

**FINITE p -GROUPS IN WHICH SOME SUBGROUPS ARE
GENERATED BY ELEMENTS OF ORDER p**

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ABSTRACT. We prove that if a p -group G of exponent $p^e > p$ has no subgroup H such that $|\Omega_1(H)| = p^p$ and $H/\Omega_1(H)$ is cyclic of order $p^{e-1} \geq p$ and H is regular provided $e = 2$, then G is either absolutely regular or of maximal class. This result supplements the fundamental theorem of Blackburn on p -groups without normal subgroups of order p^p and exponent p . For $p > 2$, we deduce even stronger result than (respective result for $p = 2$ is unknown) a theorem of Božikov and Janko.

The 2-groups all of whose nonmetacyclic subgroups are generated by involutions, are classified in [Boz.J]. Corollary 5 contains a stronger version of main theorem from [Boz.J] for $p > 2$.

In what follows, G is a p -group, where p is a prime. By $\text{cl}(G)$ we denote the class of G . Set $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1 \rangle$, $\mathcal{U}_n(G) = \langle x^{p^n} \mid x \in G \rangle$, $\mathcal{U}_n^*(G) = \langle x \in G \mid o(x) = p^n \rangle$, (n is a positive integer). By G' , $Z(G)$, $\Phi(G)$ we denote the derived subgroup, the center and the Frattini subgroup of G , respectively. A p -group G is said to be *absolutely regular* if $|G/\mathcal{U}_1(G)| < p^p$ [Bla1]. A p -group G is said to be *regular* if for any $x, y \in G$ there exists $z \in \langle x, y \rangle'$ such that $(xy)^p = x^p y^p z^p$ [Hal]. By $K_n(G)$ we denote the n -th member of the lower central series of G . A p -group G is said to be of *maximal class* if $|G| = p^m$, $m > 2$, and $\text{cl}(G) = m - 1$ (so p -groups of maximal class are nonabelian). Next, Γ_1 is the set of all maximal subgroups of G .

DEFINITION 1. A p -group G is said to be an L_n -group, if $\Omega_1(G)$ is of order p^n and exponent p and $G/\Omega_1(G)$ is cyclic of order greater than p .

In what follows we consider L_n -groups only for $n = p$.

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Let G be an L_p -group of exponent p^e ; then $e > 2$ and $|G| = p^{p+e-1}$. Let us prove that $\mathcal{U}_1(G)$ is cyclic of order p^{e-1} . We have

$$p^e = \exp(G) \leq \exp(\mathcal{U}_1(G)) \exp(G/\mathcal{U}_1(G)) = p \cdot \exp(\mathcal{U}_1(G))$$

so $|\mathcal{U}_1(G)| \geq \exp(\mathcal{U}_1(G)) \geq p^{e-1}$. Therefore, it suffices to show that $|\mathcal{U}_1(G)| = p^{e-1}$. If G is regular, then $|\mathcal{U}_1(G)| = |G/\Omega_1(G)| = p^{e-1}$. If G is irregular, then, by Hall's first regularity criterion [Ber2, Theorem 9.8(a)], we have $|G/\mathcal{U}_1(G)| \geq p^p$ so $|\mathcal{U}_1(G)| \leq p^{-p}|G| = p^{e-1}$, and we conclude that $|\mathcal{U}_1(G)| = p^{e-1}$, as required. It follows that $\exp(\Omega_{e-1}(G)) = p^{e-1}$ so we get $\Omega_e^*(G) = \langle G - \Omega_{e-1}(G) \rangle = G$.

Our main result is the following

THEOREM 2. *Suppose that a p -group G of exponent $p^e > p$ is neither absolutely regular nor of maximal class. Then G contains a subgroup H of order p^{p+e-1} such that $|\Omega_1(H)| = p^p$, $H/\Omega_1(H)$ is cyclic of order p^{e-1} and $\Omega_2(H)$ is regular (so, if $e > 2$, then H is an L_p -group).*

Our proof of Theorem 2 is based on Blackburn's theory of p -groups of maximal class [Bla2] and some its consequences (see also [Ber1, §7] or [Hup, §III.13]). We also use some theorems from [Ber2, §13] (proofs of some of them we reproduce below; see Lemmas J(d), 6 and 7).

