $\frac{|G-A|}{p(p-1)} = p^n$ (indeed, if $B$ is such a subgroup, then $|B - A| = p(p-1)$ and all such $B$ cover the set $G - A$). These $p^n$ subgroups together with $A$ cover $G$ and this covering is irredundant. If $T$ is one of such abelian subgroups, take $x \in T - \mathrm{Z}(G)$. So obtained set of cardinality $p^n + 1$ is contained in $\Lambda(G)$. It is easy to see that $\gamma(G) = p^n + 1$.

Remark 4.9. Let $G$ be a nonabelian $p$-group. If $x \in G$ and $A_x$ is a maximal abelian subgroup of $\mathrm{C}_G(x)$, then $A_x$ is also a maximal abelian subgroup of $G$. Let $\mathcal{M} \in \Lambda(G)$. Take in $\mathrm{C}_G(x)$ a maximal abelian subgroup $A_x$ for every $x \in \mathcal{M}$ (indeed, if $B > A_x$ is abelian, then, by the choice, $B \not\leq \mathrm{C}_G(x)$, a contradiction). Then $|\{A_x \mid x \in \mathcal{M}\}| = |\mathcal{M}|$. It follows that $G$ has at least $\gamma(G)$ maximal abelian subgroups. If $G$ has exactly $p + 1$ maximal abelian subgroups, say $A_1, \ldots, A_{p+1}$, they cover $G$. In that case, $A_1, \ldots, A_{p+1}$ are maximal in $G$ (Lemma 1.3(b)) and $G$ has the structure described in Theorem 2.3(b).

Remark 4.10. Let $B_1, \ldots, B_n$ be all maximal abelian subgroups of a nonabelian group $G$. Then $\gamma(G) \leq n$ since $G = \bigcup_{i=1}^n B_i$ (it is possible that this covering may be redundant). If $B_i \cap B_j = \mathrm{Z}(G)$ for all $i \neq j$ (in this case, the considered covering is irredundant), then $\gamma(G) = n$. Indeed, take $x_i \in B_i - \mathrm{Z}(G)$ for all $i$. We claim that $\{x_1, \ldots, x_n\} \in \Lambda(G)$.[3] For example, let $G$ be a Sylow 2-subgroup of the simple Suzuki group $\mathrm{Sz}(q)$, where $q = 2^{2m+1}$. Then $\mathrm{Z}(G) = \Phi(G)$ has index $2^{2m+1}$ in $G$. If $A < G$ is maximal abelian, then $|A : \mathrm{Z}(G)| = 2$. It follows that there is $\mathcal{M} \in \Lambda(G)$ of cardinality $2^{2m+1} - 1$; moreover, all members of the set $\Lambda(G)$ have the same cardinality.

Remark 4.11. Let $G = A * B$ be a central product of nonabelian subgroups $A$ and $B$, $\mathcal{M} = \{a_1, \ldots, a_m\} \in \Lambda(A)$ and $\mathcal{N} = \{b_1, \ldots, b_n\} \in \Lambda(B)$. Then $m - 1 + n$ elements of the following set

$$\mathcal{M}_1 = \{a_2, \ldots, a_m, a_1b_1, a_1b_2, \ldots, a_1b_n\}$$

are pairwise non-commuting. We claim that $\mathcal{M}_1 \in \Lambda(G)$. Assume that there is $x \in G - \mathcal{M}_1$ such that all elements of the set $\mathcal{M}_1 \cup \{x\}$ are pairwise non-commuting. We have $x = ab$, where $a \in A$ and $b \in B$. For $i > 1$, we have

[3]Indeed, if $\{x, x_1, \ldots, x_n\} \in \Lambda(G)$ and $x \in B$, where $B < G$ is maximal abelian and $x \neq x_i$ for all $i$, then $B \notin \{B_1, \ldots, B_n\}$.

