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COMPACTIFICATIONS OF $[0,\infty)$ WITH UNIQUE HYPERSPACE $F_n(X)$

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Abstract. Given a metric continuum X, $F_n(X)$ denotes the hyperspace of nonempty subsets of X with at most n elements. In this paper we show the following result. Suppose that X is a metric compactification of $[0,\infty)$, Y is a continuum and $F_n(X)$ is homemorphic to $F_n(Y)$. Then: (a) if $n \neq 3$, then X is homeomorphic to Y, (b) if $n = 3$ and the remainder of X is an ANR, then X is homeomorphic to Y. The question if the result in (a) is valid for $n = 3$ remains open.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a continuum X, we consider the following hyperspaces of X:

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \text{ and for each } n \in \mathbb{N}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}, \\ F_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\}. \end{aligned}$$

All these hyperspaces are considered with the Hausdorff metric H ([9, Theorem 2.2, p. 11]).

The continuum X is said to have *unique hyperspace* $F_n(X)$ provided that the following implication holds: if Y is a continuum and $F_n(X)$ is homeomorphic to $F_n(Y)$, then X is homeomorphic to Y.

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It is known that if X is either a finite graph or a dendrite (locally connected continuum containing no simple closed curves) with closed set of end points, then X has unique hyperspace $F_n(X)$ (see [1] and [7]).

A lot of work has been done on determining continua X for which some of the hyperspaces 2^X, $C_n(X)$ and $C(X)$ is unique (see, for example [3] and [8]).

The subspace of the real line $[0, \infty)$ is called *the ray*. In this paper we prove:

Theorems 3.1 and 4.1 If X is a metric compactification of the ray and $n \neq 3$, then X has unique hyperspace $F_n(X)$.

Theorem 5.6. If X is a metric compactification of the ray and the remainder of X is an ANR, then X has unique hyperspace $F_3(X)$.

We do not know if in Theorem 5.6 the hypothesis that the remainder of X is an ANR can be removed.

2. Auxiliary Results

An *n-cell* is a space homeomorphic to $[0, 1]^n$. The manifold boundary of an n-cell M is denoted by $\partial(M)$. A *map* is a continuous function. A *simple triod* is a continuum T which is the union of three arcs α_1, α_2 and α_3 and T contains a point p, called *the vertex of* T, such that p is an end point of each α_i and $\alpha_i \cap \alpha_j = \{p\}$, if $i \neq j$. Given a continuum X and a subset C of X, let

$$F_n(C) = \{A \in F_n(X) : A \subset C\}$$

and

$$\mathcal{E}_n(C) = \{A \in F_n(C) : A \text{ has a neighborhood in } F_n(X) \text{ which is an } n\text{-cell}\}.$$

For each $p \in X$ and $\varepsilon > 0$, let $B(\varepsilon, p)$ be the open ε-neighborhood in X around p and let $N(\varepsilon, C) = \bigcup\{B(\varepsilon, x) : x \in C\}$. Given subsets $A_1, \ldots, A_m$ of X let $\langle A_1, \ldots, A_m\rangle_n = \{A \in F_n(X) : A \subset A_1 \cup \cdots \cup A_m$ and $A \cap A_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}\}$. It is easy to prove that if the sets $A_1, \ldots, A_m$ are closed (resp., open) in X, then $\langle A_1, \ldots, A_m\rangle_n$ is closed (resp., open) in $F_n(X)$. If the sets $A_1, \ldots, A_n$ are closed (or open) and pairwise disjoint, then $A_1 \times \cdots \times A_n$ is homeomorphic to $\langle A_1, \ldots, A_n\rangle_n$ by the map that sends $(a_1, \ldots, a_n)$ to $\{a_1, \ldots, a_n\}$.

Given a topological space Z and $n \in \mathbb{N}$, define

$$\Delta_n(Z) = \{z \in Z : z \text{ has a neighborhood } M \text{ in } Z \text{ such that } M \text{ is an } n\text{-cell and } z \in \partial(M)\}.$$

Given a metric compactification X, of the ray $[0, \infty)$, we denote by $S_X \subset X$ the topological copy of $[0, \infty)$, we call 0_X to the end point of S_X and we denote the remainder $X - S_X$ of X by R_X.

The proof of the following lemma can be made with similar arguments as those in Lemmas 4.2, 4.3 and 4.5 of [3].

LEMMA 2.1. *Let X be a metric compactification of the ray and $n \in \mathbb{N}$. Then:*

(a) $F_n(S_X) - F_{n-1}(S_X) \subset \mathcal{E}_n(S_X)$.

(b) *If $A \in F_{n-1}(S_X)$ and $n \geq 4$, then no neighborhood of A in $F_n(X)$ can be embedded in $[0,1]^n$.*

(c) *If $A \in F_n(S_X)$ and $n \geq 4$, then $A \in F_1(X)$ if and only if $A \notin \mathcal{E}_n(S_X)$ and A has a basis of neighborhoods $\mathcal{B}$ in $F_n(X)$ such that $\mathcal{U} \cap \mathcal{E}_n(S_X)$ is arcwise connected for each $\mathcal{U} \in \mathcal{B}$.*

THEOREM 2.2. *Suppose that X is a metric compactification of the ray and $F_n(X)$ is homeomorphic to $F_n(Y)$, where Y is a continuum. Then Y is a metric compactification of the ray.*

PROOF. Define $\mathcal{U} = F_n(S_X) - F_{n-1}(S_X)$. By Lemma 2.1(a), $\mathcal{U} \subset \mathcal{E}_n(S_X)$. We prove some properties of $\mathcal{U}$.

A. $\mathcal{U}$ is a locally arcwise connected open subset of $F_n(X)$.

Let $A = \{x_1, \ldots, x_n\} \in \mathcal{U}$ and let $\mathcal{V}$ be an open subset of $F_n(X)$ such that $A \in \mathcal{V}$. Since $A \subset S_X$, we can choose pairwise disjoint arcs $J_1, \ldots, J_n$ in S_X such that $x_i \in \operatorname{int}_X(J_i)$, for each $i \in \{1, \ldots, n\}$, and $A \in \langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n \subset \mathcal{V}$. Notice that $A \in \langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n \subset \mathcal{U}$. Thus $\mathcal{U}$ is open in $F_n(X)$. Now we prove that $\langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n$ is pathwise connected. Take $B = \{y_1, \ldots, y_n\} \in \langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n$. We may assume that $y_i \in \operatorname{int}_X(J_i)$, for each $i \in \{1, \ldots, n\}$. Given $i \in \{1, \ldots, n\}$, since $\operatorname{int}_X(J_i)$ is homeomorphic to an interval of the real line, there exists a map $\alpha_i : [0,1] \to \operatorname{int}_X(J_i)$ such that $\alpha_i(0) = x_i$ and $\alpha_i(1) = y_i$. So, the function $\alpha : [0,1] \to \langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n$ given by $\alpha(t) = \{\alpha_1(t), \ldots, \alpha_n(t)\}$ is continuous, $\alpha(0) = A$ and $\alpha(1) = B$. We have shown that $\langle \operatorname{int}_X(J_1), \ldots, \operatorname{int}_X(J_n)\rangle_n$ is pathwise connected. This completes the proof of property A.

B. $\mathcal{U}$ is a connected and dense subset of $F_n(X)$.

It is easy to show that any two elements of $\mathcal{U}$ can be joined by an arc inside $\mathcal{U}$. In order to show that $\mathcal{U}$ is dense in $F_n(X)$, take a nonempty open set $\mathcal{V}$ in $F_n(X)$. Then there exists $m \leq n$ and nonempty open subsets $U_1, \ldots, U_m$ of X such that $\langle U_1, \ldots, U_m\rangle_n \subset \mathcal{V}$. Since S_X is dense in X, for each $i \in \{1, \ldots, m\}$ we can choose a point $x_i \in U_i \cap S_X$. Choose points $x_{m+1}, , \ldots, x_n$ in $U_m \cap S_X$ such that the points $x_1, \ldots, x_n$ are pairwise different. Thus $\{x_1, \ldots, x_n\} \in \langle U_1, \ldots, U_m\rangle_n \cap \mathcal{U}$. Hence $\mathcal{U}$ is dense in $F_n(X)$.

Let $h : F_n(X) \to F_n(Y)$ be a homeomorphism. Define $\mathcal{W} = h(\mathcal{U})$. So, $\mathcal{W}$ is a connected, locally arcwise connected, dense open subset of $F_n(Y)$. Define $W = \bigcup \mathcal{W}$. We prove some properties of W.

C. W is a connected, locally arcwise connected, dense open subset of Y.

It is easy to prove that W is open. By [4, Theorem 6.3], W is locally arcwise connected. In order to show that W is dense, let V be a nonempty subset of Y. Then $\langle V\rangle_n$ is a nonempty open subset of $F_n(Y)$. By the density of $\mathcal{W}$, there exists an element $A \in \langle V\rangle_n \cap \mathcal{W}$. Take $x \in A$. Thus $x \in V \cap W$. Therefore W is dense in Y. We show that W is connected. By ([4, Lemma 2.1]), W has at most n components. Since W is open in Y and it has a finite number of components, each component of W is open in Y. Let C be a component of W. Then $D = W - C$ is open in Y. Note that $\mathcal{W} \subset \langle C\rangle_n \cup \langle D, W\rangle_n$, the sets $\langle C\rangle_n$ and $\langle D, W\rangle_n$ are disjoint open subsets of $F_n(Y)$. Since $\langle C\rangle_n$ is a nonempty open subset of $F_n(Y)$ and $\mathcal{W}$ is dense in $F_n(Y)$, $\langle C\rangle_n \cap \mathcal{W} \neq \emptyset$. The connectedness of $\mathcal{W}$ implies that $\mathcal{W} \subset \langle C\rangle_n$. This implies that $W \subset C$. Thus $W = C$. Therefore, W is connected. This completes the proof of property C.

D. W contains no simple triods.

Suppose, to the contrary, that W contains a simple triod T, with vertex p. Then there exists an element $B \in \mathcal{W}$ such that $p \in B$. Let $A \in \mathcal{U}$ be such that $h(A) = B$. Suppose that $B = \{p_1, \ldots, p_m\}$, where $p_1, \ldots, p_m$ are pairwise different, $p_1 = p$ and $m \leq n$. Since $\mathcal{U}$ is open in $F_n(X)$, there exists $\varepsilon > 0$ such that, if $C \in F_n(Y)$ and $H(B, C) < \varepsilon$, then $C \in \mathcal{W}$. Let d_Y be a metric for Y. Since W is connected, dense in Y and locally arcwise connected, we can construct pairwise disjoint arcs $\beta_2, \ldots, \beta_m$ in W such that $p_i \in \beta_i$, for each $i \in \{1, \ldots, m\}$. Shortening T if it were necessary, we can assume that $T \cap (\beta_2 \cup \cdots \cup \beta_m) = \emptyset$ and each one of the sets $T, \beta_2, \ldots, \beta_m$ is of diameter less than ε. Choose pairwise different points $p_{m+1}, \ldots, p_n$ in $T - \{p\}$. Let $T_1 \subset T$ be a simple triod such that $p_1 \in T_1 \subset T - \{p_{m+1}, \ldots, p_n\}$. Choose pairwise disjoint arcs $\beta_{m+1}, \ldots, \beta_n$ in $T - T_1$ such that $p_i \in \beta_i$, for each $i \in \{m+1, \ldots, n\}$. Thus $h^{-1}(\{p_1, \ldots, p_n\}) \in h^{-1}(\langle T_1, \beta_2, \ldots, \beta_n\rangle_n) \subset \mathcal{U}$. Notice that each neighborhood of $\{p_1, \ldots, p_n\}$ in $F_n(Y)$ contains a copy of the space $T_1 \times \beta_2 \times \cdots \times \beta_n$ and the same happens for $h^{-1}(\{p_1, \ldots, p_n\})$ (in $F_n(X)$). Using the Invariance Domain Theorem ([11, Theorem 16, Sec. 7, Ch. 4]), it can be shown that $T_1 \times \beta_2 \times \cdots \times \beta_n$ cannot be embedded in $[0, 1]^n$. This implies that $h^{-1}(\{p_1, \ldots, p_n\}) \notin \mathcal{U}$, a contradiction. Therefore, W contains no simple triods.