If a p -group G is either absolutely regular or of maximal class, then it has no subgroup H such as in conclusion of Theorem 2 (in the second case this follows from Lemma J(h)) so absolutely regular p -groups and p -groups of maximal class are excluded from hypothesis of Theorem 2.

COROLLARY 3. *Suppose that a p -group G of exponent greater than p is not absolutely regular. If all proper not absolutely regular subgroups of G are generated by elements of order p , then one and only one of the following holds:*

- (a) $|G| = p^{p+1}$ and $|\Omega_1(G)| > p^{p-1}$ so, if G is regular, then $|\Omega_1(G)| = p^p$.
- (b) $|G| = p^{p+1}$, $\text{cl}(G) = p$, $|\Omega_1(G)| = p^{p-1}$ (in that case, all proper subgroups of G are absolutely regular).
- (c) G is of maximal class, $|G| > p^{p+1}$ and every irregular member of the set Γ_1 has two distinct subgroups of order p^p and exponent p (so, if $p = 2$, then G is dihedral).

COROLLARY 4. *Let G be an irregular p -group, $p > 2$. Suppose that, whenever $H < G$ is neither absolutely regular nor of maximal class, then $\Omega_1(H) = H$. Then G is of maximal class.*

COROLLARY 5. *Let $p > 2$ and let M be a maximal metacyclic subgroup of a nonmetacyclic p -group G , where $|M| > p^2$. Suppose that, whenever $M < N \leq G$ and $|N : M| = p$, then $\Omega_1(N) = N$. Then $p = 3$ and G is of maximal class.*

In particular, if $p > 2$ and a p -group G of exponent greater than p is neither metacyclic nor minimal nonmetacyclic and such that all proper nonmetacyclic subgroups of G are generated by elements of order p , then one and only one of the following assertions holds:

- (a) G is regular of order p^4 and $|\Omega_1(G)| = p^3$.
- (b) G is of maximal class and order 3^4 , $|\Omega_1(G)| \geq 3^3$.
- (c) $p = 3$, G is of maximal class, $|G| > 3^4$ and every irregular member of the set Γ_1 has two distinct (nonabelian) subgroups of order 3^3 and exponent 3.

Note that if a 3-group of maximal class has elementary abelian subgroup of order 3^3 , then G is isomorphic to a Sylow 3-subgroup of the symmetric group of degree 3^2 ([Ber1, Theorem 5.2]). Therefore, a group of Corollary 5(c) has no abelian subgroups of order 3^3 and exponent 3.

In Lemma J we collected known results which are used in what follows.

