ON AUTOMORPHISMS OF ORDER $p$ OF METACYCLIC $p$-GROUPS WITHOUT CYCLIC SUBGROUPS OF INDEX $p$

Yakov Berkovich
University of Haifa, Israel

Abstract. Let $L$ be a metacyclic $p$-group, $p > 2$, without cyclic subgroups of index $p$ and let $a \in \text{Aut}(L)$ be of order $p$. We show that either $a$ centralizes $\Omega_1(L)$ or $p = 3$ and the natural semidirect product $(a) \cdot L$ is of maximal class so the subgroup $L$ has very specific structure. This improves Lemma 4.9 from [MS].

According to [MS, Lemma 4.9], if $p > 3$ is a prime and $a$ is an automorphism of order $p$ of abelian group $L$ of type $(p^2, p^2)$, then $a$ centralizes $\Omega_1(L)$ (the proof of this result is also reproduced in [AS, Lemma A.1.30]). The same conclusion is true provided $L$ is abelian of type $(p^m, p^n)$, $p > 3$ and $m \geq n > 2$ (it suffices to consider the restriction of $a$ on $\Omega_2(L)$). Our aim is to improve this result as follows:

Theorem 1. Suppose that $L$ is a metacyclic $p$-group without cyclic subgroup of index $p$, $p > 2$. An element $a \in \text{Aut}(L)$ of order $p$ does not centralize $\Omega_1(L)$ if and only if $p = 3$ and the natural semidirect product $G = (a) \cdot L$ is a 3-group of maximal class.\textsuperscript{1}

By Theorem 1, if $p > 3$ and $L$ is a metacyclic $p$-group without cyclic subgroup of index $p$, then $\Omega_1(L)$ is centralized by $A$, where $A$ is the subgroup generated by all elements of $\text{Aut}(L)$ of order $p$. We claim that, if $W = A \cdot L$ is the natural semidirect product, then $|W : C_W(L)|$ is a power of $p$. Indeed, if $b$ is a $p'$-element of $W$, then $b$, as an element of $A$, centralizes $\Omega_1(L)$ and so

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\textsuperscript{1}The proof of this theorem shows that if $L$ has a cyclic subgroup of index $p$, then either $G = (a) \cdot L$ is a group of maximal class and order $p^4$ or a group (b2) of Lemma 3(b).

It is known that an outer automorphism of $L$ of order $p$ exists; see, for example, [Hup, Satz III.19.1]
the natural semidirect product \( b \cdot L \) has no minimal non-nilpotent subgroups (see [B2, Theorem 10.8]) so it is nilpotent [I, Theorem 9.18]; in that case \( b \) centralizes \( L \).

Our proof of Theorem 1 uses fairly deep results of finite \( p \)-group theory and so it is essentially differed from the proof of [MS, Lemma 4.9] which is based on intricate computations with elements of \( L \) and the given automorphism \( a \) of \( L \) of order \( p \).

**Corollary 2.** Suppose that \( p > 2 \) and \( L \) is an abelian group of type \( (p^m, p^n) \), \( m > 1, n > 1 \). An element \( a \in \text{Aut}(L) \) of order \( p \) does not centralize \( \Omega_1(L) \) if and only if \( |m - n| \leq 1 \), \( p = 3 \) and the natural semidirect product \( G = \langle a \rangle \cdot L \) is a 3-group of maximal class.

To deduce Corollary 2 from Theorem 1, it suffices to apply Remark 4, below.

We use standard notation of finite \( p \)-group theory (see [B1–B5]).

In Lemma 3 we gathered all known results which are used in what follows.

**Lemma 3.** Let \( G > \{1\} \) be a \( p \)-group.

(a) If \( G \) is regular, then \( \exp(\Omega_1(G)) = p \) and \( |G/\Omega_1(G)| = |\Omega_1(G)| \).

(b) Blackburn; see also [B1, Theorem 6.1]. If \( p > 2 \) and \( G \) has no normal elementary abelian subgroup of order \( p^3 \), then one of the following holds:

(b1) \( G \) is metacyclic.

(b2) \( G = C\Omega_1(G) \), where \( \Omega_1(G) \) is nonabelian of order \( p^3 \) and exponent \( p \) and \( G/\Omega_1(G) \) is cyclic (in particular, \( G/\Omega_1(G) \) is cyclic and \( \Omega_1(G) \leq Z(G) \)).

(b3) \( G \) is a 3-group of maximal class not isomorphic to a Sylow 3-subgroup of the symmetric group of degree \( 3^2 \).

