ON THE ORDER STRUCTURE ON THE SET OF COMPLETELY MULTI-POSITIVE LINEAR MAPS ON $C^*$-ALGEBRAS

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Abstract. In this paper we characterize the order relation on the set of all completely $n$-positive linear maps on $C^*$-algebras in terms of the representation associated to each completely $n$-positive linear map given by Suen’s construction.

1. Introduction and Preliminaries

Completely positive linear maps are an often used tool in operator algebras theory and quantum information theory [1, 3, 5, 7, 10].

In the mathematical framework of quantum information theory, all admissible devices are modelled by the so-called quantum operations (that is, completely positive linear maps on the algebra of observables ($C^*$-algebra) of the physical system under consideration). A good analysis of completely multi-positive maps between $C^*$-algebras involves understanding and solving certain problems in quantum information theory and understanding the infinite dimensional non-commutative structure of topological $*$-algebras [2, 5, 7, 10]. The theorems on the structure of completely linear maps and Radon-Nikodym type theorems for completely positive linear maps are an extremely powerful and veritable tool for problems involving characterization and comparison of quantum operations.

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Given a $C^*$-algebra $A$ and a positive integer $n$, we denote by $M_n(A)$ the $C^*$-algebra of all $n \times n$ matrices over $A$ with the algebraic operations and the topology obtained by regarding it as a direct sum of $n^2$ copies of $A$.

**Definition 1.1.** A linear map $\rho: A \to B$ between two $C^*$-algebras is completely positive if the linear maps $\rho^{(n)}: M_n(A) \to M_n(B)$ defined by

$$\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$$

are positive for any positive integer $n$.

**Definition 1.2.** Let $A$ and $B$ be two $C^*$-algebras. An $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ of linear maps from $A$ to $B$ can be regarded as a linear map $\rho$ from $M_n(A)$ to $M_n(B)$ defined by

$$\rho([a_{ij}]_{i,j=1}^n) = [\rho_{ij}(a_{ij})]_{i,j=1}^n.$$\n
We say that $[\rho_{ij}]_{i,j=1}^n$ is a completely $n$-positive linear map from $A$ to $B$ if $\rho$ is a completely positive linear map from $M_n(A)$ to $M_n(B)$.

We shall denote by $\text{CP}_n(A, B)$ the set of all completely positive linear maps from $A$ to $B$ and by $\text{CP}_n(A, B)$ the set of all completely $n$-positive linear maps from $A$ to $B$.

In [9], Suen showed that any completely $n$-positive linear map from a $C^*$-algebra $A$ to $L(H)$, the $C^*$-algebra of all bounded linear operators on a Hilbert space $H$, is of the form $[V^*T_1\Phi(\cdot)V]_{i,j=1}^n$, where $\Phi$ is a representation of $A$ on a Hilbert space $K$, $V \in L(H,K)$ and $[T_1]_{i,j=1}^n$ is a positive element in $M_n(\Phi(A)^\prime)$ ($\Phi(A)^\prime$ denotes the commutant of $\Phi(A)$ in $L(K)$).

**Theorem 1.3** ([9, 4]). Let $A$ be a $C^*$-algebra, let $H$ be a Hilbert space and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely $n$-positive linear map from $A$ to $L(H)$. Then there is a representation $\Phi_\rho$ of $A$ on a Hilbert space $H_\rho$, $V_\rho \in L(H_\rho)$ and a positive element $T_\rho = [T_\rho_{ij}]_{i,j=1}^n$ in $M_n(\Phi_\rho(A)^\prime)$ with $\sum_{k=1}^n T_{kk}^\rho = n\text{id}_{L(H_\rho)}$ such that:

i. $\{\Phi_\rho(a)V_\rho \xi; a \in A, \xi \in H\}$ spans a dense subspace in $H_\rho$;

ii. $\rho_{ij}(a) = V_\rho^* T_\rho_{ij} \Phi_\rho(a)V_\rho$, for all $a \in A$ and for all $i, j = 1, \ldots, n$.

The quadruple $(\Phi_\rho, H_\rho, V_\rho, T_\rho)$ will be called the Suen’s construction associated with $\rho$ and it is unique up to unitary equivalence [4, Theorem 2.3].