LEMMA J. *Let $G > \{1\}$ be a p -group, $p > 2$.*

- (a) (*P. Hall* [Hal]) *If G is regular, then $\exp(\Omega_1(G)) = p$ and $|\Omega_1(G)| = |G/\mathcal{U}_1(G)|$. Absolutely regular p -groups, groups of exponent p and p -groups of class $< p$ are regular.*
- (b) [Ber2, Exercise 9.1(b)] *If G is of maximal class and order p^m , then it has exactly one normal subgroup of order p^i for all $i < m - 1$.*
- (c) (*Suzuki*; see [Ber2, Proposition 1.8]) *If $H < G$ is of order p^2 and $|C_G(H)| = p^2$, then G is of maximal class.*
- (d) [Ber2, Exercise 13.10(a)] *If $A < G$ and all subgroups of G of order $p|A|$ and containing A are of maximal class (so that $|A| > p$), then G is also of maximal class.*
- (e) [Ber1, Theorem 7.4] *Suppose that $L = L_1 \in \Gamma_1$ is irregular of maximal class. If G is not of maximal class, then $G/K_p(G)$ is of order p^{p+1} and exponent p and $\Gamma_1 = \{L = L_1, \dots, L_{p^2}, T_1, \dots, T_{p+1}\}$, where all L_i 's are of maximal class and all T_j 's are not of maximal class (T_j are also not absolutely regular since $|G/\mathcal{U}_1(G)| \geq p^{p+1}$), $\eta(G) = \bigcap_{i=1}^{p+1} T_i$ has index p^2 in G and $G/\eta(G)$ is abelian of type (p, p) so $\bigcup_{i=1}^{p+1} T_i = G$ so that $\exp(T_i) = \exp(G)$ for some i .*
- (f) ([Bla2]; see also [Ber1, Theorem 7.5]) *If G is neither absolutely regular nor of maximal class and $H \in \Gamma_1$ is absolutely regular, then $G = H\Omega_1(G)$, where $\Omega_1(G)$ is of order p^p and exponent p (in particular, $|\Omega_1(H)| = p^{p-1}$).*
- (g) [Ber1, Remark 7.2] and [Bla3]. *If G is neither absolutely regular nor of maximal class, then the number of subgroups of order p^p and exponent p in G is $\equiv 1 \pmod{p}$.*
- (h) (*Blackburn* [Bla2]) *If G is of maximal class and order greater than p^{p+1} , then $\Gamma_1 = \{G_1, G_2, \dots, G_{p+1}\}$, where G_1 is absolutely regular with $|\Omega_1(G_1)| = p^{p-1}$ (G_1 is called the fundamental subgroup of G),*

and G_2, \dots, G_{p+1} are irregular of maximal class. All p -groups of maximal class and order p^{p+1} are irregular.

A p -group G of maximal class and order greater than p^{p+1} has no normal subgroup of order p^p and exponent p . Assume that this is false, and let $R \triangleleft G$ be of order p^p and exponent p . Then, by Lemma J(b), $R \leq \Phi(G) < G_1$, a contradiction since the fundamental subgroup G_1 is absolutely regular.

If a p -group G satisfies $\exp(\Omega_1(G)) > p$, then it is irregular (Lemma J(a)).

To facilitate the proof of Theorem 2, we prove the following three assertions.

LEMMA 6. *If $H < G$ is such that $N = N_G(H)$ is of maximal class, then G is also of maximal class.*

PROOF. We use induction on $|G|$. One may assume that $N < G$; then H is not characteristic in N (otherwise, $N = G$). In that case, by Lemma J(b), we have $|N : H| = p$. Since $|Z(N)| = p$ and $Z(G) < N$, we get $Z(G) = Z(N)$ so $|Z(G)| = p$ and $Z(G) \leq \Phi(N) < H$. Set $\bar{G} = G/Z(G)$. If $|\bar{H}| = p$, then $C_{\bar{G}}(\bar{H}) = \bar{N}$ is of order p^2 so \bar{G} is of maximal class (Lemma J(c)). Now let $|\bar{H}| > p$; then \bar{N} is of maximal class so \bar{G} is also of maximal class, by induction, and we are done since $|Z(G)| = p$. \square

PROOF OF LEMMA J(D). In view of Lemma 6, it suffices to show that $N = N_G(A)$ is of maximal class, so one may assume that $|N : A| > p$ and $N = G$; then $A \triangleleft G$ and $|G : A| > p$. Let $D < A$ be G -invariant of index p^2 and set $C = C_G(A/D)$; then $|G : C| \leq p$ so $A < C$ since $|G : A| > p$. Let $B/A \leq C/A$ be of order p ; then $|B| = p|A|$. Since B/D is abelian of order p^3 , B is not of maximal class, a contradiction. \square

LEMMA 7. *Let G be a p -group. Suppose that $A \in \Gamma_1$ is absolutely regular and $M < G$ is irregular of maximal class. Then G is of maximal class.*

PROOF. Assume that G is not of maximal class. Then, by Lemma J(d), $M < H \leq G$, where $|H : M| = p$ and H is not of maximal class. In that case, by Lemma J(e), $H/K_p(H)$ is of order p^{p+1} and exponent p . It follows that H has no absolutely regular maximal subgroups. However, $A \cap H$ is an absolutely regular maximal subgroup of H , and this is a contradiction. \square

LEMMA 8. *All proper subgroups of an L_p -group G are regular; in particular, $\Omega_2(G)$ is regular.*

PROOF. It suffices to show that all maximal subgroups of G are regular. Take $M \in \Gamma_1$.

