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ON THE SECOND HOMOTOPY GROUP OF $SC(Z)$

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Abstract. In our earlier paper we introduced a cone-like space $SC(Z)$. In the present note we establish some new algebraic properties of $SC(Z)$.

1. Introduction

In our earlier paper ([1]) we introduced a new, cone-like construction of a space $SC(Z)$ using the topologist sine curve and proved that $SC(Z)$ is simply connected for every path-connected space Z. In another paper ([3]) we proved that the singular homology $H_2(SC(Z);\mathbb{Z})$ is non-trivial if $\pi_1(Z)$ is non-trivial. In the present paper we prove its converse, that is:

Theorem 1.1. *Let Z be any path-connected space, $z_0 \in Z$. If $\pi_1(Z, z_0)$ is trivial, then $\pi_2(SC(Z), z_0)$ is also trivial.*

Consequently, we get the following:

Corollary 1.2. *For any path-connected space Z, $z_0 \in Z$, the following statements are equivalent:*

(1) *$\pi_1(Z, z_0)$ is trivial;*
(2) *$\pi_2(SC(Z), z_0)$ is trivial; and*
(3) *$H_2(SC(Z);\mathbb{Z})$ is trivial.*

We also use this opportunity to correct the proof of Lemma 3.2 in [2] (see Section 3).

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2. Proof of Theorem 1.1

In order to describe the homotopies we shall need to introduce some notations. The unit interval $[0,1]$ is denoted by $\mathbb{I}$. For a map $f : [a_0, b_0] \times [c, d] \to X$, define $f^- : [a_0, b_0] \times [c, d] \to X$ by:

$$f^-(x, y) = f(a_0 + b_0 - x, y)$$

and $f_{[a,b]} : [a, b] \times [c, d] \to X$ by:

$$f_{[a,b]}(x, y) = f(a_0 + (b_0 - a_0)(x - a)/(b - a), y).$$

We follow the notation in [1] for the space $SC(Z)$ and the projection $p : SC(Z) \to \mathbb{I}^2$. In particular, a point of the subspace $\mathbb{I}^2$ of $SC(Z)$ is denoted by $(x; y)$. The following figure denotes the part $\mathbb{I}^2$ of $SC(Z)$, where the polygonal line $A_1B_1A_2B_2 \cdots AB$ is the piecewise linear version of the topologists sine curve in Figure 1.

Definition 2.1 ([4, Definition 2.1]). *A continuous map $f : \mathbb{I}^2 \to p^{-1}(\mathbb{I} \times \{0\})$ with $f(\partial\mathbb{I}^2) = \{A\}$ is said to be standard, if $f([1/(m+1), 1/m] \times \mathbb{I}) \subset [A, A_m] \cup p^{-1}(\{A_m\})$, and $f(\partial([1/(m+1), 1/m] \times \mathbb{I})) = \{A\}$ for each $m < \omega$.*

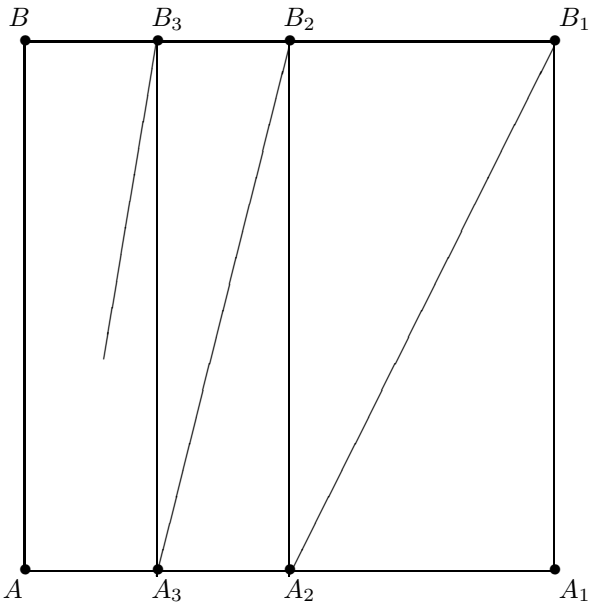


Figure 1

Proof of Theorem 1.1. Recall that $p : SC(Z) \to \mathbb{I}^2$ and $p(z) = (p_1(z); p_2(z))$. Let $f : \mathbb{I}^2 \to SC(Z)$ with $f(\partial\mathbb{I}^2) = A = (0, 0)$. First we claim that f is homotopic to a map f_1 in $p^{-1}(\mathbb{I} \times \{0\})$ relative to $\partial\mathbb{I}^2$.

