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BIMORPHISMS OF A *pro**-CATEGORY

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Abstract. Every morphism of an abstract coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ can be viewed as a morphism of the category pro^*-$\mathcal{D}$ (defined on the class of inverse systems in $\mathcal{D}$), where $\mathcal{D}$ is dense in $\mathcal{C}$. Thus, the study of coarse shape isomorphisms reduces to the study of isomorphisms in the appropriate category pro^*-$\mathcal{D}$. In this paper bimorphisms in a category pro^*-$\mathcal{D}$ are considered, for various categories $\mathcal{D}$. We discuss in which cases pro^*-$\mathcal{D}$ is a balanced category (category in which every bimorphism is an isomorphism). We are interested in the question whether the fact that one of the categories: $\mathcal{D}$, pro-$\mathcal{D}$ and pro^*-$\mathcal{D}$ is balanced implies that the other two categories are balanced. It is proved that if pro^*-$\mathcal{D}$ is balanced then $\mathcal{D}$ is balanced. Further, if $\mathcal{D}$ admits sums and products and pro^*-$\mathcal{D}$ is balanced then pro-$\mathcal{D}$ is balanced. In particular, pro^*-$\mathcal{C}$ is balanced for $\mathcal{C} = Set$ (the category of sets and functions) and $\mathcal{C} = Grp$ (the category of groups and homomorphisms).

1. Introduction

The *coarse shape theory* was introduced and studied by N. Uglešić and the author in the joint paper [4]. In that paper, among others, the *coarse shape category* of topological spaces $Sh^* \equiv Sh^*_{(HTop,HPol)}$ has been constructed. The corresponding classification of topological spaces, induced by isomorphisms of Sh^*, is strictly coarser than the standard shape type classification. One can apply the same construction for any category pair $(\mathcal{C},\mathcal{D})$, where $\mathcal{D}$ is dense in $\mathcal{C}$ (in the shape-theoretical sense [6]), to obtain an *abstract coarse shape category* $Sh^*_{(\mathcal{C},\mathcal{D})}$, having $\mathcal{C}$-objects for the object class. The category $Sh^*_{(\mathcal{C},\mathcal{D})}$ is constructed from the category pro^*-$\mathcal{D}$ which

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is defined on the class of inverse systems in $\mathcal{D}$. For any pair X, Y of $\mathcal{C}$-objects, every coarse shape morphism $F^* \in Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y)$ is represented by a unique morphism $\boldsymbol{f}^* \in pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$ between inverse systems $\boldsymbol{X}$ and $\boldsymbol{Y}$, i.e. $Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y) \approx pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$, where $\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{q}: Y \to \boldsymbol{Y}$ are $\mathcal{D}$-expansions of objects X and Y, respectively. Thus, the category $Sh^*_{(\mathcal{C},\mathcal{D})}$ is obtained via the category *pro**-$\mathcal{D}$ in the same manner as the abstract shape category $Sh_{(\mathcal{C},\mathcal{D})}$ is obtained via the category *pro*-$\mathcal{D}$. Namely, the categories *pro*-$\mathcal{D}$ and *pro**-$\mathcal{D}$ have the same object class (inverse systems in $\mathcal{D}$), but sets of morphisms are much larger in *pro**-$\mathcal{D}$. Since a certain faithful functor $\underline{J} \equiv \underline{J}_{\mathcal{D}} : pro\text{-}\mathcal{D} \to pro^*\text{-}\mathcal{D}$, keeping the objects fixed, has been constructed, one may consider *pro*-$\mathcal{D}$ to be a subcategory of *pro**-$\mathcal{D}$. Therefore, we may write

$$\mathcal{D} \subseteq pro\text{-}\mathcal{D} \subseteq pro^*\text{-}\mathcal{D}. \tag{1}$$

In any category the problem of detecting isomorphisms is essential. It can be readily seen that the discussion about coarse shape isomorphisms (isomorphisms in $Sh^*_{(\mathcal{C};\mathcal{D})}$) reduces to the studying of isomorphisms in the category *pro**-$\mathcal{D}$. For many familiar categories (*Set* - the category of sets and functions, *Grp* - the category of groups and homomorphisms, *Cpt* - the category of compact Hausdorff spaces) every morphism which is simultaneously a monomorphism and an epimorphism (called *bimorphism*) is an isomorphism. Such a category is called *balanced*. Notice that if a category is balanced, then, generally, its subcategory (or supercategory) needs not to be balanced. In this paper we are interested whether the fact that one of the categories in (1) is balanced implies that the other categories in (1) are balanced.

2. Preliminaries

Let us recall the basic facts about *pro*-categories (see [6]) as well as *pro**-categories (see [4]).