Suppose that $\Omega_1(G) \not\leq M$; then $\Omega_1(M) = M \cap \Omega_1(G)$ is of order p^{p-1} (and exponent p). Since M is not of maximal class (indeed, $M/\Omega_1(M) \cong G/\Omega_1(G)$ is cyclic of order $> p$), it follows that M is absolutely regular so regular (Lemma J(g,a)).

Now let $\Omega_1(G) < M$. Let $D < \Omega_1(G)$ be a G -invariant subgroup of index p^2 . Set $C = C_G(\Omega_1(G)/D)$; then $|G : C| \leq p$. Take in $C/\Omega_1(G)$ a subgroup $H/\Omega_1(G)$ which is maximal in $G/\Omega_1(G)$; then H/D is abelian so $\text{cl}(H) < p$, and hence H is regular (Lemma J(a)). Since $G/\Omega_1(G)$ has only one maximal subgroup, we get $H = M$. \square

Suppose that $p > 2$ and G is an irregular L_p -group of exponent p^e . Assume that there is in G a normal cyclic subgroup C of order p^e . Set $D = C \cap \Omega_1(G)$; then $|D| = p$ and $G/D = (C/D) \times (\Omega_1(G)/D)$ so $d(G) > 2$. By Lemma 8, all proper subgroups of G are regular. It follows that $d(G) = 2$, contrary to what has just been proved. Thus, all cyclic subgroups of order p^e are not normal in G .

PROOF OF THEOREM 2. Let $p = 2$. Since G is not of maximal class, there is in G a normal abelian subgroup R of type $(2, 2)$ (Lemma J(g)). Put $C = C_G(R)$; then $|G : C| \leq 2$ so $\exp(C) \geq 2^{e-1}$. Suppose that $\exp(C) = 2^e$; then there is in $C - R$ an element x of order 2^e . In that case, $A = \langle x, R \rangle$ is abelian of type $(2^e, 2)$ or $(2^e, 2, 2)$. In any case, A contains an abelian subgroup H of type $(2^e, 2)$, and H is the desired subgroup. Now let $\exp(C) = 2^{e-1}$. Take $y \in G - C$ of order 2^e . Suppose that $U = C_G(y)$ is cyclic; then $C_G(U) = U$ and $Z(G)$ is cyclic. If $e = 2$, then G is of maximal class (Lemma J(c)), contrary to the hypothesis. Thus, $e > 2$. In that case, $U \cap R$ is of order 2 so $H = RU$ is an L_2 -subgroup of exponent 2^e . If $C_G(y)$ is noncyclic, then there is an involution $x \in C_G(y) - \langle y \rangle$; in that case, $H = \langle x, y \rangle = \langle x \rangle \times \langle y \rangle$ is abelian of type $(2^e, 2)$ so H is the desired subgroup.

In what follows we assume that $p > 2$. We use induction on $|G|$. Take an element $x \in G$ of order p^e . Let $x \in L < G$, where L is either absolutely regular or irregular of maximal class such that if $L < M \leq G$, then M is neither absolutely regular nor of maximal class (L exists, by hypothesis). Let $L < F \leq G$ and $|F : L| = p$; then F is neither absolutely regular nor of maximal class, by the choice of L . Therefore, if $F < G$, then F contains the desired subgroup H . Next assume that $F = G$; then $|G : L| = p$.