E. Y is a compactification of the ray.

First we show that each element p in W has a basis of neighborhoods $\mathcal{D}$ in W such that each element of $\mathcal{D}$ is an arc. Let V be an open subset of W such that $p \in V$. By property C there exists an arc α in W such that $p \in \alpha$. In the case that there exists an arc $\beta \subset V$, with end points a and b such that $p \in \beta - \{a, b\}$, by property C, there exists an arcwise connected neighborhood Z of p in W such that $p \in Z \subset V - \{a, b\}$. Given a point $z \in Z - \{p\}$, by property D, each arc in Z connecting z and p is contained in β. Thus $Z \subset \beta$. Thus, β is a neighborhood of p. Now, suppose that there are no arcs $\beta \subset V$, with end points a and b such that $p \in \beta - \{a, b\}$. We may assume that $\alpha \subset V$. In this case p is an end point of α. Let q be the other end point of α. By property C, there exists an arcwise connected neighborhood R of p in W such that $p \in R \subset V - \{q\}$. Given a point $r \in R - \{p\}$, by property D, each arc in

R connecting r and p is contained in α. Thus $R \subset \alpha$. This ends the proof of the claim. So, we have proved that W is a connected 1-dimensional manifold. By the Theorem of Classification of 1-dimensional manifolds ([6, Appendix 2, p. 208]), W is homeomorphic to one of the following spaces: $[0,1]$, the unitary circle S^1 in $\mathbb{R}^2$, $[0,\infty)$ or $\mathbb{R}$.

In the case that W is compact, we obtain $W = Y$. If W is homeomorphic to S^1, by [3, Corollary 5.8], X is also homeomorphic to S^1, a contradiction. If W is homeomorphic to $[0,1]$, then $Y = W$ is a compactification of the ray and we are done. If W is homemorphic to $[0,\infty)$, then Y is a compactification of the ray and we are done. Thus we suppose that W is homeomorphic to $\mathbb{R}$. We identify W with the interval $(-\infty,\infty)$. Let $R = \mathrm{cl}_Y([0,\infty)) - [0,\infty)$ and $L = \mathrm{cl}_Y((-\infty,0]) - (-\infty,0]$. Then R and L are nonempty and compact and $Y = L \cup (-\infty,\infty) \cup R$. In the case that L is degenerate and $L \cap R = \emptyset$, we have that $L \cup (-\infty,\infty)$ is open in Y, Y is a compactification of this set and this set is homeomorphic to $[0,\infty)$. Thus, in this case, we are done. In the case that both sets R and L are degenerate, Y is homeomorphic either to $[0,1]$ or to S^1. Therefore, we may assume that either both sets L and R are nondegenerate or one of them is nondegenerate and $L \cap R \neq \emptyset$. In both cases W coincides with the set of points of local connectedness of Y. We are going to obtain a contradiction. We analyze three cases.

CASE 1. $n \geq 4$.

Fix an element $A \in F_n(S_X)$ such that A contains exactly n elements and $0_X \in A$. Let $A = \{p_1, \ldots, p_n\}$, where $p_1 = 0_X$. Choose pairwise disjoint subarcs $\alpha_1, \ldots, \alpha_n$ of S_X such that $p_i \in \mathrm{int}_X(\alpha_i) \subset S_X$, for each $i \in \{1, \ldots, n\}$. Notice that p_1 is an end point of α_1, $\langle \alpha_1, \ldots, \alpha_n \rangle_n$ is a neighborhood of A in $F_n(X)$, $\langle \alpha_1, \ldots, \alpha_n \rangle_n$ is an n-cell (it is homeomorphic to $\alpha_1 \times \cdots \times \alpha_n$) and $A \in \partial(\langle \alpha_1, \ldots, \alpha_n \rangle_n)$. Since $A \in \mathcal{U} \subset \mathcal{E}_n(X)$ and h is a homeomorphism, $h(A) \in \mathcal{E}_n(Y)$. By definition $h(A) \subset W$.

Then there exists an arc β in W such that $h(A) \subset \mathrm{int}_Y(\beta) \subset W$. If $h(A) \in F_{n-1}(Y)$, by the arguments given in [3, Lemma 4.3], no neighborhood of $h(A)$ in $F_n(Y)$ can be embedded in $\mathbb{R}^n$, this is a contradiction with the fact that $h(A) \in \mathcal{E}_n(Y)$. Therefore, $h(A)$ contains exactly n elements $q_1, \ldots, q_n$. Since W is open in Y and h is a homeomorphism, there are pairwise disjoint arcs $\gamma_1, \ldots, \gamma_n$ in W such that, for each $i \in \{1, \ldots, n\}$, $q_i \in \mathrm{int}_Y(\gamma_i) \subset W$ and $\langle \gamma_1, \ldots, \gamma_n \rangle_n \subset h(\langle \alpha_1, \ldots, \alpha_n \rangle_n)$. Notice that q_i is not an end point of γ_i, for each $i \in \{1, \ldots, n\}$, $\langle \gamma_1, \ldots, \gamma_n \rangle_n$ is an n-cell containing $h(A)$ and $h(A) \notin \partial(\langle \gamma_1, \ldots, \gamma_n \rangle_n)$. Thus $A \in h^{-1}(\langle \gamma_1, \ldots, \gamma_n \rangle_n - \partial(\langle \gamma_1, \ldots, \gamma_n \rangle_n)) \subset \langle \alpha_1, \ldots, \alpha_n \rangle_n$. This contradicts the Invariance Domain Theorem ([11, Theorem 16, Sec. 7, Ch. 4]) and completes the analysis for this case.

CASE 2. $n = 3$.

It is known (see [2, pp. 264 and 265]) that $F_3([0,1])$ is a 3-cell and $\partial(F_3([0,1])) = \{A \in F_3([0,1]) : A \cap \{0,1\} \neq \emptyset\}$. Given an element $B \in F_3(W)$, there exists an arc β in W such that $B \subset \mathrm{int}_Y(\beta) \subset W$ and B does not contain any of the end points of β. Then $F_3(\beta)$ is a 3-cell, $F_3(\beta)$ is a neighborhood of B in $F_3(Y)$ and $B \in F_3(\beta) - \partial(F_3(\beta))$. This implies that B has a basis of neighborhoods $\mathcal{B}$ in $F_3(Y)$ such that, for each $\mathcal{R} \in \mathcal{B}$, $\mathcal{R}$ is a 3-cell and $B \notin \partial(\mathcal{R})$. Proceeding as in the first paragraph of the previous case, there exists an element $A \in \mathcal{U}$ such that there exists a 3-cell $\mathcal{S}$ that is a neighborhood of A in $F_3(X)$ and $A \in \partial(\mathcal{S})$. Making $B = h(A)$ we obtain a contradiction with the Invariance Domain Theorem ([11, Theorem 16, Sec. 7, Ch. 4]).

CASE 3. $n = 2$.

First we show that $h(F_2(S_X)) = F_2(W)$. By [4, Theorem 6.3], the set of elements of local connectedness of $F_2(X)$ is $F_2(S_X)$ and the set of elements of local connectedness of $F_2(Y)$ is $F_2(W)$. Thus $F_2(W) = h(F_2(S_X))$. It is easy to see that $\Delta_2(F_2(S_X)) = F_1(S_X) \cup \langle\{0_X\}, S_X\rangle_2$ and $\Delta_2(F_2(W)) = F_1(W)$. Note that $\Delta_2(F_2(W)) = h(\Delta_2(F_2(S_X)))$. Hence, we may assume that $h(\{0_X\}) = \{0\}$, $h(F_1(S_X)) = F_1((-\infty, 0])$ and $h(\langle\{0_X\}, S_X\rangle_2) = F_1([0, \infty))$. Note that $\mathrm{cl}_{F_2(X)}(\langle\{0_X\}, S_X\rangle_2) - \langle\{0_X\}, S_X\rangle_2 = \langle\{0_X\}, R_X\rangle_2$. Then $h(F_1(R_X)) = h(\mathrm{cl}_{F_2(X)}(F_1(S_X)) - F_1(S_X)) = \mathrm{cl}_{F_2(Y)}(F_1((-\infty, 0])) - F_1((-\infty, 0]) = F_1(L)$ and $h(\langle\{0_X\}, R_X\rangle_2) = h(\mathrm{cl}_{F_2(X)}(\langle\{0_X\}, S_X\rangle_2) - \langle\{0_X\}, S_X\rangle_2) = F_1(R)$. Since $F_1(R_X)$ and $\langle\{0_X\}, R_X\rangle_2$ are disjoint and homeomorphic, L and R are disjoint and homeomorphic (and nondegenerate).

Fix a point $p \in R_X$ and a sequence of different points $\{p_k\}_{k=1}^{\infty}$ in S_X such that $\lim p_k = p$ and $p_1 = 0_X$. For each $k \in \mathbb{N}$, let L_k be the unique arc in S_X that joins 0_X and p_k.

We claim that, for each $k \in \mathbb{N}$, $h(\{p, p_k\}) \subset R$. In order to do this, it is enough to show that, for each $x \in L_k$, $h(\{p, x\}) \subset R$. Note that $h(\{p, 0_X\}) \in h(\langle\{0_X\}, R_X\rangle_2) \subset F_1(R)$. Let $h(\{p, 0_X\}) = \{y_0\}$. Consider the arc in $F_2(Y)$, $\mathcal{L} = \{h(\{p, x\}) : x \in L_k\}$. By [5, Lemma 2.2] and [4, Lemma 2.1], the set $G = \bigcup\{K : K \in \mathcal{L}\}$ is a locally connected subcontinuum of Y. Since $y_0 \in G \cap R$ and G is arcwise connected, we obtain that $G \subset R$. Thus $h(\{p, p_k\}) \subset R$.

Hence $h(\{p\}) = \lim h(\{p, p_k\}) \subset R$. On the other hand $h(\{p\}) \in F_1(L)$. This is a contradiction since three paragraphs above we obtained that $L \cap R = \emptyset$.

With this, we finish the proof of property E. So the theorem is proved. ∎

3. THE CASE $n = 2$

THEOREM 3.1. *If X is a metric compactification of the ray, then X has unique hyperspace $F_2(X)$.*

PROOF. Let $X = R_X \cup S_X$ be a compactification of the ray and let Y be a continuum such that $F_2(X)$ is homeomorphic to $F_2(Y)$. By Theorem 2.2, Y is a compactification of the ray. Since $[0, 1]$ has unique hyperspace

$F_2([0,1])$, we suppose that X (and Y) is not an arc. Thus R_X and R_Y are nondegenerate continua.

The following facts are easy to show.

A. The set of points of local connectedness of X is S_X.

B. The set of elements of local connectedness of $F_2(X)$ is $F_2(S_X)$ (see [4, Lemma 6.3]).

C. $\Delta_2(F_2(X)) = F_1(S_X) \cup \langle\{0_X\}, S_X\rangle_2$.

D. $\mathrm{cl}_{F_2(X)}(\Delta_2(F_2(X))) - \Delta_2(F_2(X)) = F_1(R_X) \cup \langle\{0_X\}, R_X\rangle_2$, the sets $F_1(R_X)$ and $\langle\{0_X\}, R_X\rangle_2$ are disjoint and they are homeomorphic to R_X.

E. $\mathrm{cl}_{F_2(X)}(F_1(S_X)) = F_1(S_X) \cup F_1(R_X)$ is homeomorphic to X.

F. $\mathrm{cl}_{F_2(X)}(\langle\{0_X\}, S_X\rangle_2) = \langle\{0_X\}, S_X\rangle_2 \cup \langle\{0_X\}, R_X\rangle_2$ is homeomorphic to X.