**Remark 1.4.** The triple $(\Phi_\rho, H_\rho, V_\rho)$ is the Stinespring representation associated with $\tilde{\rho} = \frac{1}{n} \sum_{k=1}^n \rho_{kk}$ (see, [4, the proof of Theorem 2.3]).

In this paper we characterize the order relation on the set of all completely $n$-positive linear maps on $C^*$-algebras in terms of the representation associated to each completely $n$-positive linear map given by Suen’s construction [9].
We also give sufficient conditions for that a completely $n$-positive linear map from a unital $C^*$-algebra $A$ to $L(H)$ to be an extreme point in the set of all completely $n$-positive linear maps $[\rho_{ij}]_{i,j=1}^n$ from $A$ to $L(H)$ such that $[\rho_{ij}(1_A)]_{i,j=1}^n = T^0$ for some $T^0 \in M_n(L(H))$.

2. The main results

Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $C_{\infty,\sum}^n(A, L(H))$ and let $(\Phi_\rho, H_\rho, V_\rho, T^\rho)$ be the construction associated to $\rho$ given by Theorem 1.3.

Lemma 2.1. Let $S = [S_{ij}]_{i,j=1}^n$ be a positive element in $M_n(\Phi_\rho(A'))$. The map $\rho_S = [\rho_{S,ij}]_{i,j=1}^n$ from $M_n(A)$ to $M_n(L(H))$ defined by

$$\rho_S([a_{ij}]_{i,j=1}^n) = [V_\rho^* S_{ij} \Phi_\rho(a_{ij}) V_\rho]_{i,j=1}^n$$

is a completely $n$-positive linear map from $A$ to $L(H)$.

Proof. It is not difficult to see that $\rho_S$ is an $n \times n$ matrix of linear maps from $A$ to $L(H)$ whose $(i,j)$-entry is the linear map $\rho_{S,ij}$ from $A$ to $L(H)$ defined by $\rho_{S,ij}(a) = V_\rho^* S_{ij} \Phi_\rho(a) V_\rho$ for all $a \in A$ and for all $i,j = 1, \ldots, n$.

To show that $\rho_S$ is a completely $n$-positive linear map from $A$ to $L(H)$ it is sufficient to show that $\Gamma(\rho_S) \in C_{\infty,\sum}(A, M_n(L(H)))$, where $\Gamma$ is the map from $C_{\infty,\sum}^n(A, B)$ onto $C_{\infty,\sum}(A, M_n(B))$ defined by $\Gamma([\rho_{ij}]_{i,j=1}^n)(a) = [\rho_{ij}(a)]_{i,j=1}^n$ for all $a \in A$ [2, Theorem 1.4]. For this, let $m$ be a positive integer, $a_1, \ldots, a_m \in A, \xi_1 = (\xi_i^1)_{i=1}^n, \ldots, \xi_m = (\xi_i^m)_{i=1}^n \in H^n$. Then we have

$$\sum_{k,l=1}^m (\Gamma(\rho_S)(a_i^a_1 a_k) \xi_k, \xi_l) = \sum_{k,l=1}^m \langle [V_\rho^* S_{ij} \Phi_\rho(a_i^a_1 a_k) V_\rho]_{i,j=1}^n (\xi_i^k), (\xi_i^l)_{i=1}^n \rangle$$

$$= \sum_{k,l=1}^m \sum_{i,j=1}^n \langle V_\rho^* S_{ij} \Phi_\rho(a_i^a_k) V_\rho \xi_i^j, \xi_l^j \rangle$$

$$= \sum_{i,j=1}^n \sum_{k=1}^m \langle S_{ij} \sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_i^j, \sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_l^j \rangle$$

$$= \langle [S_{ij}]_{i,j=1}^n \sum_{k=1}^m (\sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_i^j)_{i=1}^n, (\sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_l^j)_{i=1}^n \rangle \geq 0$$

since $[S_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi_\rho(A'))$. From this fact we conclude that $\Gamma(\rho_S) \in C_{\infty,\sum}(A, M_n(L(H)))$ and the lemma is proved.