(i) Let L be absolutely regular. Then, by Lemma J(f), $G = L\Omega_1(G)$, where $\Omega_1(G)(< G)$ is of order p^p and exponent p . Set $H = \langle x, \Omega_1(G) \rangle$, where $x \in L$ has order p^e ; then $\Omega_1(H) = \Omega_1(G)$ is of order p^p and $|H| = p^{p+e-1}$, by the product formula. If $e > 2$, then H is an L_p -group. It remains to show that if $e = 2$, then H is regular. This is true provided $H = G$ since, by hypothesis, G is not of maximal class, so $\text{cl}(G) < p$ (Lemma J(a)). Now let $H < G$ and assume, by the way of contradiction, that H is irregular. Then H is of maximal class since $|H| = p^{p+1}$ (Lemma J(a) again) and, by assumption, L is absolutely regular of index p in G (indeed, $L \cap H = \Omega_1(L)$ is of order p^{p-1} so $|G| = p|L|$, by the product formula). It follows from Lemma 7 that G is of maximal class, contrary to the hypothesis. Thus, H is the desired subgroup.

(ii) Now let L be irregular of maximal class (indeed, since $\exp(L) = p^e > p$ and L is not absolutely regular, we get $|L| \geq p^{p+1}$ so it is irregular, by Lemma J(h), since it is of maximal class). Then, by Lemma J(e), we have $\Gamma_1 = \{L = L_1, L_2, \dots, L_{p^2}, T_1, \dots, T_{p+1}\}$, where L_1, \dots, L_{p^2} are of maximal class and T_1, \dots, T_{p+1} are neither absolutely regular nor of maximal class and $G = \bigcup_{i=1}^{p+1} T_i$. It follows that one of T_i 's, say T_1 , has exponent p^e . Therefore, by induction, there is $H \leq T_1$ of exponent p^e such that $\Omega_1(H)$ is of order p^p and exponent p , $H/\Omega_1(H)$ is cyclic of order p^{e-1} , and H is regular if $e = 2$. \square

LEMMA 9. *Suppose that A is a proper absolutely regular subgroup of a p -group G , $\exp(A) > p$ and, whenever $A < B \leq G$ and $|B : A| = p$, then $\Omega_1(B) = B$. Then G is of maximal class.*

PROOF. By Lemma J(a), B is irregular so G is also irregular. Assume that G is not of maximal class. Let $|G : A| = p$; then $\Omega_1(G) = G$, by hypothesis. However, by Lemma J(f), $G = A\Omega_1(G)$, where $|\Omega_1(G)| = p^p < |G| = |\Omega_1(G)|$, a contradiction. Now let $|G : A| > p$. If $A < B < G$, where $|B : A| = p$, then by what has just been proved, B is of maximal class so G is also of maximal class, by Lemma J(d). \square

PROOF OF COROLLARY 3. Suppose that G is regular and set $L = \Omega_1(G)$; then $|L| \geq p^p$ since G is not absolutely regular. By Lemma J(a), $|G : L| = p$ so G is as in (a) (indeed, $|L| = p^p$ since every maximal subgroup of G not containing L is absolutely regular because of it is not generated by elements of order p).

Next we assume that G is irregular. Since G has a proper absolutely regular subgroup of composite exponent, it follows from Lemma 9 that G is of maximal class. If all maximal subgroups of G are absolutely regular, then G is as in (b), by Lemma J(h). Obviously, every group of maximal class and order p^{p+1} satisfies the hypothesis. Now let $|G| > p^{p+1}$. If $M \in \Gamma_1$ is irregular, then $\Omega_1(M) = M$, by hypothesis. If $L = \Omega_1(\Phi(G))$, then L is of order p^{p-1} and exponent p (Lemma J(h,a,b)). Take an element $x \in M - L$ of order p and set $U_1 = \langle x, L \rangle$. Take an element $y \in M - U_1$ of order p and set $U_2 = \langle y, L \rangle$. Then U_1 and U_2 are distinct of order p^p and exponent p (Lemma J(a)), and the proof is complete since $U_1, U_2 < M$. \square

PROOF OF COROLLARY 4. Assume that G is not of maximal class. Then there is in G a subgroup H such as in Theorem 2. Since H is neither absolutely regular nor of maximal class and $\Omega_1(H) \neq H$, we get a contradiction. \square

PROOF OF COROLLARY 5. Let M be as in the statement of the corollary. Then M is absolutely regular since $p > 2$ so, by Lemma 9, G is of maximal class. Let $M < N \leq G$, where $|N : M| = p$. Then N is irregular of maximal class (Lemma 9). Since $p^{p-1} = |\Omega_1(M)| \leq p^2$ and $p > 2$, we get $p = 3$. \square