G. $\mathrm{cl}_{F_2(X)}(\Delta_2(F_2(X)))$ is a compactification of the real line $(-\infty,\infty)$ and, if we identify $\Delta_2(F_2(X))$ with the line $(-\infty,\infty)$, then each one of the spaces $\mathrm{cl}_{F_2(X)}((-\infty,0])$ and $\mathrm{cl}_{F_2(X)}([0,\infty))$ is homeomorphic to X.

Let $h : F_2(X) \to F_2(Y)$ be a homeomorphism. Then

$$h(cl_{F_2(X)}(\Delta_2(F_2(X)))) = cl_{F_2(Y)}(\Delta_2(F_2(Y))).$$

Since Y and $\Delta_2(F_2(Y))$ satisfy the corresponding properties A-G, we conclude that X and Y are homeomorphic. □

4. The case $n \geq 4$

Theorem 4.1. *If X is a metric compactification of the ray and $n \geq 4$, then X has unique hyperspace $F_n(X)$.*

Proof. Let $X = R_X \cup S_X$ be a compactification of the ray and let Y be a continuum such that $F_n(X)$ is homeomorphic to $F_n(Y)$. By Theorem 2.2, Y is a compactification of the ray. Since $[0,1]$ has unique hyperspace $F_n([0,1])$ ([3, Corollary 5.8]), we suppose that R_X and R_Y are nondegenerate continua.

Let $h : F_n(X) \to F_n(Y)$ be a homeomorphism. Since the set of elements of local connectedness of $F_n(X)$ (resp., $F_n(Y)$) is $F_n(S_X)$ (resp., $F_n(S_Y)$), we have that $h(F_n(S_X)) = F_n(S_Y)$. This implies that $h(\mathcal{E}_n(F_n(S_X))) = \mathcal{E}_n(F_n(S_Y))$. By Lemma 2.1(c), $h(F_1(S_X)) = F_1(S_Y)$. So, $F_1(X) = \mathrm{cl}_{F_n(X)}(F_1(S_X))$ is homeomorphic to $F_1(Y) = \mathrm{cl}_{F_n(Y)}(F_1(S_Y))$ and X is homeomorphic to Y. □

5. The case $n = 3$

Given a topological space Z, define

$$LC(Z) = \{z \in Z : Z \text{ is locally connected at } z\}$$

and

$$N(Z) = \mathrm{Cl}_{F_3(Z)}(\Delta_3(F_3(Z))) - \Delta_3(F_3(Z)).$$

Given a subset A of Z and $p \in Z$, we say that p is *arcwise accessible* from A provided that $p \notin A$ and there exists an arc α in Z such that $p \in \alpha$ and $\alpha - \{p\} \subset A$.

A subcontinuum A of a continuum Z is said to be *terminal* provided that for each subcontinuum B of Z satisfying $B \cap A \neq \emptyset$ we have that $A \subset B$ or $B \subset A$. It is easy to prove that, if $Z = R_Z \cup S_Z$ is a compactification of the ray, then R_Z is a terminal subcontinuum of Z.

In [2, pp. 264 and 265] it is shown that $[0,1]^3$ is a model for $F_3([0,1])$ ($[0,1]^3$ is homeomorphic to $F_3([0,1])$). In the following lemma we show models for some subsets of $F_3([0,1])$.

LEMMA 5.1. (a) *$F_3([0,1))$ is homeomorphic to $[0,1) \times [0,1]^2$.*

(b) *$\Delta_3(F_3([0,1))) = \{A \in F_3([0,1)) : 0 \in A\} = \langle\{0\},[0,1)\rangle_3$ and $\Delta_3(F_3([0,1)))$ is homeomorphic to an open disc in the Euclidean plane.*

PROOF. Let R be the solid triangle in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$ and $(\frac{1}{2},1,0)$. Let S be $R -$ (convex segment in $\mathbb{R}^3$ joining the points $(1,0,0)$ and $(\frac{1}{2},0,0)$). Let $T =$ (convex segment in $\mathbb{R}^3$ joining $(0,0,0)$ and $(\frac{1}{2},1,0)) - \{(\frac{1}{2},1,0)\}$. Define $g : F_3([0,1]) \to R$ be given by $g(A) = (\frac{\max(A)+\min(A)}{2}, \max(A) - \min(A), 0)$. It is easy to prove that g is continuous, $g(\{A \in F_3([0,1]) : 0 \in A \text{ and } 1 \notin A\}) = T$ and $g(\{A \in F_3([0,1]) : 1 \notin A\}) = S$. Let $\mathcal{R}$ and $\mathcal{S}$ be the solids of revolution obtained by rotating the triangles R and S, respectively, around the x-axis. Define $f : F_3([0,1]) \to \mathcal{R}$ as follows, given $A = \{p,q,r\} \in F_3([0,1])$, with $p \leq q \leq r$, define $f(A) = (\frac{p+r}{2}, (r-p)\cos(2\pi(\frac{q-p}{r-p})), (r-p)\sin(2\pi(\frac{q-p}{r-p})))$, if $p < r$, and $f(A) = (p,0,0)$, if $p = r$. It is easy to prove that f is a homeomorphism and $f(F_3([0,1))) = \mathcal{S}$. So $F_3([0,1))$ is homeomorphic to $[0,1) \times [0,1]^2$. Moreover, $f(\Delta_3(F_3([0,1)))) = \Delta_3(\mathcal{S})$. Notice that $\Delta_3(\mathcal{S})$ is the surface of revolution in $\mathbb{R}^3$ obtained by rotating the set T around the x-axis. Thus $\Delta_3(F_3([0,1)))$ is homeomorphic to an open disc in the Euclidean plane and $\Delta_3(F_3([0,1))) = \{A \in F_3([0,1)) : 0 \in A\} = \langle\{0\},[0,1)\rangle_3$. □

LEMMA 5.2. *Let $Z = S_Z \cup R_Z$ be a compactification of the ray with nondegenerate remainder. Then*

(a) *$LC(F_3(Z)) = F_3(S_Z)$,*
(b) *$\Delta_3(F_3(Z)) = \langle\{0_Z\},S_Z\rangle_3$,*
(c) *$\mathrm{Cl}_{F_3(Z)}(F_3(S_Z)) = F_3(Z)$,*
(d) *$N(Z) = \langle\{0_Z\},R_Z,Z\rangle_3$.*

PROOF. Since $LC(Z) = S_Z$, by [4, Lemma 6.3], we obtain that $LC(F_3(Z)) = F_3(S_Z)$. Let $A \in F_3(Z)$ be such that $A \cap R_Z \neq \emptyset$. Notice that each small neighborhood of A in $F_3(Z)$ is disconnected. This implies $\Delta_3(F_3(Z)) \subset F_3(S_Z)$. Thus $\Delta_3(F_3(Z)) = \Delta_3(F_3(S_Z))$. By Lemma 5.1(b), $\Delta_3(F_3(S_Z)) = \langle\{0_Z\},S_Z\rangle_3$. Therefore, $\Delta_3(F_3(Z)) = \langle\{0_Z\},S_Z\rangle_3$. Property

(c) is immediate from the density of S_Z in Z and property (d) follows from (b). ◻

LEMMA 5.3. *Let $Z = S_Z \cup R_Z$ be a compactification of the ray where R_Z is nondegenerate. Then*

(a) $LC(N(Z)) = \{\{p, q, 0_Z\} \in F_3(Z) : q \in S_Z - \{0_Z\},\ p \in R_Z$ *and* R_Z *is locally connected at* $p\}$.

(b) *An element $A \in F_3(Z)$ is in the set of elements in $N(Z)$ that are arcwise accessible from $LC(N(Z))$ if and only if A is of one of the following two forms: (1) $A = \{p, 0_Z\}$, where $p \in R_Z$ and either R_Z is locally connected at p or p is arcwise accessible from $LC(R_Z)$ or (2) $A = \{p, q, 0_Z\}$, where $p \in R_Z$, $q \in S_Z - \{0_Z\}$ and p is arcwise accessible from $LC(R_Z)$.*

(c) *$N(Z)$ is arcwise connected if and only if R_Z is arcwise connected.*

PROOF. (a) Let $A \in LC(N(Z)) \subset \langle\{0_Z\}, R_Z, Z\rangle_3$ (Lemma 5.2(d)). Let $A = \{p, q, 0_Z\}$, where $p \in R_Z$ and $q \neq 0_Z$ could be equal to p. First we show that $q \notin R_Z$. Suppose to the contrary that $q \in R_Z$.

Let $0 < \varepsilon < \frac{\text{diameter}(R_Z)}{2}$ be such that $B(2\varepsilon, 0_Z) \cap R_Z = \emptyset$ and, in the case that $p \neq q$, $B(\varepsilon, p) \cap B(\varepsilon, q) = \emptyset$. Let $\mathcal{U}$ be a connected open subset of $N(Z)$ such that $A \in \mathcal{U}$ and $H(A, B) < \frac{\varepsilon}{2}$ for each $B \in \mathcal{U}$. Let $C = \bigcup\{B : B \in \text{cl}_{F_3(Z)}(\mathcal{U})\}$. Since $A \in \mathcal{U}$, by [4, Lemma 2.1] C is compact and it has at most three components (C has at most two components when $p = q$). For each $B \in \text{cl}_{F_3(Z)}(\mathcal{U})$, $H(A, B) < \varepsilon$. Thus $C \subset N(\varepsilon, A) = B(\varepsilon, p) \cup B(\varepsilon, q) \cup B(\varepsilon, 0_Z)$. Since $B(\varepsilon, p) \cup B(\varepsilon, q)$ and $B(\varepsilon, 0_Z)$ are disjoint, we have that the sets $C_1 = C \cap B(\varepsilon, p)$, $C_2 = C \cap B(\varepsilon, q)$ and $C_3 = C \cap B(\varepsilon, 0_Z)$ are the components of C and they are subcontinua of Z ($C_1 = C_2$, if $p = q$). Notice that $p \in C_1$ and diameter$(C_1) \leq 2\varepsilon$, so $R_Z \nsubseteq C_1$. Since $\mathcal{U}$ is open in $N(Z)$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and, if $B \in N(Z)$ and $H(A, B) < \delta$, then $B \in \mathcal{U}$. By the density of S_Z in Z, we can take an element $x \in B(\delta, p) \cap S_Z$. Then the set $B = \{q, x, 0_Z\} \in \mathcal{U}$, so $x \in C_1$. Thus $C_1 \cap R_Z \neq \emptyset$, $R_Z \nsubseteq C_1$ and $C_1 \nsubseteq R_Z$. This contradicts the fact that R_Z is terminal in Z and proves that $q \notin R_Z$. Thus $q \in S_Z - \{0_Z\}$.