(c) Blackburn; see also [B2, Theorems 9.5 and 9.6]. Let a \( p \)-group \( G \) of maximal class be of order greater than \( p^6 \). Then \( G \) is irregular, \( \Omega_1(\Phi(G)) \) is of order \( p^{p-1} \) and exponent \( p \) and \( |G/\Omega_1(G)| = p^6 \). If, in addition, \( |G| > p^{p+1} \), then there is in \( G \) a unique regular maximal subgroup, say \( G_1 \), and it is absolutely regular; all other maximal subgroups of \( G \) are of maximal class.

(d) A \( p \)-group of maximal class and order \( > p^3 \) has no normal cyclic subgroup of order \( p^2 \), unless \( p = 2 \).

(e) Blackburn; see also [B3, Theorem 7.5]. Suppose that a non-absolutely regular \( p \)-group \( G \) has an absolutely regular maximal subgroup \( H \). Then

\[ \text{A Sylow 3-subgroup of the symmetric group of degree } 3^2 \text{ is the unique 3-group of maximal class that contains an elementary abelian subgroup of order } 3^3. \]

\[ \text{A } p \text{-group } X \text{ is absolutely regular if } |X/\Omega_1(X)| < p^p; \text{ then } X \text{ is regular, by Hall’s regularity criterion [B2, Theorem 9.8(a)]}. \] It follows that, if \( p > 2 \), then metacyclic \( p \)-groups are absolutely regular.
either $G$ is irregular of maximal class or $G = H\Omega_1(G)$, where $\Omega_1(G)$ is of order $p^2$ and exponent $p$.

(f) Blackburn; see also [J, Theorem 7.1] and [BJ, Theorem 7.1]. If a 2-group $G$ is minimal nonmetacyclic, then $G$ is one of the following groups: (i) $E_8$, (ii) $Q_8 \times C_2$, (iii) $D_8 \times C_4$ (central product) of order 16.

(g) [B4, Proposition 19(a)]. If $B$ is a nonabelian subgroup of order $p^3$ of a $p$-group $G$ such that $C_G(B) < B$, then $G$ is of maximal class.

(h) If a metacyclic $p$-group $G$ has a nonabelian subgroup $B$ of order $p^3$, then either $G$ is a 2-group of maximal class or $G = B$.

(i) [BJ, Lemma 3.2(a)]. If $G$ is a nonabelian two-generator $p$-group and $G' \leq \Omega_1(Z(G))$, then $G$ is minimal nonabelian.

(j) Blackburn; see also [B3, Theorem 7.6]. If a $p$-group $G$ has no normal subgroup of order $p^6$ and exponent $p$, then it is either absolutely regular or of maximal class.

(k) Huppert; see also [B5, Corollary 13]. If $p > 2$ and $G$ is such that $|G/\Omega_1(G)| \leq p^2$, then $G$ is metacyclic.

(l) Redei ([R]); see also [B2, Exercise 1.8a]. If $G$ is a metacyclic minimal nonabelian $p$-group of order $p^m$, then either $G \cong Q_8$ or $G = \langle a, b \mid a^{p^m} = b^p = 1, a^k = a^{1+p^{m-1}} \rangle$. If $G$ is nonmetacyclic minimal nonabelian of order $> p^3$, then $\Omega_1(G) \cong E_{p^3}$.

Let us prove Lemma 3(d). Let $p > 2$ and $X$ a $p$-group of maximal class and order $> p^3$. Then $X$ has only one normal subgroup of order $p^2$; since this subgroup is abelian of type $(p, p)$ [B2, Lemma 1.4], we are done.

Let us prove Lemma 3(h). Assume that $|G| > p^3$ and $C_G(B) \not\leq B$, where $B$ is nonabelian of order $p^3$ and $G$ is metacyclic. If $F \leq C_G(B)$ is of order $p^2$, then $d(BF) > 2$ so $BF$ is a nonmetacyclic subgroup of a metacyclic group $G$, a contradiction. Thus, $C_G(B) < B$. Then, by Lemma 3(g), $G$ is of maximal class so $|G : G'| = p^2$ which is impossible for metacyclic $p$-groups of order $> p^3$ with $p > 2$; in case $p = 2$, our $G$ is of maximal class (Taussky).