REMARK 10. Here we offer another proof of Theorem 2 in the case $\exp(G) = p^e > p^2$. We have to prove that G contains an L_p -subgroup of order p^{p+e-1} . We use induction on $|G|$. By Lemma J(g), there is $M \triangleleft G$ of order p^p and exponent p . Take a cyclic $X < G$ of order p^e and set $F = MX$; then F of exponent $p^e > p^2$ is neither absolutely regular nor of maximal class. Therefore, if $F < G$, the result follows by induction. Now let $F = G$. Suppose that $X \cap M > \{1\}$; then $|G| = p^{p+e-1}$. We claim that G is an L_p -group. It suffices to show that $\Omega_1(G) = M$. Assume that this is false. Since G/M is cyclic, we conclude that $|\Omega_1(G)| \leq p^{p+1}$. Assume that $|\Omega_1(G)| = p^{p+1}$. Then $X \cap \Omega_1(G)$ is cyclic of order p^2 so $\Omega_1(G)$ is irregular and we conclude that it is of maximal class (Lemma J(a)). Since G is not of maximal class, the number $e_p(G)$ of subgroups of order p^p and exponent p in G is congruent with 1 modulo p (Lemma J(g)), and all these subgroups lie in $\Omega_1(G)$. Since $e_p(G) > 1$ and $d(\Omega_1(G)) = 2$, it follows that all maximal subgroups of $\Omega_1(G)$ have exponent p , so $\exp(\Omega_1(G)) = p$ and $\Omega_1(G)$ is regular (Lemma J(a)), a contradiction. Thus, in the case under consideration, G is an L_p -group. Now let $X \cap M = \{1\}$. Let $R < M$ be a G -invariant subgroup of index p . Set $H = RX$. Let us show that H is an L_p -group. Indeed, H is not absolutely regular since $\Omega_1(X)R$ is of order p^p and exponent p (Lemma J(a)). Next, H/R is cyclic of order $p^e > p^2$ so H is not of maximal class. If $K/R < H/R$ is of order p , then $\Omega_1(H) \leq K \leq \Omega_1(H)$ so $\Omega_1(H) = \Omega_1(X)R = K$ is of order p^p and exponent p whence H is an L_p -subgroup.

THEOREM 11. *Let H be a normal absolutely regular subgroup of a p -group G , $|H| > p^{p-1}$ and $\Omega_1(G) \not\leq H$.*

- (a) *If for every $z \in G - H$ of order p , the subgroup $V = \langle z, H \rangle$ is of maximal class, then G is also of maximal class.*
- (b) *If, in addition, $|H| > p^p$ and, for every $z \in G - H$ of order p , we have $\Omega_1(\langle z, H \rangle) = \langle z, H \rangle$, then G is of maximal class.*

PROOF. By hypothesis, $\exp(H) > p$. Assume that (a) and (b) are not true. Let $z \in G - H$ be of order p and $V = \langle z, H \rangle$. Then, by Lemma J(a), V is irregular. In (a), if $V = G$, then, by hypothesis, G is of maximal class. In (b), if $V = G$, then, by Lemma J(f), G is of maximal class. Assume that this is false. Then there is $R \triangleleft G$ of order p^p and exponent p . Let $R_0 \leq R$ be G -invariant and minimal such that $R_0 \not\leq H$. Then $F = R_0H = \langle x, H \rangle$ for any $x \in R_0 - H$. By hypothesis, $\Omega_1(F) = F$ so F is irregular of maximal class, contrary to Lemma J(b) (indeed, $|R_0| < p^p$ since F is of maximal class and order $> p^{p+1}$, and so $\Omega_1(H)$ and R_0 are F -invariant of indices greater than p and non-incident). Therefore, in both cases, one may assume that $V < G$ so $|G : H| > p$.