Now we check that R_Z is locally connected at p. Let $\varepsilon > 0$ be such that the sets $B(\varepsilon, p)$, $B(\varepsilon, q)$ and $B(\varepsilon, 0_Z)$ are pairwise disjoint and $R_Z \cap (B(\varepsilon, q) \cup B(\varepsilon, 0_Z)) = \emptyset$. Let $\mathcal{U}$ be a connected open subset of $N(Z)$ such that $A \in \mathcal{U}$ and $H(A, B) < \varepsilon$ for each $B \in \mathcal{U}$. Let $U = \bigcup\{D : D \in \mathcal{U}\}$. By [4, Lemma 2.1], U has at most three components, so the components of U are $U \cap B(\varepsilon, p)$, $U \cap B(\varepsilon, q)$ and $U \cap B(\varepsilon, 0_Z)$. Let $z \in U \cap B(\varepsilon, p)$. Let $D \in \mathcal{U} \subset N(Z) = \langle\{0_Z\}, R_Z, Z\rangle_3$ be such that $z \in D$. Then $0_Z \in D$. Notice that $\mathcal{U} \subset \langle B(\varepsilon, p), B(\varepsilon, q), B(\varepsilon, 0_Z)\rangle_3$. Thus there exists a point $w \in D \cap B(\varepsilon, q) \subset D - (R_Z \cup \{0_Z\})$. Since $D \in \langle\{0_Z\}, R_Z, Z\rangle_3$, we have that $z \in R_Z$. Since $\mathcal{U}$ is open in $N(Z)$, there exists $\delta > 0$ such that $\delta < \varepsilon$, $B(\delta, z) \subset B(\varepsilon, p)$ and, if $B \in N(Z)$ and $H(D, B) < \delta$, then $B \in \mathcal{U}$. Given a point $x \in R_Z \cap B(\delta, z)$, the set $B = \{x, w, 0_Z\}$ belongs to $N(Z)$ and $H(B, D) < \delta$, so $B \in \mathcal{U}$ and $x \in U \cap B(\varepsilon, p)$. This shows that $R_Z \cap B(\delta, z) \subset U \cap B(\varepsilon, p)$. We have shown that $U \cap B(\varepsilon, p)$ is a connected open subset of R_Z containing p. This proves that R_Z is locally connected at p.

In order to prove the opposite inclusion in (a), let $A = \{p, q, 0_Z\}$, where $q \in S_Z - \{0_Z\}$, $p \in R_Z$ and R_Z is locally connected at p. Let $\varepsilon > 0$ be such that $B(\varepsilon, p)$, $B(\varepsilon, q)$ and $B(\varepsilon, 0_Z)$ are pairwise disjoint. Let U and V be open connected subsets of R_Z and $S_Z - \{0_Z\}$, respectively, such that $p \in U \subset B(\varepsilon, p)$ and $q \in V \subset B(\varepsilon, q)$. Let $\mathcal{U} = \langle U, V, \{0_Z\}\rangle_3$. Clearly, $\mathcal{U}$ is a connected subset of $N(Z)$, $A \in \mathcal{U}$ and $H(A, B) < \varepsilon$ for each $B \in \mathcal{U}$. In order to show that $\mathcal{U}$ is open in $N(Z)$. Let $B = \{x, y, 0_Z\} \in \mathcal{U}$, where $x \in U$ and $y \in V$. Let $\delta > 0$ be such that $\delta < \varepsilon$, $B(\delta, x) \subset B(\varepsilon, p)$, $B(\delta, y) \subset B(\varepsilon, q)$, $B(\delta, x) \cap R_Z \subset U$ and $B(\delta, y) \subset V$ ($S_Z - \{0_Z\}$ is open in Z). Let $C \in N(Z)$ be such that $H(B, C) < \delta$. Then $0_Z \in C$ and there exist points $u, v \in C$ such that $u \in B(\delta, x)$ and $v \in B(\delta, y)$. Since $C \in N(Z)$, $C \cap R_Z \neq \emptyset$. Notice that $v \in S_Z - \{0_Z\}$, so $u \in R_Z$ and $u \in U$. Hence $C \in \mathcal{U}$. This completes the proof that $\mathcal{U}$ is open in $N(Z)$. Therefore $N(Z)$ is locally connected at A. We have proved (a).

(b) Let $A \in N(Z)$ be such that A is arcwise accessible from $LC(N(Z)) \subset \langle\{0_Z\}, R_Z, S_Z - \{0_Z\}\rangle_3 \subset \langle\{0_Z\}, R_Z, Z\rangle_3$. Since $\langle\{0_Z\}, R_Z, Z\rangle_3$ is closed in $F_3(Z)$, $A \in \langle\{0_Z\}, R_Z, Z\rangle_3$. Let $\alpha : [0, 1] \to F_3(Z)$ be a one-to-one map such that $\alpha(1) = A$ and $\alpha([0, 1)) \subset LC(N(Z))$. First, we show that $A \cap R_Z$ is a one-point set. Suppose to the contrary that $A = \{0_Z, x, y\}$, where $x \neq y$ and $x, y \in R_Z$. Let U, V be open subsets of Z such that $x \in U$, $y \in V$, $\mathrm{cl}_Z(U) \cap \mathrm{cl}_Z(V) = \emptyset$ and $0_Z \notin \mathrm{cl}_Z(U) \cup \mathrm{cl}_Z(V)$. Since $\alpha(1) = A$, there exists $t_1 < 1$ such that $\alpha([t_1, 1)) \subset \langle U, V, S_Z\rangle_3 \cap LC(N(Z)) \subset \langle U, V, \{0_Z\}\rangle_3$. Since $\alpha(t_1) \in LC(N(Z))$, we may assume that $\alpha(t_1) = \{0_Z, p_1, q_1\}$, where $p_1 \in R_Z \cap U$ and $q_1 \in (S_Z - \{0_Z\}) \cap V$. Let $E = \bigcup\{\alpha(s) : s \in [t_1, 1]\}$. Then $E \in \langle U, V, \{0_Z\}\rangle_3$ and, by [4, Lemma 2.1], E is closed and it has at most three components. Thus the components of E are $E \cap U$, $E \cap V$ and $\{0_Z\}$. So, $E \cap V$ is a subcontinuum of Z with the following properties: $E \cap V \cap R_Z \neq \emptyset$, $E \cap V \cap S_Z \neq \emptyset$ and $R_Z \not\subset E \cap V$. This contradicts the fact that R_Z is terminal in Z. Therefore, $A \cap R_Z$ is a one-point set. Suppose that $A \cap R_Z = \{p\}$. We analyze two cases.

CASE 1. $A = \{p, 0_Z\}$.

Let $\varepsilon > 0$ be such that $(B(\varepsilon, p) \cup R_Z) \cap B(\varepsilon, 0_Z) = \emptyset$ and $R_Z \not\subset B(2\varepsilon, p)$. Let $t_0 \in [0, 1)$ be such that $H(A, \alpha(t)) < \varepsilon$ for each $t \in [t_0, 1]$. Let $G = \bigcup\{\alpha(s) : s \in [t_0, 1]\}$. Notice that $G \subset B(\varepsilon, p) \cup B(\varepsilon, 0_Z)$. Since $A = \alpha(1)$, by [4, Lemma 2.1], G is a compact subset of Z and it has at most two components. Therefore, the components of G are the sets $G_1 = G \cap B(\varepsilon, p)$ and $G_2 = G \cap B(\varepsilon, 0_Z)$. Hence G_1 is a subcontinuum of Z such that $G_1 \cap R_Z \neq \emptyset$ and $R_Z \not\subset G_1$. Since R_Z is terminal in Z, we obtain that $G_1 \subset R_Z$. Given $t \in [t_0, 1]$, by (a), $\alpha(t) = \{p_t, q_t, 0_Z\}$, where $p_t \in R_Z$, $q_t \in S_Z - \{0_Z\}$ and R_Z

is locally connected at p_t. Since $G_1 \subset R_Z$, $q_t \in B(\varepsilon, 0_Z)$. Now it is easy to show that the function $\beta : [t_0, 1] \to R_Z$ be given by $\beta(t) = p_t$ is continuous. Thus, if R_Z is not locally connected at p, then p is arcwise accessible from $LC(R_Z)$. This proves that A is of the form described in (1).

CASE 2. $A = \{p, q, 0_Z\}$, where $q \notin \{p, 0_Z\}$.

In this case $q \in S_Z - \{0_Z\}$. Since $A \notin LC(N(Z))$, by (a), R_Z is not locally connected at p. Thus, proceeding as in Case 1, it is possible to prove that p is arcwise accessible from $LC(R_Z)$.

This completes the proof that, if $A \in N(Z)$ and A is arcwise accessible from $LC(N(Z))$, then A is of one of the forms described in (1) and (2).

Now take an element $A = \{p, 0_Z\}$, where $p \in R_Z$ and either R_Z is locally connected at p or p is arcwise accessible from $LC(R_Z)$. By (a), $A \notin LC(N(Z))$. Fix a point $q \in S_Z - \{0_Z\}$ and take a one-to-one map $\alpha : [0,1] \to S_Z$ such that $\alpha(0) = q$ and $\alpha(1) = 0_Z$. In the case that R_Z is locally connected at p. Define $\gamma : [0,1] \to N(Z) = \langle\{0_Z\}, R_Z, Z\rangle_3$ by $\gamma(t) = \{p, 0_Z, \alpha(t)\}$. Then γ is continuous, $\operatorname{Im}\gamma$ is an arc, $\gamma(1) = A$ and, by (a), $\gamma([0,1)) \subset LC(N(Z))$. Hence A is arcwise accessible from $LC(N(Z))$. In the case that p is arcwise accessible from $LC(R_Z)$, let $\beta : [0,1] \to R_Z$ be a one-to-one map such that $\beta(1) = p$ and $\beta([0,1)) \subset LC(R_Z)$. In this case define $\lambda : [0,1] \to N(Z)$ by $\lambda(t) = \{\beta(t), \alpha(t), 0_Z\}$. Then λ is continuous, $\operatorname{Im}\lambda$ is an arc, $\lambda(1) = A$ and, by (a), $\lambda([0,1)) \subset LC(N(Z))$. Thus A is arcwise accessible from $LC(N(Z))$.

Finally, let $A = \{p, q, 0_Z\}$, where $p \in R_Z$, $q \in S_Z - \{0_Z\}$ and p is arcwise accessible from $LC(R_Z)$. Since R_Z is not locally connected at p, by (a), $A \notin LC(N(Z))$. Let $\beta : [0,1] \to R_Z$ be a one-to-one map such that $\beta(1) = p$ and $\beta([0,1)) \subset LC(R_Z)$. Define $\sigma : [0,1] \to N(Z)$ by $\sigma(t) = \{\beta(t), q, 0_Z\}$. Then σ is continuous, $\operatorname{Im}\sigma$ is an arc, $\sigma(1) = A$ and, by (a), $\sigma([0,1)) \subset LC(N(Z))$. This proves that A is arcwise accessible from $LC(N(Z))$ and ends the proof of (b).

(c) By Lemma 5.2(d), $N(Z) = \langle\{0_Z\}, R_Z, Z\rangle_3$. First, suppose that R_Z is arcwise connected. Fix a point $p_0 \in R_Z$. Let $A_0 = \{0_Z, p_0\}$. Let $A = \{0_Z, p, q\} \in \langle\{0_Z\}, R_Z, Z\rangle_3$, where $p \in R_Z$ and it could be that $p = q$. Let $\alpha : [0,1] \to R_Z$ be a map such that $\alpha(0) = p_0$ and $\alpha(1) = p$. We show that there exists a map $\gamma : [0,1] \to \langle\{0_Z\}, R_Z, Z\rangle_3$ such that $\gamma(0) = A_0$ and $\gamma(1) = A$. We consider two cases. If $q \in S_Z$, then let $\beta : [0,1] \to S_Z$ be a map such that $\beta(0) = 0_Z$ and $\beta(1) = q$. Then define $\gamma(t) = \{0_Z, \alpha(t), \beta(t)\}$. If $q \in R_Z$, let $\lambda : [0,1] \to R_Z$ be such that $\lambda(0) = p_0$ and $\lambda(1) = q$. In this case, define $\gamma(t) = \{0_Z, \alpha(t), \lambda(t)\}$. Hence, $N(Z)$ is arcwise connected.