**Remark 4** (Blackburn). Suppose that $G$ is a 3-group of maximal class and order $> 3^4$ and $G_1 < G$ is absolutely regular; then $G_1$ is noncyclic (Lemma 3(c)) and metacyclic (Lemmas (c,k)). Assume that $G_1$ has a cyclic subgroup of index 3. In that case, $\Omega_2(\Omega_1(G_1))$ is cyclic of order $3^2$, contrary to Lemma 3(d). Suppose that $G_1$ is abelian of type $(3^m, 3^n)$ with $m > n$. Then $\Omega_n(G_1)$ is $G$-invariant and cyclic of order $3^{m-n}$ so $m - n \leq 1$ (Lemma 3(d)). Now suppose that $G_1$ is nonabelian. Then $G_1'$ is cyclic and $G$-invariant so $|G_1'| = 3$ (Lemma 3(d)). In that case, $G_1$ is minimal nonabelian and $G_1 = \langle a, b \mid a^{3^m} = b^{3^n} = 1, a^b = a^{1+3^{m-1}} \rangle$ (Lemma 3(i,l)). The center $Z(G_1)$ is abelian of type $(3^{m-1}, 3^{n-1})$ and $G$-invariant. Let $k = \min \{m - 1, n - 1\}$. Then $\Omega_k(Z(G_1))$ is $G$-invariant and cyclic of order $3^{m-n}$ so $|m - n| \leq 1$ (Lemma 3(d)).
Let $G$ be a 3-group of maximal class and order $> 3^4$ and let $L < G$ be absolutely regular maximal subgroup of $G$ (Lemma 3(c)). By Remark 4, $L$ is either abelian or minimal nonabelian; in addition, $L$ has no cyclic subgroup of index 3. In any case, the abelian subgroup $\Omega_1(L)$ of type $\{3, 3\}$ is contained in $Z(L)$ (see Lemma 3(l)) so $C_G(\Omega_1(L)) = L$ since $|Z(G)| = 3$. Therefore, if $x \in G - L$ is of order 3 (note that, in general, such $x$ need not exist), then $x$ does not centralize $\Omega_1(L)$, and then such pair $\{x, L\}$ satisfies the hypothesis of Theorem 1.

**Proof of Theorem 1.** By Remark 4 and the paragraph following the remark, it suffices to prove that the natural semidirect product $G = \langle a \rangle \cdot L$ is a 3-group of maximal class (obviously, this semidirect product is not metacyclic). We have $|L| \geq p^4$ since the metacyclic subgroup $L$ has no cyclic subgroup of index $p$.

Suppose that an element $a \in \text{Aut}(L)$ of order $p$ does not centralize $\Omega_1(L)$. Let $G$ be defined as in the previous paragraph. By Lemma 3(a), $\Omega_1(L)$ and $L/\Omega_1(L)$ are abelian of type $(p, p)$, and $\Omega_1(L) \triangleleft G$. Since $p > 2$, the subgroup $H = \langle a, \Omega_1(L) \rangle$ is nonabelian of order $p^3$ and exponent $p$, by assumption. We have $G = LH$ since $H \not\leq L$ and $L$ is maximal in $G$. Clearly, $G$ has no subgroup of order $p^3$ and exponent $p$ (otherwise, the intersection of that subgroup with $L$ will be of order $> p^2$ and exponent $p$, which is impossible).

Assume that $G$ is regular. Then $\exp(\Omega_1(G)) = p$ (Lemma 3(a)) so, by the previous paragraph, $|\Omega_1(G)| = p^3 = |H|$ hence $\Omega_1(G) = H$. It follows that $G$ has no elementary abelian subgroup of order $p^3$ so $G$ is as in part (b2) of Lemma 3(b) (the group (b3) of Lemma 3(b) is irregular, by Lemma 3(c)). In that case, however, every metacyclic subgroup of that group has a cyclic subgroup of index $p$, contrary to the hypothesis.

Thus, $G$ is irregular. In view of Remark 4 and the paragraph following it, one may assume that $G$ is not a 3-group of maximal class. It follows from Lemma 3(c) that $G$ is not of maximal class for all $p > 3$ (indeed, $\Phi(G) < L$ and $\Omega_1(\Phi(G))$ is of exponent $p$ and order $p^{p−1} > p^2 = |\Omega_1(L)|$). As we have noticed, $L$ is absolutely regular. Therefore, by Lemma 3(c), $G = LH_1(G)$, where $\Omega_1(G)$ is of order $p^2$ and exponent $p$. Since $L \cap \Omega_1(G) = \Omega_1(L)$ is abelian of order $p^2$, we get $p = 3$. It follows that $\Omega_1(G) = H = \langle x, \Omega_1(L) \rangle$ is nonabelian of order $p^3$ and exponent $p$ so $G$ has no elementary abelian subgroup of order $p^3$. In that case, $G$ is an irregular 3-group of maximal class (since, as we have noticed, any group of part (b2) of Lemma 3(b) has no such a subgroup as $L$), contrary to the assumption.