Remark 2.2. It is not difficult to check that:

1. $\rho T^\rho = \rho$;
2. $\rho_{\alpha S} = \alpha \rho_S$, for all positive numbers $\alpha$ and for all positive elements $S$ in $M_n(\Phi_\rho(A'))$;
3. $\rho_{S_1 + S_2} = \rho_{S_1} + \rho_{S_2}$, for all positive elements $S_1, S_2$ in $M_n(\Phi_\rho(A'))$;
4. $\rho_{S_1} \leq \rho_{S_2}$ if and only if $S_1 \leq S_2$, where $S_1, S_2$ are positive elements in $M_n(\Phi_\rho(A'))$.

Let $\rho, \theta \in CP^\infty_n(A, L(H))$. We say that $\rho$ dominates $\theta$, and we write $\theta \leq \rho$, if $\rho - \theta \in CP^\infty_n(A, L(H))$.

For $\rho \in CP^\infty_n(A, L(H))$, we put:

$$[0, \rho] = \{ \theta = [\theta_{ij}]_{i,j=1}^n \in CP^\infty_n(A, L(H)) : \theta \leq \rho \}$$

and

$$[0, T^\rho] = \{ S = [S_{ij}]_{i,j=1}^n \in M_n(\Phi_\rho(A')) : 0 \leq S \leq T^\rho \}.$$  

**Theorem 2.3.** The map $S \rightarrow \rho_S$ is an affine order isomorphism from $[0, T^\rho]$ to $[0, \rho]$.

**Proof.** By Lemma 2.1 and Remark 2.2, the map $S \rightarrow \rho_S$ from $[0, T^\rho]$ to $[0, \rho]$ is well-defined and moreover, it is affine.

To show that the map is injective, let $S = [S_{ij}]_{i,j=1}^n$ be an element in $[0, T^\rho]$ such that $\rho_S = 0$. Then $[\rho_S]_{i,j=1}^n = 0$, that is $V^*_\rho S_{ij} \Phi_\rho(a)V_\rho = 0$, for all $a \in A$ and for all $i,j = 1, \ldots, n$.

For each $a, b \in A, \xi, \eta \in H$ and $i,j = 1, \ldots, n$, we have

$$\langle S_{ij} \Phi_\rho(a)V_\rho \xi, \Phi_\rho(b)V_\rho \eta \rangle = \langle V^*_\rho \Phi_\rho(b)^* S_{ij} \Phi_\rho(a)V_\rho \xi, \eta \rangle = \langle V^*_\rho S_{ij} \Phi_\rho(b^*a)V_\rho \xi, \eta \rangle = 0.$$

From this fact and taking into account that $\{ \Phi_\rho(a)V_\rho \xi : a \in A, \xi \in H \}$ spans a dense subspace of $H_\rho$, we conclude that $S_{ij} = 0$. Hence $S = 0$ and the map $S \rightarrow \rho_S$ is injective.

It remains to show that the map $S \rightarrow \rho_S$ from $[0, T^\rho]$ to $[0, \rho]$ is surjective.

Let $\sigma = [\sigma_{kl}]_{k,l=1}^n$ be an element in $[0, \rho]$. By [4, the proof of Theorem 2.3] (see also [6]),

$$\frac{1}{n} \sigma_{kk} + \frac{1}{n} \Re \sigma_{kl} \pm \frac{1}{n} \Im \sigma_{kl} \in [0, \tilde{\rho}],$$

where $\tilde{\rho} = \frac{1}{n} \sum_{j=1}^n \rho_{jj}$ and $\tilde{\sigma} = \frac{1}{n} \sum_{j=1}^n \sigma_{jj}$, for all $k, l = 1, \ldots, n$ with $k \neq l$. Let $(\Phi_\rho, H_\rho, V_\rho, T^\rho)$ be the Stinespring construction associated with $\rho$. By Remark 1.4, $(\Phi_\rho, H_\rho, V_\rho)$ is the Stinespring representation of $A$ associated with $\tilde{\rho}$. Then by [1, Theorem 1.4.6], for each $j = 1, \ldots, n$, there is a positive element $S_{jj} \in \Phi_\rho(A')$ such that