Now suppose that $\langle\{0_Z\}, R_Z, Z\rangle_3$ is arcwise connected. Fix a point $p_0 \in R_Z$ and let $p \in R_Z$. By hypothesis there exists a map $\gamma : [0,1] \to \langle\{0_Z\}, R_Z, Z\rangle_3$ such that $\gamma(0) = \{p_0, 0_Z\}$ and $\gamma(1) = \{p, 0_Z\}$. By [5, Lemma 2.2] and [4, Lemma 2.1], the set $B = \bigcup\{\gamma(t) : t \in [0,1]\}$ is a compact, locally connected subspace of Z, with at most two components C_1 and C_2. Then C_1

and C_2 are locally connected subcontinua of Z. Suppose that $0_Z \in C_2$, since C_2 is an arcwise connected subset of Z and R_Z is terminal in Z, $p_0, p \in C_1$. Thus there exists a map $\sigma : [0,1] \to C_1 \subset Z$ such that $\sigma(0) = p_0$ and $\sigma(1) = p$. Since R_Z is terminal in Z, $\operatorname{Im}\sigma \subset R_Z$. Therefore, R_Z is arcwise connected. □

THEOREM 5.4. *Let $X = R_X \cup S_X$ be a metric compactification of the ray such that R_X is a locally connected nondegenerate continuum. Let Y be a continuum such that there exists a homeomorphism $h : F_3(X) \to F_3(Y)$. Then:*

(a) *The set of elements $A \in N(X)$ that are arcwise accessible from $LC(N(X))$ is $\{\{0_Z, p\} : p \in R_X\}$, so this set is homeomorphic to R_X and it is compact.*

(b) *Y is a compactification of the ray, $h(\{\{0_Z, p\} : p \in R_X\}) = \{\{0_Z, q\} : q \in R_Y\}$ and the function that assigns, to each $p \in R_X$, the unique point in R_Y satisfying $h(\{0_Z, p\}) = \{0_Y, q\}$, is a homeomorphism. In particular, R_X and R_Y are homeomorphic.*

PROOF. By Theorem 2.2, Y is a compactification of the ray. By [3, Corollary 5.9], R_Y is nondegenerate. Given a continuum Z, the definition of $\Delta_3(Z)$ involves only topological properties, so $h(N(X)) = N(Y)$. By Lemma 5.3(c), R_Y is arcwise connected. Since R_X is locally connected, the set of points in R_X that are arcwise accessible from $LC(R_X)$ is empty, so Lemma 5.3(b), implies that (a) holds.

(b) Since h is a homeomorphism, $h(LC(N(X))) = LC(N(Y))$. Let $\mathcal{A}(X) = \{A \in N(X) : A$ is arcwise accessible from $LC(N(X))\}$ and $\mathcal{A}(Y) = \{B \in N(Y) : B$ is arcwise accessible from $LC(N(Y))\}$. Notice that $h(\mathcal{A}(X)) = \mathcal{A}(Y)$. By (a), $\mathcal{A}(Y)$ is compact. Now we show that there is no point q in R_Y such that q is arcwise accessible from $LC(R_Y)$. Suppose to the contrary that there exists $q \in R_Y$ such that q is arcwise accessible from $LC(R_Y)$. By Lemma 5.3(b), for each $y \in S_Y - \{0_Y\}$, the set $A_y = \{q, y, 0_Z\}$ belongs to $\mathcal{A}(Y)$. Fix a point $q_0 \in R_Y - \{q\}$ and choose a sequence $\{y_n\}_{n=1}^{\infty}$ in $S_Y - \{0_Y\}$ such that $\lim y_n = q_0$. Then $\lim A_{y_n} = \{q, q_0, 0_Z\}$. By the compactness of $\mathcal{A}(Y)$, $\{q, q_0, 0_Z\} \in \mathcal{A}(Y)$. This contradicts Lemma 5.3(b) and completes the proof that no point in R_Y is arcwise accessible from $LC(R_Y)$. By Lemma 5.3(b), we conclude that $\mathcal{A}(Y) = \{\{q, 0_Y\} : q \in R_Y$ and R_Y is locally connected at $q\}$.

We check that $LC(R_Y)$ is open in R_Y. Let $q \in LC(R_Y)$. Fix a point $q_0 \in S_Y - \{0_Y\}$. By Lemma 5.3(a), the set $B = \{q, q_0, 0_Y\}$ belongs to $LC(N(Y))$. Since $h(LC(N(X))) = LC(N(Y))$, there exist $p \in LC(R_X)$ and $p_0 \in S_X - \{0_X\}$ such that, if $A = \{p, p_0, 0_X\}$, then $h(A) = B$. Let $\varepsilon > 0$ be such that the sets $B(p, \varepsilon)$, $B(p_0, \varepsilon)$ and $B(0_X, \varepsilon)$ are pairwise disjoint and $(B(p_0, \varepsilon) \cup B(0_X, \varepsilon)) \cap R_X = \emptyset$. Since the set $\mathcal{G} = h(\langle B(p, \varepsilon), B(p_0, \varepsilon), B(0_X, \varepsilon)\rangle_3)$ is an open subset of $F_3(Y)$ containing $h(A) =$

B, there exists $\delta > 0$ such that, if $D \in F_3(Y)$ and $H(D,B) < \delta$, then $D \in \mathcal{G}$. Given a point $y \in R_Y \cap B(\delta, q)$, $H(\{y, q_0, 0_Y\}, B) < \delta$ and $\{y, q_0, 0_Y\} \in N(Y)$ (Lemma 5.2(d)), so there exist $x \in B(p,\varepsilon)$, $q_1 \in B(p_0,\varepsilon)$ and $u \in B(0_X,\varepsilon)$ such that $h(\{x, q_1, u\}) = \{y, q_0, 0_Y\}$ and $\{x, q_1, u\} \in N(X)$. This implies (see Lemma 5.2(d)) that $x \in R_X$ and $u = 0_X$. Since R_X is locally connected, by Lemma 5.3(a), $\{x, q_1, u\} \in LC(N(X))$. Thus $\{y, q_0, 0_Y\} \in LC(N(Y))$. Applying again Lemma 5.3(a), we obtain that R_Y is locally connected at y. We have shown that $R_Y \cap B(\delta, q) \subset LC(R_Y)$. Therefore, $LC(R_Y)$ is open in R_Y.

Now, we show that R_Y is locally connected. Since $\mathcal{A}(X)$ is nonempty, $\mathcal{A}(Y)$ is nonempty. This implies that $LC(R_Y)$ is nonempty. If R_Y is not locally connected, choose points $q, y \in R_Y$ such that $q \in LC(R_Y)$ and $y \notin LC(R_Y)$. Since R_Y is arcwise connected, there exists a one-to-one map $\alpha : [0,1] \to R_Y$ such that $\alpha(0) = q$ and $\alpha(1) = y$. Let $t_0 = \min \alpha^{-1}(R_Y - LC(R_Y))$. Then $0 < t_0$ and $\alpha(t_0)$ is arcwise accessible from $LC(R_Y)$. This is a contradiction since we proved before that no point in R_Y is arcwise accessible from $LC(R_Y)$. Therefore, R_Y is locally connected.

Hence $\mathcal{A}(Y) = \{\{q, 0_Y\} : q \in R_Y\}$ and $\mathcal{A}(X) = \{\{p, 0_X\} : p \in R_X\}$. Thus $h(\{\{0_X, p\} : p \in R_X\}) = \{\{0_Y, q\} : q \in R_Y\}$. For each $p \in R_X$, define $f(p)$ as the unique point in R_Y such that $h(\{0_X, p\}) = \{0_Y, f(p)\}$. Clearly, f is a homeomorphism from R_X onto R_Y. □

LEMMA 5.5. *Let $f : [0,\infty) \to [0,\infty)$ be a map such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$ and let $t = \{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,\infty)$ such that $0 = t_1 < t_2 < \cdots$ and $\lim t_n = \infty$. Define, recursively,*

$$m_1 = \min(f^{-1}(t_2)), \quad M_1 = \max(f^{-1}(t_2)),$$

$$m_{n+1} = \min([M_n, \infty) \cap f^{-1}(t_{n+2})) \text{ and } M_{n+1} = \max(f^{-1}(t_{n+2})).$$

Then there exists a continuous function $k(t,f) : [0,\infty) \to [0,\infty)$ with the following properties:

(a) $k(t,f)(0) = 0$ *and* $\lim_{x\to\infty} k(t,f)(x) = \infty$.
(b) *For each $n \in \mathbb{N}$, $(k(t,f))^{-1}(t_{n+1}) = [m_n, M_n]$, $(k(t,f))^{-1}([0, t_{n+1})) = [0, m_n)$ and $(k(t,f))^{-1}((t_{n+1}, \infty)) = (M_n, \infty)$.*

PROOF. Note that $0 < m_1 \le M_1 < m_2 \le M_2 < m_3 \le M_3 < \cdots$. We show that $\lim M_n = \infty = \lim m_n$. Take $K \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that, for each $n \ge N$, $t_n > \max(f([0,K]))$. Given $n \ge N$, $f(M_{n+1}) = t_{n+2} > \max(f([0,K]))$. Hence, $M_{n+1} > K$ for each $n \ge N$. Therefore, $\lim M_n = \infty$ and $\lim m_n = \infty$.

Define $k(t,f) : [0,\infty) \to [0,\infty)$ as follows

$$k(t,f)(x) = \begin{cases} t_{n+1}, & \text{if } x \in [m_n, M_n] \text{ for some } n \in \mathbb{N}, \\ f(x), & \text{if } x \notin [m_n, M_n] \text{ for every } n \in \mathbb{N}. \end{cases}$$

Note that, for each $n \in \mathbb{N}$, $f(m_n) = k(t,f)(m_n)$ and $f(M_n) = k(t,f)(M_n)$. Since $\lim M_n = \infty = \lim m_n$ and $0 < m_1 \le M_1 < m_2 \le M_2 < m_3 \le M_3 < \cdots$, the family $\{[m_n, M_n] : n \in \mathbb{N}\}$ is locally finite and the boundary of the set $\bigcup\{[m_n, M_n] : n \in \mathbb{N}\}$ is the set $\{m_n : n \in \mathbb{N}\} \cup \{M_n : n \in \mathbb{N}\}$. This implies that $k(t,f)$ is continuous.

We show property (a). Note that $k(t,f)(0) = 0$. Given $K \in \mathbb{R}$, let $L \in \mathbb{N}$ and $R \in \mathbb{R}$ be such that, if $R \le x$ and $L \le n$, then $K \le f(x)$ and $K \le t_n$. Given $x \in [0,\infty)$ such that $\max\{R, M_L\} < x$, we have that either $x \in [m_n, M_n]$, for some $n \ge L$ or $x \notin \bigcup\{[m_n, M_n] : n \in \mathbb{N}\}$. In the first case, $K \le t_{n+1} = k(t,f)(x)$ and, in the second case, $K \le f(x) = k(t,f)(x)$. We have shown that $\lim_{x\to\infty} k(t,f)(x) = \infty$. Therefore, property (a) holds.

Now, we show property (b). Take $n \in \mathbb{N}$. If $x \in (M_n, \infty)$ and $x \notin \bigcup\{[m_r, M_r] : r \in \mathbb{N}\}$, then $k(t,f)(x) = f(x)$. If $f(x) \le t_{n+1}$, then by the Intermediate Value Theorem, there would be $u \in [x,\infty)$ such that $f(u) = t_{n+1}$; which is impossible given the fact that $\max(f^{-1}(t_{n+1})) = M_n < u$. Hence $k(t,f)(x) = f(x) > t_{n+1}$. We have proved that $(M_n,\infty) - (\bigcup\{[m_r, M_r] : r \in \mathbb{N}\}) \subset (k(t,f))^{-1}((t_{n+1},\infty))$. If $x \in (M_n,\infty) \cap (\bigcup\{[m_r, M_r] : r \in \mathbb{N}\})$, then there is $r > n$ such that $x \in [m_r, M_r]$. Hence, $k(t,f)(x) = t_{r+1} > t_{n+1}$. This completes the proof that $(M_n,\infty) \subset (k(t,f))^{-1}((t_{n+1},\infty))$.