Since $p^{-1}(\mathbb{I}\times\{0\})$ is a retraction of $SC(Z)\setminus\bigcup\{p^{-1}(\{B_n\}) : n\in\mathbb{N}\}$, it suffices to show that f is homotopic to a map in $SC(Z)\setminus\bigcup\{p^{-1}(\{B_n\}) : n\in\mathbb{N}\}$ relative to $\partial\mathbb{I}^2$. The idea to remove $p^{-1}(\{B_n\})$ from the image of f is basically the same as that in the proof of the simple connectivity of $SC(Z)$. Since the boundary of an open connected subset of the square is complicated in comparison with those of the interval, some more care is needed.

Here we also use the simple connectivity of Z. Using this property we get the simple connectivity of a certain small open subset containing $p^{-1}(\{B_n\})$ in $SC(Z)$, i.e., $p^{-1}(U_n)$ according to the following notation. Let U_n be a square neighborhood of B_n in $\mathbb{I}^2$ which does not contain any B_i for $i\neq n$, and choose a point $u_n\in U_n\cap T$ satisfying $u_n\neq B_n$.

We have finitely many polygonal connected open sets O_i in $\mathbb{I}^2$ such that $(p\circ f)^{-1}(\{B_n\})\subseteq\bigcup_i O_i\subseteq(p\circ f)^{-1}(U_n)$, where O_i may fail to be simply connected. Observe that $p^{-1}(U_n)$ is homotopy equivalent to Z and so it is simply connected. Hence, working in each O_i, we have $g_1:\mathbb{I}^2\to SC(Z)$ satisfying:

(1) g_1 is homotopic to f relative to $\mathbb{I}^2\setminus O_i$,
(2) there exist finitely many simply connected polygonal pairwise disjoint subsets P_{ij} of O_i such that $(p\circ g_1)^{-1}(\{B_n\}))\subseteq\bigcup_j P_{ij}$,
(3) $g(\partial P_{ij})=\{u_n\}$.

We remark that the range of the homotopy between f and g_1 is contained in $p^{-1}(U_n)$. Since $p^{-1}(\{u_n\})$ is a strong deformation retract of $p^{-1}(U_n)$, we have $g_2:\mathbb{I}^2\to SC(Z)$ such that $g_2(\bigcup_j P_{ij})\subseteq p^{-1}(\{u_n\})\cup\mathbb{I}\times\mathbb{I}$, g_2 is homotopic g_1 relative to $\mathbb{I}^2\setminus\bigcup_j P_{ij}$, and the range of the homotopy is contained in $p^{-1}(U_n)$.

Observe that there are only finitely many O_i for each B_n and that $p^{-1}(U_n)$ converge to B. Working on each B_n successively, we obtain maps homotopic to f. Let f_0 be the limit of these maps.

Since $f(O_i)\subseteq p^{-1}(U_n)$ and $p^{-1}(U_n)$ converge to B, f_0 is continuous and f_0 is homotopic to f relative to $\partial\mathbb{I}^2$ and also $f_0(\mathbb{I}^2)$ does not intersect with any $p^{-1}(\{B_n\})$, as desired. Hence we have $f_1:\mathbb{I}^2\to p^{-1}(\mathbb{I}\times\{0\})$ which is homotopic to f relative to $\partial\mathbb{I}^2$.

