

## CONSTRUCTING NEAR-EMBEDDINGS OF CODIMENSION ONE MANIFOLDS WITH COUNTABLE DENSE SINGULAR SETS

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ABSTRACT. We present, for all  $n \geq 3$ , very simple examples of continuous maps  $f: M^{n-1} \rightarrow M^n$  from closed  $(n-1)$ -manifolds  $M^{n-1}$  into closed  $n$ -manifolds  $M^n$  such that even though the singular set  $S(f)$  of  $f$  is countable and dense, the map  $f$  can nevertheless be approximated by an embedding, i.e.  $f$  is a *near-embedding*. In dimension 3 one can get even a piecewise-linear approximation by an embedding.

### 1. INTRODUCTION

Denote the *singular* set of an arbitrary continuous mapping  $f: X \rightarrow Y$  between topological spaces by  $S(f) = \{x \in X \mid f^{-1}(f(x)) \neq x\}$ . A manifold which is connected, compact and has no boundary is said to be *closed*. The following conjecture was proposed by the first author in the mid 1980's:

CONJECTURE 1.1. *Let  $f: M^{n-1} \rightarrow M^n$  be any continuous (possibly surjective) map from a closed  $(n-1)$ -manifold  $M^{n-1}$  into a closed  $n$ -manifold  $M^n$ ,  $n \geq 3$ , such that  $\dim S(f) = 0$ . Then  $f$  is a near-embedding, i.e., for every  $\varepsilon > 0$  there exists an embedding  $g: M^{n-1} \rightarrow M^n$  such that for every  $x \in M^{n-1}$ ,  $d(f(x), g(x)) < \varepsilon$ .*

For  $n = 3$  it has since been shown to be equivalent to the Bing Conjecture from the 1950's (cf. [3]) and it is closely related to the 3-dimensional Recognition Problem, one of the central problems of geometric topology (cf. [8]). In particular, it is closely related to the general position of 3-manifolds, called the

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*light map separation property LMSP\**, introduced by Daverman and Repovš (cf. [7, Conjecture 5.4]).

In the case when  $n = 3$ , a very special case of Conjecture 1.1 was verified by Anderson ([1]) in 1965. Then in 1992 Brahm ([4]) proved Conjecture 1.1 for the case when  $n = 3$  and the closure of the singular set is 0-dimensional,  $\dim(\text{Cl}S(f)) = 0$ . However, in general the second property needs not be satisfied – it was shown in [5] that for  $n = 3$  it can happen that  $\dim(\text{Cl}S(f)) = n - 1$ .

Since the construction in [5] is very technical, there has been for a long time an open question if there is an *elementary* example of a continuous map  $f : M^2 \rightarrow M^3$  such that  $\dim S(f) = 0$  whereas  $0 < \dim \text{Cl}(S(f)) \leq 3$ . The purpose of this note is to present such an example – it has a very simple construction and the verification of all asserted properties is straightforward. Moreover, unlike [5], our methods evidently generalize in a direct manner to yield continuous maps  $f : M^{n-1} \rightarrow M^n$  of closed codimension one manifolds into closed  $n$ -manifolds, with properties analogous to (i) and (ii) below, for every  $n \geq 3$ .

**THEOREM 1.2.** *For every  $n \geq 3$ , there exists a continuous map  $f : S^{n-1} \rightarrow S^n$  such that:*

- (i) *the singular set  $S(f)$  of  $f$  is countable and dense (hence 0-dimensional and nonclosed); and*
- (ii)  *$f$  is a near-embedding.*

*Remark.* A question when a light map is a near-embedding is also interesting in view of the classical *Monotone-Light Factorization Theorem* (cf. e.g. [9]) which asserts that every continuous mapping  $f : X \rightarrow Y$  from any compact space  $X$  to any space  $Y$  can be factorized as a product  $f = l \circ m$  of a monotone map  $m : X \rightarrow Z$  (i.e. each point inverse  $m^{-1}(z)$  is connected) and a light map  $l : Z \rightarrow Y$  (i.e. each point inverse  $l^{-1}(y)$  is totally disconnected).