Let $x \in [0, m_n)$. Put $M_0 = 0$. Then there exists $1 \le r \le n$ such that $x \in [M_{r-1}, M_r]$. If $x \in [M_{r-1}, m_r)$, since $m_r = \min([M_{r-1},\infty) \cap f^{-1}(t_{r+1}))$ and $f(M_{r-1}) = t_r < t_{r+1}$, by the Intermediate Value Theorem, $f(x) < t_{r+1}$. Since $k(t,f)(x) = f(x)$, we obtain that $k(t,f)(x) < t_{r+1} \le t_{n+1}$. If $x \in [m_r, M_r]$, then $k(t,f)(x) = t_{r+1}$. Since $x \notin [m_n, M_n]$, $r < n$, so $t_{r+1} < t_{n+1}$. In any case, $k(t,f)(x) < t_{n+1}$. We have shown that $[0, m_n) \subset (k(t,f)(x))^{-1}([0, t_{n+1}))$.

Finally, since $[m_n, M_n] \subset (k(t,f))^{-1}(t_{n+1})$, we conclude that property (b) holds. ∎

THEOREM 5.6. *Let $X = R_X \cup S_X$ be a compactification of the ray such that R_X is an ANR. If Y is a continuum such that $F_3(X)$ is homeomorphic to $F_3(Y)$, then X is homeomorphic to Y.*

PROOF. By Theorem 2.2, Y is a compactification of the ray. By [3, Corollary 5.9] we may assume that R_X and R_Y are nondegenerate. Let $h : F_3(X) \to F_3(Y)$ be a homeomorphism. We identify S_X (resp., S_Y) with the interval $[0_X, \infty)$ (resp., $[0_Y, \infty)$). First we show that R_X is a retract of X. Since R_X is an ANR, there exist an open subset U of X, with $R_X \subset U$, and a retraction $r_1 : U \to R_X$. Then there exists $a \in [0_X, \infty)$ such that $[a, \infty) \subset U$. To obtain the desired retraction, define $r : X \to R_X$ by

$$r(p) = \begin{cases} r_1(p), & \text{if } p \in [a,\infty) \cup R_X, \\ r_1(a), & \text{if } p \in [0_X, a]. \end{cases}$$

We are going to define a map $f : X \to Y$ which will be the base to define a homeomorphism from X onto Y.

By Theorem 5.4(b), $h(\{\{0_Z, p\} : p \in R_X\}) = \{\{0_Z, q\} : q \in R_Y\}$ and the function $f_1 : R_X \to R_Y$ that assigns, to each $p \in R_X$, the unique point $f_1(p)$ in R_Y satisfying $h(\{0_Z, p\}) = \{0_Y, f_1(p)\}$, is a homeomorphism.

Since h is a homeomorphism, $h(LC(N(X))) = LC(N(Y))$. Thus, Lemma 5.3(a) implies that $h(\{\{p, x, 0_Z\} \in F_3(X) : p \in (0_X, \infty) \text{ and } x \in R_X\}) = \{\{q, y, 0_Y\} \in F_3(Y) : q \in (0_Y, \infty) \text{ and } y \in R_Y\}$. So, given $p \in (0_X, \infty)$, define $f(p) \in (0_Y, \infty)$ and $f_0(p) \in R_Y$ to be the unique points that satisfy that $h(\{p, r(p), 0_X\}) = \{f(p), f_0(p), 0_Y\}$.

We show that the function $f : (0_X, \infty) \to (0_Y, \infty)$ is continuous. Take a sequence $\{p_n\}_{n=1}^{\infty}$ in $(0_X, \infty)$ such that $\lim p_n = p \in (0_X, \infty)$. Since r is continuous, $\lim r(p_n) = r(p)$. Since h is continuous, $\{f(p), f_0(p), 0_Y\} = h(\{p, r(p), 0_X\}) = \lim h(\{p_n, r(p_n), 0_X\}) = \lim\{f(p_n), f_0(p_n), 0_Y\}$. Since each $f_0(p_n)$ belongs to R_Y and R_Y is closed, we conclude that $\lim f(p_n) = f(p)$. Therefore, f is continuous.

Extend the function f by defining $f(0_X) = 0_Y$. We show that f is continuous at 0_X. Let $\varepsilon > 0$ be such that $B(\varepsilon, 0_Y) \cap B(\varepsilon, f_1(r(0_X))) = \emptyset$ and $R_Y \not\subseteq B(\varepsilon, f_1(r(0_X)))$. Since h is a homeomorphism, there exists $\delta > 0$ such that $H(A, B) < \delta$ implies $H(h(A), h(B)) < \varepsilon$. Fix an element $p \in (0_X, \infty)$ such that diameter$([0_X, p]) < \delta$ and, for each $x \in [0_X, p]$, $r(x) \in B(\delta, r(0_X))$. Given $x \in [0_X, p]$, we have that $H(\{x, r(x), 0_X\}, \{r(0_X), 0_X\}) < \delta$. This implies that $H(h(\{x, r(x), 0_X\}), h(\{r(0_X), 0_X\})) < \varepsilon$. So,

$$H(h(\{x, r(x), 0_X\}), \{f_1(r(0_X)), 0_Y\}) < \varepsilon.$$

Thus $h(\{x, r(x), 0_X\}) \in \langle B(\varepsilon, 0_Y), B(\varepsilon, f_1(r(0_X)))\rangle_3$. Let

$$G = \bigcup\{h(\{x, r(x), 0_X\}) : x \in [0_X, p]\}.$$

Then $G \in \langle B(\varepsilon, 0_Y), B(\varepsilon, f_1(r(0_X)))\rangle_3$. Since

$$h(\{0_X, r(0_X), 0_X\}) = \{f_1(r(0_X)), 0_Y\},$$

by [4, Lemma 2.1], G has at most two components. Therefore, the components of G are the sets $G_1 = G \cap B(\varepsilon, f_1(r(0_X)))$ and $G_2 = G \cap B(\varepsilon, 0_Y)$. Hence, G_1 is a subcontinuum of Y such that $G_1 \cap R_Y \neq \emptyset$ and $R_Y \not\subseteq G_1$. Since R_Y is terminal in Y, $G_1 \subset R_Y$. Given $x \in (0_X, p]$, $\{f(x), f_0(x), 0_Y\} = h(\{x, r(x), 0_X\}) \in G_1 \cup G_2$. Since $f(x) \in S_Y$, $f(x) \notin G_1$. Thus $f(x) \in G_2 \subset B(\varepsilon, 0_Y)$. Hence, $f(x) \in B(\varepsilon, 0_Y)$ for each $x \in (0_X, p]$. Therefore, f is continuous at 0_X.

We have defined a homeomorphism $f_1 : R_X \to R_Y \subset Y$ and a map $f : [0_X, \infty) \to [0_Y, \infty) \subset Y$. Since R_X and $[0_X, \infty)$ are disjoint, there exists a well defined common extension of the functions f_1 and f. This common extension will be denoted by $f : X \to Y$.

In order to complete the proof that f is continuous, take a sequence $\{p_n\}_{n=1}^{\infty}$ in $[0_X,\infty)$ such that $\lim p_n = p$, for some $p \in R_X$. Note that $\lim\{p_n, r(p_n), 0_X\} = \{p, r(p), 0_X\} = \{p, 0_X\}$. Thus, $\lim\{0_Y, f(p_n), f_0(p_n)\} = \lim h(\{p_n, r(p_n), 0_X\}) = h(\{p, 0_X\}) = \{0_Y, f_1(p)\} = \{0_Y, f(p)\}$. This implies that the only limit points that the sequences $\{f(p_n)\}_{n=1}^{\infty}$ and $\{f_0(p_n)\}_{n=1}^{\infty}$ can have are 0_Y and $f(p)$. Since $f_0(p_n) \in R_Y$ for each $n \in \mathbb{N}$, we have that $\lim f_0(p_n) = f(p)$. We need to prove that $\lim f(p_n) = f(p)$. Suppose to the contrary that 0_Y is an accumulation point of this sequence. We may assume that $\lim f(p_n) = 0_Y$. Let $\varepsilon > 0$ be such that $B(\varepsilon, 0_X) \cap B(\varepsilon, p) = \emptyset$ and $R_X \not\subseteq B(\varepsilon, p)$. Let $\delta > 0$ be such that $H(A, B) < \delta$ implies $H(h^{-1}(A), h^{-1}(B)) < \varepsilon$. By Theorem 5.4(b), R_Y is locally connected. Let S be a connected and compact neighborhood of $f(p)$ in the space R_Y such that diameter$(S) < \delta$. Fix $m \in \mathbb{N}$ such that diameter $([0_Y, f(p_m)]) < \delta$, $p_m \in B(\varepsilon, p)$ and $f_0(p_m) \in S$. Given points $z \in S$ and $w \in [0_Y, f(p_m)]$, $H(\{z, w, 0_Y\}, \{f(p), 0_Y\}) < \delta$, so $H(h^{-1}(\{z, w, 0_Y\}), \{p, 0_X\}) < \varepsilon$. Let $K = \bigcup\{h^{-1}(\{z, w, 0_Y\}) : z \in S$ and $w \in [0_Y, f(p_m)]\}$. Then K is a compact subset of X. Since $h^{-1}(\{f(p), 0_Y\}) = \{p, 0_X\}$ has two elements, by [4, Lemma 2.1], K has at most two components. Note that $K \subset B(\varepsilon, 0_X) \cup B(\varepsilon, p)$. Thus the components of K are the sets $K_1 = K \cap B(\varepsilon, 0_X)$ and $K_2 = K \cap B(\varepsilon, p)$. Since R_X is terminal in X and K_2 is a continuum containing $p \in R_X$ and $R_X \not\subseteq K_2$, we obtain that $K_2 \subset R_X$. Note that $\{p_m, r(p_m), 0_X\} = h^{-1}(\{0_Y, f(p_m), f_0(p_m)\}) \subset K$. So, $p_m \in K_2$. Thus $p_m \in R_X$. This contradicts the choice of the sequence $\{p_n\}_{n=1}^{\infty}$ and proves that $\lim f_n(p) = f(p)$. Therefore, f is continuous.

Now we prove an important property of the function f which will help us to "straighten it out" to obtain a homeomorphism from X onto Y.

Claim 1. Let $\{x_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences in $[0_X,\infty)$. Suppose that $\lim x_n = \infty = \lim p_n$ (as sequences in $[0_X,\infty)$) and, for each $n \in \mathbb{N}$, $0_X < x_n \leq p_n$ and $f(x_n) = f(p_n)$. Then $\lim(\text{diameter}([x_n, p_n])) = 0$ and $\lim(\text{diameter}(f([x_n, p_n]))) = 0$, where the diameters are taken in the spaces X and Y, respectively.

To prove Claim 1. Suppose to the contrary that $\{\text{diameter}([x_n, p_n])\}_{n=1}^{\infty}$ does not converge to 0. Then there exists $\varepsilon_0 > 0$ such that $B(4\varepsilon_0, 0_X) \cap R_X = \emptyset$, diameter$(R_X) > 2\varepsilon_0$ and $2\varepsilon_0 <$ diameter$([x_n, p_n])$, for infinitely many numbers n. Thus, we may assume that $2\varepsilon_0 <$ diameter$([x_n, p_n])$ for every $n \in \mathbb{N}$. By the compactness of X, we also may assume that $\lim x_n = x$ and $\lim p_n = p$, for some $x, p \in R_X$. Since $f(x_n) = f(p_n)$, for each $n \in \mathbb{N}$, we have that $f(x) = f(p)$. Since $\lim\{0_Y, f(p_n), f_0(p_n)\} = \lim h(\{p_n, r(p_n), 0_X\}) = h(\{p, 0_X\}) = \{f(p), 0_Y\}$ and each point $f_0(p_n)$ belongs to R_Y, we obtain that $\lim f_0(p_n) = f(p)$. Similarly, $\lim f_0(x_n) = f(x)$.