The next procedure is similar to the proof of [4, Lemma 2.2], which is rather long but each step is simple. Using the commutativity of π_2 and the simple connectivity of Z again and also using the fact that $p^{-1}(\mathbb{I}\times\{0\})$ is locally strongly contractible at points $(x;0)$ with $(x;0)\notin\{A_n : n\in\mathbb{N}\}\cup\{A\}$, we get a standard map $f_2:\mathbb{I}^2\to p^{-1}(\mathbb{I}\times\{0\})$ which is homotopic to f_1 relative to $\partial\mathbb{I}^2$.

The proof that f_2 is null-homotopic is the 2-dimensional version of procedures II and III in the proof of the simple connectivity of $SC(Z)$ ([1, Theorem 1.1]). We outline these procedures. We concentrate on description of the null homotopy of $f_2\,|\,[1/(k+1),1/k]\times\mathbb{I}$.

Fix $k \in \mathbb{N}$. For $m \in \mathbb{N}$, define $h_m : [1/(k+1), 1/k] \times \mathbb{I} \to SC(Z)$ by:

$$h_m(x,y) = \begin{cases} (ku/(k+m-1); 0) & \text{if } f_2(x,y) = (u;0) \\ (A_{k+m-1}, z) & \text{if } f_2(x,y) = (A_k, z). \end{cases}$$

Next define $g_{k,m} : [1/(k+1) + 1/((m+1)k(k+1)), 1/(k+1) + 1/(mk(k+1))] \times \mathbb{I} \to SC(Z)$ by:

$$\begin{aligned} g_{k,2m-1} &= (h_m)_{[1/(k+1)+1/(2mk(k+1)),1/(k+1)+1/((2m-1)k(k+1))]}, \\ g_{k,2m} &= (h^-_{m+1})_{[1/(k+1)+1/((2m+1)k(k+1)),1/(k+1)+1/(2mk(k+1))]}. \end{aligned}$$

Let $g_k : [1/(k+1), 1/k] \times \mathbb{I} \to SC(Z)$ be the unique continuous extension of

$$\bigcup_{m \in \mathbb{N}} g_{k,m},$$

i.e., $g_k(1/(k+1), y) = A$.

Since the images of $g_{k,m}$ converge to A and g_{2m+1} is the homotopy inverse of g_{2m} in $p^{-1}(\mathbb{I} \times \{0\})$ for each m, g_k is continuous and is homotopic relative to the boundary to the restriction $f_2 \,|\, [1/(k+1), 1/k] \times \mathbb{I}$, and the homotopy can be taken in $p^{-1}([A, A_k])$. Hence f_2 is homotopic relative to the boundary to $g : \mathbb{I} \times \mathbb{I} \to SC(Z)$ which is the unique continuous extension of $\bigcup\{g_k : k \in \mathbb{N}\}$, i.e., $g(0, y) = A$.

For the next step we do not care for the boundary for a while. We push up the ranges of $g_{k,2m-1}$ along $A_{k+m}B_{k+m}$ for $m \geq 0$ and $g^-_{k,2m}$ along $A_{k+m}B_{k+m-1}$ so that the y-coordinates of $p(u)$ for u in each of the ranges are the same. Then the resulting map is defined in $p^{-1}(\mathbb{I} \times \{1\})$ and we couple $g_{k,1}$ and $g_{k,2}$, and generally $g_{k,2m-1}$ and $g_{k,2m}$. Since these homotopies of couplings converge to B, we see that the resulting map is null-homotopic. We can perform these procedures uniformly in k, and we have a homotopy from f_2 to the constant map B.

In order to obtain the desired homotopy to the constant A relative to the boundary, we can modify the homotopy above to the desired one, because we have homotopies in the pushing up procedure above, so that the y-coordinates of $p(u)$ for u in the ranges are the same, even uniformly in k. □

3. Correction of the proof of Lemma 3.2 of [2]

In our earlier paper ([2, Lemma 3.2]) we used the following auxiliary result:

Lemma 3.1. *Let p_0, p_1, p^* be distinct points in a Hausdorff space X and let f be a loop in X with the base point p_1 such that $f^{-1}(\{p_0\}) = \emptyset$ and $f^{-1}(\{p^*\})$ is a singleton.*

If f is null-homotopic relative to end points, then there exists a loop f' in X with the base point p_1 in $X \setminus \{p_0, p^*\}$ such that f and f' are homotopic relative to end points in $X \setminus \{p_0\}$.