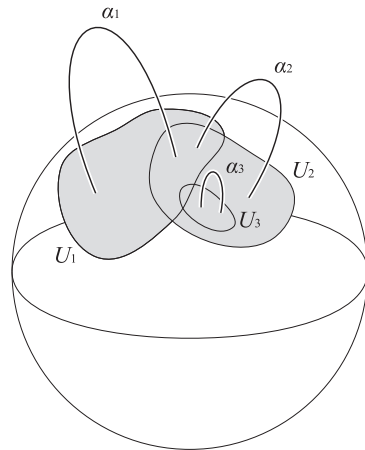
## 2. PROOF OF THEOREM 1.2

Let  $n \geq 3$  and choose a countable basis  $\{U_i\}_{i \in \mathbb{N}}$  of open sets for  $S^{n-1}$  (which is considered as the standardly embedded  $(n - 1)$ -sphere in  $S^n$ ). We shall inductively construct a sequence of pairwise disjoint *tame* PL arcs  $\alpha_i$  in  $S^n$  (i.e. for every  $i \geq 1$  there is a homeomorphism  $h_i : S^n \rightarrow S^n$  such that  $h_i(\alpha_i) \subset S^1 \subset S^n$ ) with the property that:

1. for every  $i$ ,  $\alpha_i \cap S^{n-1} = \partial\alpha_i \subset U_i$ ;
2. for every  $i$ ,  $\text{diam}(\alpha_i) < \frac{1}{2^i}$ .

Begin with a tame PL arc  $\alpha_1 \subset S^n$  such that  $\partial\alpha_1 \subset U_1$  and  $\text{diam}(\alpha_1) < 1/2$ . Assume inductively, that we have already constructed pairwise disjoint tame PL arcs  $\alpha_1, \dots, \alpha_{n-1} \subset S^n$  with all required properties. We can then

clearly find a tame PL arc  $\alpha_n \subset S^n$  such that  $\partial\alpha_n \subset U_n$  and  $\alpha_n$  is disjoint with  $\alpha_1 \cup \dots \cup \alpha_{n-1}$ .



By our construction,  $\{\alpha_n\}_{n \in \mathbb{N}}$  is a *null-sequence*, i.e.  $\lim_{i \rightarrow \infty} \text{diam } \alpha_i = 0$ . The decomposition  $G = \{\alpha_i\}_{i \in \mathbb{N}}$  of  $S^n$  into points and arcs is clearly *cellular* (i.e., each element of the decomposition  $G$  is the intersection of a *nested* sequence of closed  $n$ -cells  $\{B_k^n\}_{k \in \mathbb{N}}$  in  $S^n$ ) (i.e., for ever  $k, B_{k+1}^n \subset \text{Int} B_k^n$ ) and *upper semicontinuous* (i.e. the quotient map  $\pi : S^n \rightarrow S^n/G$  is closed).

Therefore it follows by [6, Theorem 7, page 56] that the decomposition  $G$  is *shrinkable* (i.e., the map  $\pi$  is approximable by homeomorphisms). In particular, the quotient space  $S^n/G$  is homeomorphic to  $S^n$ . The desired mapping  $f : S^{n-1} \rightarrow S^n$  is now defined as the compositum  $f = \pi \circ i$  of the inclusion  $i : S^{n-1} \rightarrow S^n$  and the decomposition quotient mapping  $\pi : S^n \rightarrow S^n/G$ .

It follows by construction that the singular set  $S(f) = \bigcup_i \partial\alpha_i$  is countable and dense. It is also clear that this map can be approximated arbitrarily closely by embeddings of  $S^{n-1}$  into  $S^n$  (by not shrinking the arcs all the way but only as much as it is necessary to make them sufficiently small).

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