Let $\mu : C(Y) \to [0, 1]$ be a Whitney map, where $\mu(Y) = 1$ (see [9, Theorem 13.4]). For each $n \in \mathbb{N}$, let $A_n, B_n \in C(R_Y)$ be such that $f(p), f_0(p_n) \in A_n$, $f(p), f_0(x_n) \in B_n$, $\mu(A_n) = \min\{\mu(A) : A \in C(R_Y)$ and $f(p), f_0(p_n) \in A\}$ and $\mu(B_n) = \min\{\mu(B) : B \in C(R_Y)$ and $f(p), f_0(x_n) \in B\}$. By Theorem

5.4(b), R_Y is locally connected. This implies that $\lim A_n = \lim B_n = \{f(p)\}$ (in $C(R_Y)$). Thus $\lim \langle A_n \cup B_n, \{f(p_n)\}, \{0_Y\}\rangle_3 = \{\{f(p), 0_Y\}\}$ (in the space $C(F_3(Y))$). So, we have that $\{h^{-1}(\langle A_n \cup B_n, \{f(p_n)\}, \{0_Y\}\rangle_3)\}_{n=1}^{\infty}$ is a sequence of subcontinua of $F_3(X)$ that converges to $\{\{p, 0_X\}\}$ (in $C(F_3(X))$). Then there exists $m \in \mathbb{N}$ such that $x_m \in B(\varepsilon_0, x)$, $p_m \in B(\varepsilon_0, p)$ and, if $\mathcal{C}_m = h^{-1}(\langle A_m \cup B_m, \{f(p_n)\}, \{0_Y\}\rangle_3)$, then the set $C_m = \bigcup\{C : C \in \mathcal{C}_m\}$ is contained in $N(\varepsilon_0, \{p, 0_X\}) = B(\varepsilon_0, p) \cup B(\varepsilon_0, 0_X)$. By [4, Lemma 2.1], C_m has at most three components. Note that the set $\langle A_m \cup B_m, \{f(p_m)\}, \{0_Y\}\rangle_3$ is contained in $N(Y)$ (Lemma 5.2(d)). Thus $\mathcal{C}_m \subset N(X)$. Also note that $\{f_0(x_m), f(x_m), 0_Y\} \in \langle A_m \cup B_m, \{f(p_m)\}, \{0_Y\}\rangle_3$. Hence $\{x_m, r(x_m), 0_X\} \in \mathcal{C}_m$ and $x_m, 0_X \in C_m$. Similarly, $p_m \in C_m$.

Since $x_m, p_m \notin B(\varepsilon_0, 0_X)$, we have that $x_m, p_m \in B(\varepsilon_0, p)$. Let A and B be the components of C_m such that $x_m \in A$ and $p_m \in B$. Then $A \cup B \subset B(\varepsilon_0, p)$. Let C be the component of C_m such that $0_X \in C$. The connectedness of C implies that $C \subset B(\varepsilon_0, 0_X)$. This implies that $C \cap R_X = \emptyset$ and $x_m, p_m \notin C$. Since $\mathcal{C}_m \subset N(X)$, $C_m \cap R_X \neq \emptyset$. Hence there exists a component D of C_m such that $D \cap R_X \neq \emptyset$. Since R_X is terminal in X, $D \subset R_X$ or $R_X \subset D$. If $R_X \subset D$, then $2\varepsilon_0 <$ diameter$(R_X) \leq$ diameter(D). Since D is connected, we have that $D \subset B(\varepsilon_0, p)$, so diameter$(D) \leq 2\varepsilon_0$, a contradiction. Thus $D \subset R_X$. Hence $p_m, x_m \notin D$. Thus C and D are different components of C_m and $\{x_m, p_m\} \cap (C \cup D) = \emptyset$. Since C_m has at most three components, $A = B$.

We have that A is a connected subset of X containing both x_m and p_m. Hence $[x_m, p_m] \subset A \subset B(\varepsilon_0, p)$. So, $2\varepsilon_0 <$ diameter$([x_m, p_m]) \leq$ diameter$(A) \leq 2\varepsilon_0$. This contradiction establishes the proof that

$$\lim(\text{diameter}([x_n, p_n])) = 0.$$

Since f is uniformly continuous, $\lim(\text{diameter}(f([x_n, p_n]))) = 0$.

We define a sequence $\{g_m\}_{m=0}^{\infty}$ of maps from $[0_X, \infty)$ onto $[0_Y, \infty)$. For each $m \in \mathbb{N} \cup \{0\}$, consider the sequence $t^{(m)} = \{t_i^{(m)}\}_{i=1}^{\infty}$ given by $t_i^{(m)} = \frac{i-1}{2^m}$. Define, recursively, $g_0 = f$ and, for each $m \geq 0$,

$$g_{m+1} = k(t^{(m)}, g_m),$$

where $k(t^{(m)}, g_m)$ is the map defined in Lemma 5.5.

For each $m \in \mathbb{N}$, define $\mathfrak{M}(0, m-1) = 0_X$ and, for each $n \in \mathbb{N}$, define

$$\mathfrak{M}(n, m-1) = \max(g_{m-1}^{-1}(t_{n+1}^{(m-1)}))$$

and

$$\mathfrak{m}(n, m-1) = \min([\mathfrak{M}(n-1, m-1), \infty) \cap g_{m-1}^{-1}(t_{n+1}^{(m-1)})).$$

That is, $\mathfrak{m}(n, m-1)$ and $\mathfrak{M}(n, m-1)$ are the numbers used in Lemma 5.5 to define $g_m = k(t^{(m-1)}, g_{m-1})$.

Claim 2. For each $m, n \in \mathbb{N}$, $\mathfrak{M}(n, m-1) = \mathfrak{M}(2n, m)$ and $\mathfrak{m}(n, m) = \mathfrak{m}(2n, m+1)$.

We prove Claim 2. Let $m, n \in \mathbb{N}$. By Lemma 5.5(b), we have $\mathfrak{M}(n, m-1) = \max(g_m^{-1}(t_{n+1}^{(m-1)}))$. So $\mathfrak{M}(n, m-1) = \max(g_m^{-1}(t_{n+1}^{(m-1)})) = \max(g_m^{-1}(\frac{n}{2^{m-1}})) = \max(g_m^{-1}(\frac{2n}{2^m})) = \max(g_m^{-1}(t_{2n+1}^{(m)})) = \mathfrak{M}(2n, m)$. So, $\mathfrak{M}(n, m-1) = \mathfrak{M}(2n, m)$. We prove the other equality, by Lemma 5.5(b), we have that $g_{m+1}^{-1}(t_{n+1}^{(m)}) = (k(t^{(m)}, g_m))^{-1}(t_{n+1}^{(m)}) = [\mathfrak{m}(n, m), \mathfrak{M}(n, m)]$, we have shown that $g_{m+1}^{-1}(\frac{n}{2^m}) = [\mathfrak{m}(n, m), \mathfrak{M}(n, m)]$. Also, by Lemma 5.5(b), $g_{m+1}^{-1}([0_Y, \frac{n}{2^m})) = g_{m+1}^{-1}([0_Y, t_{n+1}^{(m)})) = [0_X, \mathfrak{m}(n, m))$. Since

$$g_{m+1}^{-1}(t_{2n}^{(m+1)}) = g_{m+1}^{-1}(\frac{2n-1}{2^{m+1}}) \subset g_{m+1}^{-1}([0_Y, \frac{n}{2^m})),$$

we have that $\mathfrak{M}(2n-1, m+1) < \mathfrak{m}(n, m)$. Since $g_{m+1}^{-1}(t_{2n+1}^{(m+1)}) = g_{m+1}^{-1}(\frac{2n}{2^{m+1}}) = g_{m+1}^{-1}(\frac{n}{2^m}) = g_{m+1}^{-1}(t_{n+1}^{(m)}) = (k(t^{(m)}, g_m))^{-1}(t_{n+1}^{(m)})$. By Lemma 5.5(b), $g_{m+1}^{-1}(t_{2n+1}^{(m+1)}) = [\mathfrak{m}(n, m), \mathfrak{M}(n, m)]$. Since $\mathfrak{M}(2n-1, m+1) < \mathfrak{m}(n, m)$, $[\mathfrak{m}(n, m), \mathfrak{M}(n, m)] \subset [\mathfrak{M}(2n-1, m+1), \infty)$. Hence, $\mathfrak{m}(2n, m+1) = \min([\mathfrak{m}(n, m), \mathfrak{M}(n, m)]) = \mathfrak{m}(n, m)$. This completes the proof of Claim 2.

CLAIM 3. $|g_m(x) - g_{m+1}(x)| \leq \frac{1}{2^{m-1}}$ for every $x \in [0_X, \infty)$ and $m \in \mathbb{N}$.

We prove Claim 3. Let $x \in [0_X, \infty)$ and $m \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $x \in [\mathfrak{M}(n-1, m-1), \mathfrak{M}(n, m-1)]$. By Claim 2, $[\mathfrak{M}(n-1, m-1), \mathfrak{M}(n, m-1)] = [\mathfrak{M}(2(n-1), m), \mathfrak{M}(2n, m)]$. By Lemma 5.5(b), we have that $g_m(x) = k(t^{(m-1)}, g_{m-1})(x) \in [t_n^{(m-1)}, t_{n+1}^{(m-1)}] = [\frac{n-1}{2^{m-1}}, \frac{n}{2^{m-1}}]$ and $g_{m+1}(x) \in [t_{2n-1}^{(m)}, t_{2n+1}^{(m)}] = [\frac{2n-2}{2^m}, \frac{2n}{2^m}] = [\frac{n-1}{2^{m-1}}, \frac{n}{2^{m-1}}]$. Therefore, $|g_m(x) - g_{m+1}(x)| \leq \frac{1}{2^{m-1}}$ and Claim 3 has been proved.

By Claim 3, we have that the sequence $\{g_m\}_{m=0}^{\infty}$ is uniformly Cauchy. Thus this sequence converges uniformly to a (hence, continuous) function $g : [0_X, \infty) \to [0_Y, \infty)$. Note that $g(0_X) = 0_Y$.

CLAIM 4. g is increasing.

Take $u, x \in [0_X, \infty)$ such that $u < x$. In the case that there exist $n, m \in \mathbb{N}$ such that $u \leq \mathfrak{M}(n, m) \leq x$, by Lemma 5.5(b), $g_{m+1}(u) = k(t^{(m)}, g_m)(u) \leq t_{n+1}^{(m)} \leq k(t^{(m)}, g_m)(x) = g_{m+1}(x)$. Similarly, since by Claim 2, $u \leq \mathfrak{M}(2n, m+1) \leq x$, we have that $g_{m+2}(u) \leq t_{2n+1}^{(m+1)} \leq g_{m+2}(x)$. Following this process, we obtain that $g_r(u) \leq g_r(x)$ for each $r \geq m+1$. Therefore, $g(u) \leq g(x)$.

Now, suppose that there are no $n, m \in \mathbb{N}$ such that $u \leq \mathfrak{M}(n, m) \leq x$. Given $m \in \mathbb{N}$, let $n_m \in \mathbb{N}$ be such that $x \in [\mathfrak{M}(n_m - 1, m), \mathfrak{M}(n_m, m)]$. Our assumption gives us that $\mathfrak{M}(n_m - 1, m) \leq u < x \leq \mathfrak{M}(n_m, m)$. By Lemma 5.5(b), both numbers $g_{m+1}(u) = k(t^{(m)}, g_m)(u)$ and $g_{m+1}(x) = k(t^{(m)}, g_m)(x)$ are in the interval $[t_{n_m}^{(m)}, t_{n_{m+1}}^{(m)}] = [\frac{n_m - 1}{2^m}, \frac{n_m}{2^m}]$. Hence $|g_{m+1}(u) - g_{m+1}(x)| \leq \frac{1}{2^m}$ for each $m \in \mathbb{N}$. Therefore, $g(u) = g(x)$. This finishes the proof of Claim 4.

Define $G : X \to Y$ by

$$G(x) = \begin{cases} g(x), & \text{if } x \in [0_X,\infty), \\ f(x), & \text{if } x \in R_X. \end{cases}$$

In order to prove that G is continuous, we first prove the following claim.