We use this opportunity to correct our original proof. The assertion "$G^{-1}(\{p^*, p_0\}) \cap O$ is compact" in [2, p.92 l.5 from the bottom] is wrong. We begin by the following well-known result - see e.g. [5, p.169]:

LEMMA 3.2. *Let X be a compact space and C a closed component of X. Then C is the intersection of clopen sets containing C.* □

PROOF OF LEMMA 3.1. Since f is null-homotopic, we have a homotopy $F : \mathbb{I} \times \mathbb{I} \to X$ from f to the constant mapping to p_1, i.e.

$$F(s,0) = f(s), F(s,1) = F(0,t) = F(1,t) = p_1 \text{ for } s,t \in \mathbb{I}.$$

Let $\{s_0\}$ be the singleton $f^{-1}(\{p^*\})$. Let M_0 be the connectedness component of $F^{-1}(\{p^*\})$ containing $(s_0, 0)$, and O the connectedness component of $\mathbb{I} \times \mathbb{I} \setminus M_0$ containing $\mathbb{I} \times \{1\}$. Define $G : \mathbb{I} \times \mathbb{I} \to \mathcal{P}^*$ by:

$$G(s,t) = \begin{cases} F(s,t) & \text{if } (s,t) \in O, \\ p^* & \text{otherwise.} \end{cases}$$

Then G is also a homotopy from f to the constant mapping to p_1 and $G^{-1}(\{p_0\})$ is contained in O (observe that $\partial(\mathbb{I} \times \mathbb{I}) \setminus \{(s_0, 0)\} \subseteq O$).

Put $M_1 = (\mathbb{I} \times \mathbb{I} \setminus O) \cup \mathbb{I} \times \{0\}$. Since $G^{-1}(\{p_0\}) \cap O$ is compact and disjoint from M_1, we have a polygonal neighborhood U of M_1 whose closure is disjoint from $G^{-1}(\{p_0\})$ and also $\mathbb{I} \times \{1\}$. The boundary of U is a piecewise linear arc connecting a point in $\{0\} \times (0,1)$ and a point in $\{1\} \times (0,1)$. We want to get a piecewise linear injective path $g : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ such that

$$\mathrm{Im}(G \circ g) \subseteq X \setminus \{p_0, p^*\},\ g(0) \in \{0\} \times \mathbb{I},\ \text{ and } g(1) \in \{1\} \times \mathbb{I}$$

and $\mathrm{Im}(g)$ divides $\mathbb{I} \times \mathbb{I}$ into two components, one of which contains $G^{-1}(\{p_0\})$ and the other contains M_1.

If the boundary of U is disjoint from $G^{-1}(\{p^*\})$, then we have such a path g tracing the boundary. Otherwise, let C_0 be the intersection of the boundary of U and $G^{-1}(\{p^*\})$. Since M_0 is contained in a connected component of $G^{-1}(\{p^*\})$ which is disjoint with C_0, we have clopen sets C_1 and C_2 in $G^{-1}(\{p^*\}) \cap \overline{U}$ such that $C_1 \cap C_2 = \emptyset$, $C_0 \subseteq C_1$ and $M_0 \subseteq C_2$ by Lemma 3.2.

Observe that C_1 and C_2 are closed subset of $\overline{U}$. Then we can see that $\{0\} \times (0,1) \cap U$ and $\{1\} \times (0,1) \cap U$ belong to the same component of $U \setminus G^{-1}(\{p^*\})$. We have a piecewise linear arc between a point in $\{0\} \times (0,1)$ and a point in $\{1\} \times (0,1)$ in U which does not intersect with $G^{-1}(\{p^*\})$ and have a path g with the required properties. We now see that $G \circ g$ is the desired loop f'. □

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