$$\sigma_{jj}(a) = V^*_\rho S_{jj} \Phi_\rho(a)V_\rho$$

for all $a \in A$ and for all $k, l = 1, \ldots, n$ with $k \neq l$, there are two positive elements $S_{kl}^1, S_{kl}^2 \in \Phi_\rho(A')$ such that

$$\frac{n}{2} \sigma(a) + (\Re \sigma_{kl})(a) = V^*_\rho S_{kl}^1 \Phi_\rho(a)V_\rho + V^*_\rho S_{kl}^2 \Phi_\rho(a)V_\rho.$$
and \[ \frac{n}{2} \tilde{\sigma}(a) + \text{Im} \sigma_{kl}(a) = V^*_\rho S^2_{kl} \Phi_\rho(a) V_\rho \]
for all \( a \in A \).

From these relations, we deduce that \( \sigma_{kl}(a) = V^*_\rho S_{kl} \Phi_\rho(a) V_\rho \) for all \( a \in A \), where
\[ S_{kl} = S^1_{kl} + i S^2_{kl} - \frac{1 + i}{2} \sum_{j=1}^n S_{ij}. \]

Clearly \( S = [S_{ij}]_{i,j=1}^n \in M_n(\Phi_\rho(A)') \). Moreover, \( S \) is positive (see, for example, [4, the proof of Theorem 2.3]) and \( \sigma = \rho_S \). Since \( \sigma \leq \rho \), by Remark 2.2, \( S \in [0, T^\rho] \) and the theorem is proved. \( \square \)

**Definition 2.4.** Let \( A \) be a \( C^* \)-algebra and let \( H \) be a Hilbert space. A completely \( n \)-positive linear map \( \rho = [\rho_{ij}]_{i,j=1}^n \) from \( A \) to \( L(H) \) is said to be pure if for every completely \( n \)-positive linear map \( \theta = [\theta_{ij}]_{i,j=1}^n \in [0, \rho] \), there is a positive number \( \alpha \) such that \( \theta = \alpha \rho \).

**Proposition 2.5.** Let \( \rho = [\rho_{ij}]_{i,j=1}^n \) be an element in \( CP^*_n(A, L(H)) \). Then \( \rho \) is pure if and only if \( [0, T^\rho] = \{ \alpha T^\rho; 0 \leq \alpha \leq 1 \} \).

**Proof.** First we suppose that \( \rho \) is pure. Let \( S = [S_{ij}]_{i,j=1}^n \) be an element in \( [0, T^\rho] \). By Theorem 2.3, \( \rho_S \in [0, \rho] \) and since \( \rho \) is pure, there is a positive number \( \alpha \) such that \( \rho_S = \alpha \rho \). From this fact, Remark 2.2 and Theorem 2.3, we deduce that \( S = \alpha T^\rho \) for some \( 0 \leq \alpha \leq 1 \).

Conversely, suppose that \( [0, T^\rho] = \{ \alpha T^\rho; 0 \leq \alpha \leq 1 \} \). Let \( \theta = [\theta_{ij}]_{i,j=1}^n \) be an element in \( [0, \rho] \). By Theorem 2.3, there is \( S \in [0, T^\rho] \) such that \( \rho_S = \theta \) and since \( S = \alpha T^\rho \) for some positive number \( \alpha \), \( \theta = \alpha \rho \) and the proposition is proved. \( \square \)

Let \( A \) be a unital \( C^* \)-algebra, let \( H \) be a Hilbert space and \( \rho = [\rho_{ij}]_{i,j=1}^n \in CP^*_n(A, L(H)) \). We denote by \( CP^*_n(A, L(H), T^0) \), where
\[ T^0 = \text{diag}(V^*_\rho, \ldots, V^*_\rho)T^\rho \text{diag}(V_\rho, \ldots, V_\rho), \]
the set of all completely \( n \)-positive linear maps \( \sigma = [\sigma_{ij}]_{i,j=1}^n \) from \( A \) to \( L(H) \) such that \( \sigma_{ij}(1_A) = (T^0)_{ij} \), for all \( i, j = 1, \ldots, n \). Clearly, \( CP^*_n(A, L(H), T^0) \) is a convex set.