**Remark 5.** Here we consider a similar, but more complicated, situation for $p = 2$. Suppose that a metacyclic 2-group $L$ without cyclic subgroups of index 2 is maximal in a 2-group $G$; then $\Omega_1(L)$ is a $G$-invariant four-subgroup (this follows immediately from Lemma 3(h)), and so $G$ is not of maximal class. Let, in addition, $\Omega_1(L) \leq Z(L)$. Suppose that there is an involution
a ∈ G − L that does not centralize Ω₁(L). Since ⟨x, Ω₁(L)⟩ ∼= D₈, it follows that G is not metacyclic (otherwise, G is of maximal class, by Lemma 3(h)). By hypothesis, C_G(Ω₁(L)) = L. If E < G is elementary abelian of order 8, then L ∩ E = Ω₁(L) so C_G(Ω₁(L)) ≥ LE = G, a contradiction. Let H be a minimal nonmetacyclic subgroup of G; then H ⊈ L. Since H has no subgroup ∼= E₈, we get |H| > 8 and exp(H) = 4 (Lemma 3(f)). If Z(H) ∼= E₄, then Z(H) is contained in every abelian subgroup of H of order ≥ 8 (Lemma 3(f)) so, since H ∩ L contains an abelian subgroup of order 8 (Lemma 3(h)), we get Z(H) = Ω₁(L) and C_G(Ω₁(L)) ≥ HL = G, a contradiction. Thus, Z(H) is cyclic so, by Lemma 3(f), H ∼= D₈ * C₄ is of order 16. A similar argument shows that if A < G and A ⊈ L is minimal nonabelian, then A has a cyclic subgroup of index 2. Indeed, A is metacyclic (Lemma 3(l)) so, if |A| > 8, then |Ω₁(A)| ≤ 4 and, if Ω₁(A) ∼= E₄, then Ω₁(A) ≤ Z(A) = Φ(A)(≤ L) so Φ(A) = Ω₁(A) is cyclic.

Now we construct a group G = ⟨a, L⟩, where a ∈ G − L is an involution and L is metacyclic without cyclic subgroups of index 2 and such that Ω₁(L) ≤ Z(L) and Ω₁(L) ⊈ Z(G). Let G = Z wr C (wreath product), where Z is cyclic of order 2ⁿ > 2 and C = ⟨a⟩ is of order 2; then |G| = 2ⁿ⁺¹ and Z(G) is cyclic of order |Z| = 2ⁿ. Let L = Z × Zⁿ be the base of the wreath product G. We see that a does not centralize Ω₁(L) and L is abelian of type (2ⁿ, 2ⁿ).

Suppose that an abelian 2-group L of type (2ⁿ, 2), n > 2, is maximal in a 2-group G = ⟨a, L⟩ and involution a does not centralize Ω₁(L). Then H = ⟨a, Ω₁(L)⟩ ∼= D₈. We have C_G(Ω₁(L)) = L so G has no subgroups ∼= E₈ (see Remark 5). Let Z < L be cyclic of index 2. We claim that H ∩ Z = Z(H). Indeed, H ∩ L = Ω₁(L) is abelian of type (2, 2) so cyclic H ∩ Z < H ∩ L, and our claim follows, since Ω₁(Z) ∼= G (consider the kernel of representation of G by permutations of left cosets of Z and take into account that |G : Z| = 4 and n > 2). Thus, G = HZ, by the product formula. By the modular law, H * Ω₂(C_G(H)) is minimal nonmetacyclic of order 2ⁿ. Assume that M < G is minimal nonmetacyclic; then M is nonabelian, 2¹ < |M| ≤ 2² and exp(M) = 4 (Lemma 3(f)) so M ∩ L (of order > 4) is abelian noncyclic (Lemma 3(g)). It follows that Ω₁(L) < M ∩ L so, if Z(M) is noncyclic, we get Z(M) ≤ M ∩ L (otherwise, Ω₁(L) = Z(M) ≤ Z(G), a contradiction). It follows from Lemma 3(f) that Z(M) is cyclic so M ∼= D₈ * C₄. As in Remark 5, if A < G is minimal nonabelian, then A has a cyclic subgroup of index 2.

Suppose that a nonmetacyclic subgroup U is maximal in a p-group G = ⟨x, U⟩, where o(x) = p > 2 and Ω₁(U) ∼= E_p²; then p = 3 and U is of maximal class and order > 3³ (Lemma 3(b)). Suppose, in addition, that there is an element of order 3 in G − U, and all such elements do not centralize Ω₁(U) (if there are no such elements, then G is of maximal class, by the same Lemma 3(b)). Then C_G(Ω₁(U)) = L is maximal in G since Ω₁(U) ∉ Z(U), and G has no elementary abelian subgroups of order 3³. Therefore, L is metacyclic,
and $G$ is as in parts (b2) or (b3) of Lemma 3(b). However, a group of part (b2) has no maximal subgroup such as $U$. Thus, $G$ is a 3-group of maximal class, and $L$ is such as the subgroup $G_1$ in Remark 4.

**References**


Y. Berkovich
Department of Mathematics
University of Haifa
Mount Carmel, Haifa 31905
Israel

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