(a) Let $H \leq H_0 < G$, where H_0 is absolutely regular such that $|H_0|$ is as large as possible. By Lemma J(d), $H_0 < B \leq G$, where $|B : H_0| = p$ and B is not of maximal class. Then, by Lemma J(f), $B = H_0\Omega_1(B)$, where $\Omega_1(B)$ is

of order p^p and exponent p so there exists an element $x \in \Omega_1(B) - H_0$ of order p . Set $U = \langle H, x \rangle$; then, by hypothesis and Lemma J(a,h), U is of maximal class and order $\geq p^{p+1}$ so irregular. We have $H_0, U < B$, H_0 is absolutely regular of index p in B and U is irregular of maximal class. Therefore, by Lemma 7, B is of maximal class, contrary to its choice.

(b) Set $N_G(\Omega_1(H)) = N$. If the set $N - H$ has no elements of order p , then $\Omega_1(N) = \Omega_1(H)$ is characteristic in N so $N = G$, a contradiction since $\Omega_1(G) \not\leq H$, by hypothesis. Therefore, there is an element $y \in N - H$ of order p . By Lemma 6, N is not of maximal class. If N is absolutely regular, there is in $\Omega_1(N)/\Omega_1(H)$ an N -invariant subgroup $R/\Omega_1(H)$ of order p . In that case, HR is absolutely regular of order $p|H|$ and $\Omega_1(HR) \neq HR$, contrary to the hypothesis since $HR = \langle z, H \rangle$ for every $z \in R - H$. Otherwise, there is $S \triangleleft N$ of order p^p and exponent p (Lemma J(g)). Let S_0 be an N -invariant subgroup of S such that $S_0 \not\leq H$ and $|S_0|$ is as small as possible. Then $|S_0H : H| = p$ and $S_0H = \langle y, H \rangle$ for every $y \in S_0 - H$. Therefore, by hypothesis, $\Omega_1(S_0H) = S_0H$ so, by Lemma J(f), S_0H is of maximal class. Then, by Lemma J(b), $S_0 \leq \Phi(S_0H) < H$, contrary to the choice of S_0 . \square

PROBLEMS

Below G is a nonabelian p -group.

1. Study the 2-groups G containing a proper metacyclic subgroup M of order > 4 such that, whenever $M < N \leq G$ and $|N : M| = 2$, then $\Omega_1(N) = N$ ($d(N) = 2$).
2. Classify the 2-groups all of whose proper nonabelian subgroups that are nonmetacyclic, are generated by elements of order p (compare with [BozJ]).
3. Study the p -groups G containing an abelian subgroup M of exponent greater than p such that, whenever $M < N \leq G$ and $|N : M| = p$, then $\Omega_1(N) = N$.
4. Study the irregular p -groups G , $p > 2$, such that $|\Omega_2^*(G)| = p^{p+1}$.
5. Study the p -groups G containing a proper minimal nonabelian subgroup M of order greater than p^3 and such that, whenever $M < N \leq G$ and $|N : M| = p$, then (i) $\Omega_1(N) = N$ (in that case, $p \leq 3$, by Lemma J(g,h)), (ii) $d(N) = 2$.
6. Let A be a subgroup of index $> p^k > p$ in a p -group G . Suppose that all subgroups of G containing A as a subgroup of index p^k , are of maximal class. Is it true that G is also of maximal class (compare with Lemma J(d))?
7. Study the p -groups G containing a subgroup M of maximal class and order $> p^p$ such that, whenever $M < N \leq G$ and $|N : M| = p$, then $\Omega_1(N) = N$.

8. Study the p -groups G of order greater than p^{p+2} containing a maximal regular subgroup A of order p^{p+1} .
9. Suppose that G is an L_p -group. Is it true that G is regular if and only if $G/\mathcal{U}_2(G)$ is regular?
10. Study the irregular p -groups G containing an L_p -subgroup of index p .
11. Does there exist a positive integer n such that a p -group G , $p > 2$, is regular if and only if $G/\mathcal{U}_n(G)$ is regular?
12. Classify the 2-groups G such that $\Omega_1(G)$ is nonabelian of order 2^4 .

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