$1 \neq [ab, a_i] = [a, a_i]$ so that $a \in D = A - \bigcup_{i=2}^{n} \mathrm{C}_A(a_i)$. By Lemma 1.3(b), the subgroup $\langle D \rangle$ is abelian so $[a, a_1] = 1$ since $a_1 \in D$. For $i = 1, \dots, n$, we have $1 \neq [ab, a_1 b_i] = [ab, b_i] = [b, b_i]$. We conclude that $n + 1 > \gamma(B)$ elements $b, b_1, \dots, b_n \in B$ are pairwise non-commuting, a contradiction. In particular, if $G$ is an extraspecial group of order $p^{2m+1}$, then, by induction, $\gamma(G) \geq mp + 1$.

It follows from Remark 4.11 and Lemma 1.2(a) that if $G$ is a nonabelian $p$-group such that $\gamma(G) \leq 2p$, then $\mathrm{C}_G(H)$ is abelian for all minimal nonabelian $H < G$.

It is interesting to carry out similar considerations for infinite groups. We consider only one example. According to [SS], every infinite minimal nonabelian group $G$ coincides with its derived subgroup. Every proper noncentral subgroup of $G$ is contained in a unique maximal subgroup of $G$, its centralizer. Since $G = G'$, every maximal subgroup has infinite index in $G$ (Poincare). Let $\Gamma_1$ be the set of maximal subgroups of $G$. For every $H \in \Gamma_1$, choose $x \in H - \mathrm{Z}(G)$. Since the intersection of any two distinct members of the set $\Gamma_1$ coincides with $\mathrm{Z}(G)$ and the set $\Gamma_1$ is infinite, all so chosen elements form an infinite set of pairwise non-commuting elements. It is not true that every nonabelian infinite group possesses an infinite set of pairwise non-commuting elements. For example, $G = H \times A$, where $H$ is finite nonabelian and $A$ is infinite abelian, satisfies $\gamma(G) = \gamma(H) < \infty$.

## 5. Problems

Below we state some related problems.

1. Classify the 2-groups without five pairwise non-commuting elements. (See Lemma 1.2 and Theorem 4.4)
2. Does there exist a $p$-group $G$ admitting an irredundant covering by $n$ subgroups, where $p + 1 < n < 2p$? If 'yes', classify such the groups.
3. Describe the set of positive integers $n$ such that there is an elementary abelian $p$-group admitting an irredundant covering by $n$ maximal subgroups.
4. Let $M, N$ be groups and $\gamma(M) = m$, $\gamma(N) = n$. Then $\gamma(M \times N) = mn$. (i) Estimate $\gamma(M * N)$ in terms of $M$, $N$ and $M \cap N$. Consider the case where $M, N$ are $p$-groups of maximal class. (ii) Find $\gamma(G)$, where $G$ is an extraspecial group of order $p^{2m+1}$ (see Remark 4.11).
5. Classify the pairs groups $N \lhd G$ such that $\gamma(G/N) = \gamma(G)$.
6. Find $\gamma(\Sigma_{p^n})$, where $\Sigma_{p^n}$ is a Sylow $p$-subgroup of the symmetric group $\mathrm{S}_{p^n}$ of degree $p^n$. The same problem for $\mathrm{UT}(m, p^n)$, a Sylow $p$-subgroup of the general linear group $\mathrm{GL}(m, p^n)$.
7. Study the nonabelian $p$-groups $G$ such that $\mathrm{C}_G(H)$ is abelian for all minimal nonabelian $H \leq G$ (see the paragraph following Remark 4.11).

8. Study the nonnilpotent groups $G$ such that $\Lambda(H) \subseteq \Lambda(G)$ for all minimal nonnilpotent $H \leq G$ (compare with Lemma 2.2(b)).
9. Study the groups that are covered by (i) minimal nonnilpotent subgroups, (ii) minimal nonabelian subgroups, (iii) Frobenius subgroups.
10. Classify the $p$-groups that are covered by subgroups of maximal class.
11. Let $H$ be a proper subgroup of maximal class of a $p$-group $G$ such that $\Lambda(H) \subset \Lambda(G)$. Study the structure of $G$.
12. Find $\gamma(G)$, where $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$ (for example, $\gamma(\mathrm{A}_5) = 21$; see also [Br]).

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