CLAIM 5. $g_r(\mathfrak{M}(n,m)) = f(\mathfrak{M}(n,m))$ and $g_r(\mathfrak{m}(n,m)) = f(\mathfrak{m}(n,m))$, for every $n,m,r \in \mathbb{N}$.

Let $n,m,r \in \mathbb{N}$. To prove Claim 5, we only prove that $g_r(\mathfrak{M}(n,m)) = f(\mathfrak{M}(n,m))$, the proof of the other equality is similar. Since $g_0(\mathfrak{M}(n,m)) = f(\mathfrak{M}(n,m))$, we only need to show that $g_r(\mathfrak{M}(n,m)) = g_{r-1}(\mathfrak{M}(n,m))$. By the definition of $k(t^{(r-1)}, g_{r-1})$, in the proof of Lemma 5.5, $g_r(x) = k(t^{(r-1)}, g_{r-1})(x) = g_{r-1}(x)$ for each $x \in \mathrm{cl}_{[0_X,\infty)}([0_X,\infty) - \bigcup\{[\mathfrak{m}(s,r-1), \mathfrak{M}(s,r-1)] : s \in \mathbb{N}\})$. Hence, it is enough to show that $\mathfrak{M}(n,m) \notin \mathrm{int}_{[0_X,\infty)}(\bigcup\{[\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1)] : s \in \mathbb{N}\})$. Since $0 < \mathfrak{m}(1,r-1) \leq \mathfrak{M}(1,r-1) < \mathfrak{m}(2,r-1) \leq \mathfrak{M}(2,r-1) < \cdots$, we obtain that $\mathrm{int}_{[0_X,\infty)}(\bigcup\{[\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1)] : s \in \mathbb{N}\}) = \bigcup\{(\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1)) : s \in \mathbb{N}\}$. Suppose, by the way of contradiction, that there is $s \in \mathbb{N}$ such that $\mathfrak{M}(n,m) \in (\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1))$. We consider two cases.

CASE 1. $r-1 \leq m$.

By Claim 2, $\mathfrak{M}(n,m) \in (\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1)) = (\mathfrak{m}(2s,r),\mathfrak{M}(2s,r)) = (\mathfrak{m}(2^2s,r+1),\mathfrak{M}(2^2s,r+1)) = \cdots = (\mathfrak{m}(2^{m-r+1}s,m),\mathfrak{M}(2^{m-r+1}s,m))$. But this is impossible since $\mathfrak{M}(n,m) \notin \bigcup\{(\mathfrak{m}(i,m),\mathfrak{M}(i,m)) : i \in \mathbb{N}\}$.

CASE 2. $m < r-1$.

In this case, by Claim 2, $\mathfrak{M}(n,m) = \mathfrak{M}(2n,m+1) = \mathfrak{M}(2^2n,m+2) = \mathfrak{M}(2^{r-(m+1)}n,r-1)$. Thus $\mathfrak{M}(2^{r-(m+1)}n,r-1) \in (\mathfrak{m}(s,r-1),\mathfrak{M}(s,r-1))$, again a contradiction.

This completes the proof of Claim 5.

CLAIM 6. G is continuous.

Since g is continuous and $[0_X,\infty)$ is open in X, we have that G is continuous at every point of $[0_X,\infty)$. Since $G|_{R_X}$ is continuous, we only have to prove that, if we take a sequence $\{p_m\}_{m=1}^{\infty}$ in $[0_X,\infty)$ converging to an element $p \in R_X$, then there exists a subsequence $\{p_{m_i}\}_{i=1}^{\infty}$ of $\{p_m\}_{m=1}^{\infty}$ such that $\lim G(p_{m_i}) = G(p)$. We consider two cases.

CASE 1. For infinitely many numbers m, we have

$$p_m \in \bigcup\{[\mathfrak{m}(i,j),\mathfrak{M}(i,j)] : i,j \in \mathbb{N}\}.$$

Here, we can suppose that our assumption holds for all $m \in \mathbb{N}$. Given $m \in \mathbb{N}$, let $i,j \in \mathbb{N}$, be such that $p_m \in [\mathfrak{m}(i,j),\mathfrak{M}(i,j)] = (k(t^{(j-1)},g_{j-1}))^{-1}(t_{i+1}^{(j-1)})$. Then $g_j(p_m) = t_{i+1}^{(j-1)} = \frac{i}{2^{j-1}}$. By Claim 2, $p_m \in [\mathfrak{m}(2i,j+1),\mathfrak{M}(2i,j+1)] = (k(t^{(j)},g_j))^{-1}(t_{2i+1}^{(j)})$, so $g_{j+1}(p_m) = t_{2i+1}^{(j)} = \frac{2i}{2^j} = \frac{i}{2^{j-1}}$. Repeating this process we obtain that $g_r(p_m) = \frac{i}{2^{j-1}}$ for each $r \geq j$. Hence, $g(p_m) = \frac{i}{2^{j-1}}$. Using the same argument, we can conclude that $g_r(\mathfrak{m}(i,j)) = g_r(\mathfrak{M}(i,j)) = \frac{i}{2^{j-1}}$ for each $r \geq j$. By Claim 5, $g_r(\mathfrak{M}(i,j)) = f(\mathfrak{M}(i,j))$ and $g_r(\mathfrak{m}(i,j)) = f(\mathfrak{m}(i,j))$, for each $r \in \mathbb{N}$. Thus $f(\mathfrak{m}(i,j)) = f(\mathfrak{M}(i,j)) = g(p_m)$. For each $m \in \mathbb{N}$, let $x_m = \mathfrak{m}(i,j)$ and $y_m = \mathfrak{M}(i,j)$ (recall that i and j depend on m). Then $0 < x_m$, $f(x_m) = f(y_m)$ and, since $\lim p_m = p \in R_X$, we have that $\lim y_m = \infty$ (as a sequence in $[0_X, \infty)$). We may assume that $\lim y_m = y$, for some $y \in R_X$. Then $\lim f(y_m) = f(y) \in R_Y$. Thus $\lim f(x_m) = f(y)$. We may also assume that $\lim x_m = x$ for some $x \in X$. Since $f(x) = \lim f(x_m) = f(y)$, we conclude that $f(x) \in R_Y$, so $x \in R_X$. This implies that $\lim x_n = \infty$ (as a sequence in $[0_X, \infty)$). Thus, we may apply Claim 1 and obtain that $\lim(\text{diameter}(f([x_n, y_n]))) = 0$. Since $f(p_m) \in f([x_m, y_m])$, for each $m \in \mathbb{N}$, $f(p) = \lim f(p_m) = \lim f(x_m) = f(x)$. Recall that $f(x_m) = f(\mathfrak{m}(i,j)) = g(p_m)$. Thus $\lim g(p_m) = f(p)$. Since $p \in R_X$, $G(p) = f(p)$. Thus $\lim G(p_m) = G(p)$. This finishes the proof of Case 1.

Case 2. For infinitely many numbers m, we have

$$p_m \notin \bigcup \{[\mathfrak{m}(i,j), \mathfrak{M}(i,j)] : i, j \in \mathbb{N}\}.$$

Again, we suppose that our assumption for this case holds for every $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. Given $r \in \mathbb{N}$, $p_m \notin \bigcup\{[\mathfrak{m}(i,r), \mathfrak{M}(i,r)] : i \in \mathbb{N}\}$. By definition, $g_{r+1}(p_m) = g_r(p_m)$. This implies that $g(p_m) = f(p_m)$. Therefore, $\lim G(p_m) = \lim g(p_m) = \lim f(p_m) = f(p) = G(p)$. This finishes the proof for the Case 2 and then Claim 6 is proved.

Now, we modify the map g to obtain a continuous function $e : [0_X, \infty) \to [0_Y, \infty)$ that will be not only increasing but strictly increasing, and, also, will have an extension to X that will be a homeomorphism.

Let d_Y be a metric for Y. Given $n \in \mathbb{N}$, since the metric of the absolute value induces the same topology that d_Y on $[0_Y, \infty)$, there exists $\delta_n \in (0,1)$ such that, if $v, y \in [0_Y, n]$ and $|v - y| \leq 2\delta_n$, then $d_Y(v, y) < \frac{1}{n}$. From Lemma 5.5(a) and Claim 3, it follows that $\lim_{x\to\infty} g(x) = \infty$ (as a sequence in $[0_Y, \infty)$). Let $r_0 = 0$ and, for each $n \in \mathbb{N}$, choose $r_n \in [0_X, \infty)$ such that $g(r_n) = n$. By Claim 4, $r_n < r_{n+1}$ for each $n \in \mathbb{N}$ and $\lim r_n = \infty$. Define $e : [0_X, \infty) \to [0_Y, \infty)$ by

$$e(x) = n - 1 + (g(x) - (n-1))(1 - \delta_n) + \delta_n\left(\frac{x - r_{n-1}}{r_n - r_{n-1}}\right), \text{ if } x \in [r_{n-1}, r_n].$$

It is easy to show that e is well defined, continuous and $e(r_n) = n$, for each $n \in \mathbb{N} \cup \{0\}$.

Given $n \in \mathbb{N}$ and $u, x \in [r_{n-1}, r_n]$ such that $u < x$, since g is increasing, $(g(u) - (n-1))(1 - \delta_n) \leq (g(x) - (n-1))(1 - \delta_n)$. Also $\delta_n(\frac{u - r_{n-1}}{r_n - r_{n-1}}) < \delta_n(\frac{x - r_{n-1}}{r_n - r_{n-1}})$. Hence $e(u) < e(x)$. It follows easily that e is strictly increasing.

Given $n \in \mathbb{N}$ and $x \in [r_{n-1}, r_n]$, $n-1 = g(r_{n-1}) \leq g(x) \leq g(r_n) = n$. Then

$$e(x) - g(x) = n - 1 + (g(x) - (n-1))(1 - \delta_n) + \delta_n(\frac{x - r_{n-1}}{r_n - r_{n-1}}) - g(x)$$
$$= \delta_n(n - 1 - g(x)) + \delta_n(\frac{x - r_{n-1}}{r_n - r_{n-1}}).$$

Hence, $|e(x) - g(x)| \leq 2\delta_n$.

Define $E : X \to Y$ by

$$E(x) = \begin{cases} e(x), & \text{if } x \in [0_X, \infty), \\ f(x), & \text{if } x \in R_X. \end{cases}$$

Clearly, E is continuous at every point in $[0_X, \infty)$ and E is one-to-one. To see that E is continuous at a point $p \in R_X$, take a sequence of points $\{p_n\}_{n=1}^{\infty}$ in $[0_X, \infty)$ such that $\lim p_n = p$. We show that there exists a subsequence $\{p_{n_i}\}_{i=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that $\lim E(p_{n_i}) = E(p)$.

Since $\lim p_n = p \in R_X$, we can take a subsequence $\{p_{n_i}\}_{i=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that $r_i < p_{n_i}$ for each $i \in \mathbb{N}$. Given $i \in \mathbb{N}$, let $j_i \in \mathbb{N}$ be such that $i \leq j_i$ and $p_{n_i} \in [r_{j_i}, r_{j_i+1}]$. Hence, $|e(p_{n_i}) - g(p_{n_i})| \leq 2\delta_{j_i+1}$. By the choice of δ_{j_i+1}, we have that $d_Y(e(p_{n_i}), g(p_{n_i})) < \frac{1}{j_i+1} \leq \frac{1}{i}$. Hence $\lim e(p_{n_i}) = \lim g(p_{n_i}) = \lim G(p_{n_i}) = G(p) = f(p) = E(p)$.

We have shown that E is continuous. Since $E(0_X) = e(r_0) = 0_Y$, the image of E is Y. Therefore X and Y are homeomorphic. □

Question 5.1. *Is Theorem 5.6 still true if we remove the assumption that R_X is an ANR?*

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