**Proposition 2.6.** Let \( \rho = [\rho_{ij}]_{i,j=1}^n \) be an element in \( CP^*_n(A, L(H), T^0) \) and let \( P_{H_0} \) be the projection on the closed subspace \( H_0 \) of \( H_\rho \) generated by \( \{ V_\rho \xi; \xi \in H \} \). If the map \( S \mapsto \text{diag}(P_{H_0}, \ldots, P_{H_0})S \text{diag}(P_{H_0}, \ldots, P_{H_0}) \) from \( M_n(\Phi_\rho(A)') \) to \( M_n(L(H_{\rho})) \) is injective then \( \rho \) is an extreme point in \( CP^*_n(A, L(H), T^0) \).
Proof. Let \( \theta, \sigma \) be elements in \( CP^n_\infty(A, L(H), T^0) \) and \( \alpha \in (0, 1) \) such that \( \alpha \theta + (1 - \alpha) \sigma = \rho \). Since \( \alpha \theta \in [0, \rho] \), by Theorem 2.3 there is a positive element \( S \) in \( M_n(\Phi_\rho(A)' \big) \) such that \( \alpha \theta = \rho_S \). Then

\[
\langle P_{H_0}(S_{ij} - \alpha T_{ij}^0)P_{H_0} V_\rho \xi, V_\rho \eta \rangle = \langle S_{ij} V_\rho \xi, V_\rho \eta \rangle - \alpha \langle T_{ij}^0 V_\rho \xi, V_\rho \eta \rangle
\]

\[
= \alpha \langle \theta_{ij}(1, \xi, \eta) \rangle - \alpha \langle \rho_{ij}(1, \xi, \eta) \rangle = 0,
\]

for all \( \xi, \eta \in H \) and for all \( i, j = 1, \ldots, n \).

From this fact we deduce that \( P_{H_0}(S_{ij} - \alpha T_{ij}^0)P_{H_0} = 0 \) for all \( i, j = 1, \ldots, n \) and since the map \( S \rightarrow \text{diag}(P_{H_0}, \ldots, P_{H_0}) \) from \( M_n(\Phi_\rho(A)') \) to \( M_n(L(H_0)) \) is injective, \( S = \alpha T^0 \). Thus we showed that \( \theta = \rho \) and so \( \rho \) is an extreme point in \( CP^n_\infty(A, L(H), T^0) \). \( \square \)

By Remark 1.4, \( (\Phi_\rho, H_\rho, V_\rho) \) is the Sinespring representation of \( A \) associated to \( \tilde{\rho} \). If \( \rho = [\rho_{ij}]_{i,j=1}^n \in CP^n_\infty(A, L(H), T^0) \), then

\[
\tilde{\rho}(1_A) = \frac{1}{n} \sum_{k=1}^n \rho_{kk}(1_A) = \frac{1}{n} \sum_{k=1}^n V_\rho^* T_{kk} V_\rho = V_\rho^* V_\rho,
\]

and by [1, Theorem 1.4.6], \( \tilde{\rho} \) is an extreme point in \( CP_\infty(A, L(H), V_\rho^* V_\rho) \) if and only if the map \( S \rightarrow P_{H_0}SP_{H_0} \) from \( \Phi_\rho(A)' \) to \( L(H_\rho) \) is injective.

Corollary 2.7. Let \( \rho = [\rho_{ij}]_{i,j=1}^n \) be an element in \( CP^n_\infty(A, L(H), T^0) \). If \( \tilde{\rho} = \frac{1}{n} \sum_{k=1}^n \rho_{kk} \) is an extreme point in \( CP_\infty(A, L(H), V_\rho^* V_\rho) \), then \( \rho \) is an extreme point in \( CP^n_\infty(A, L(H), T^0) \).

Proof. Since \( \tilde{\rho} \) is an extreme point in the set \( CP_\infty(A, L(H), V_\rho^* V_\rho) \), the map \( S_0 \rightarrow P_{H_0}SP_{H_0} \) from \( \Phi_\rho(A)' \) to \( L(H_\rho) \) is injective [1, Theorem 1.4.6], and so the map \( S \rightarrow \text{diag}(P_{H_0}, \ldots, P_{H_0}) \) is injective. From this fact and Proposition 2.6, we deduce that \( \rho \) is an extreme point in \( CP^n_\infty(A, L(H), T^0) \). \( \square \)

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