Let $\mathcal{C}$ be a category. An *inverse system* in $\mathcal{C}$, denoted by $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, consists of a directed preordered set $(\Lambda, \leq)$, of $\mathcal{C}$-objects X_λ for each $\lambda \in \Lambda$, and of $\mathcal{C}$-morphisms $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ ($p_{\lambda\lambda} = 1_{X_\lambda}$), for each related pair $\lambda \leq \lambda'$ in Λ, such that $p_{\lambda\lambda'}p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, whenever $\lambda \leq \lambda' \leq \lambda''$. A *morphism of inverse systems* $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ consists of a function $f : M \to \Lambda$, and of $\mathcal{C}$-morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ for each $\mu \in M$, such that, for every related pair $\mu \leq \mu'$, there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$, such that

$$f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two inverse systems over the same index set Λ. A morphism $(1_\Lambda, f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ is called a *level morphism* of inverse systems, provided

$$f_\lambda p_{\lambda\lambda'} = q_{\lambda\lambda'} f_{\lambda'},$$

for every related pair $\lambda \leq \lambda'$.

A morphism $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *equivalent to* a morphism $(f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, provided each $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

The equivalence class $[(f, f_\mu)]$ of an (f, f_μ) is denoted by $\boldsymbol{f}$. The *composition* of equivalence classes is well defined by putting

$$\boldsymbol{gf} = [(g, g_\nu)][(f, f_\mu)] = [(g, g_\nu)(f, f_\mu)].$$

The corresponding quotient category having all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ for objects, all equivalence classes $\boldsymbol{f}$ of morphisms of inverse systems for morphisms and $\boldsymbol{1_X} = [(1_\Lambda, 1_{X_\lambda})]$ for the identity morphism on an $\boldsymbol{X}$ is denoted by *pro*-$\mathcal{C}$ and is called the *pro-category* for the category $\mathcal{C}$. We may treat each $\mathcal{C}$-morphism $f : X \to Y$ as a morphism in *pro*-$\mathcal{C}$ by putting $\boldsymbol{f} = (f) : (X) \to (Y)$, where (X) and (Y) are the rudimentary inverse systems. A morphism $\boldsymbol{f}$ is said to be *induced* by f. In this way, a category $\mathcal{C}$ can be considered as a subcategory of *pro*-$\mathcal{C}$.

An S^*-*morphism of inverse systems*, $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, consists of a function $f : M \to \Lambda$, called the *index function*, and of a set of $\mathcal{C}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that, for every related pair $\mu \leq \mu'$ in M, there exists a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f(\mu')$, and there exists an $n \in \mathbb{N}$ so that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

If the index function f is increasing and, for every pair $\mu \leq \mu'$, one may put $\lambda = f(\mu')$, then (f, f_μ^n) is said to be a *simple* S^*-morphism. If, in addition, $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\Lambda, f_\lambda^n)$ is said to be a *level* S^*-morphism.

The *composition* of S^*-morphisms of inverse systems is defined as follows: If $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z}$, then $(g, g_\nu^n)(f, f_\mu^n) = (h, h_\nu^n) : \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$. The *identity* S^*-morphism on $\boldsymbol{X}$ is an S^*-morphism $(1_\Lambda, 1_{X_\lambda}^n) : \boldsymbol{X} \to \boldsymbol{X}$, consisting of the identity function 1_Λ and of the identity morphisms $1_{X_\lambda}^n = 1_{X_\lambda}$ in $\mathcal{C}$, for every $n \in \mathbb{N}$ and every $\lambda \in \Lambda$.

An S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse systems in $\mathcal{C}$ is said to be *equivalent to* an S^*-morphism $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu^n) \sim (f', f_\mu'^n)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, and an $n \in \mathbb{N}$, such that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation among S^*-morphisms of inverse systems in $\mathcal{C}$. The equivalence class $[(f, f_\mu^n)]$ of an S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$.

The category pro^*-$\mathcal{C}$ has as objects all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ and as morphisms all equivalence classes $\boldsymbol{f}^* = [(f, f_\mu^n)]$ of S^*-morphisms (f, f_μ^n). Since the equivalence relation respects the composition of S^*-morphisms, a composition in pro^*-$\mathcal{C}$ is well defined by putting

$$\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h, h_\nu^n)],$$

where $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) = (fg, g_\nu^n f_{g(\nu)}^n)$. For every inverse system $\boldsymbol{X}$ in $\mathcal{C}$, the identity morphism in pro^*-$\mathcal{C}$ is $\boldsymbol{1}_{\boldsymbol{X}}^* = [(1_\Lambda, 1_{X_\lambda})]$.

A functor $\underline{J} \equiv \underline{J}_{\mathcal{C}} : pro\text{-}\mathcal{C} \to pro^*\text{-}\mathcal{C}$ is defined as follows. It keeps objects fixed, i.e. $\underline{J}(\boldsymbol{X}) = \boldsymbol{X}$, for every inverse system $\boldsymbol{X}$ in $\mathcal{C}$. If $\boldsymbol{f} \in pro\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ and if (f, f_μ) is any representative of $\boldsymbol{f}$, then a morphism $\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^* = [(f, f_\mu^n)] \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ is represented by an S^*-morphism (f, f_μ^n) where $f_\mu^n = f_\mu$ for all $\mu \in M$ and $n \in \mathbb{N}$. The morphism $\boldsymbol{f}^*$ is said to be *induced* by $\boldsymbol{f}$. Since the functor $\underline{J}$ is faithful, we may consider the category *pro*-$\mathcal{C}$ as a subcategory of pro^*-$\mathcal{C}$. Thus, every morphism $\boldsymbol{f}$ in *pro*-$\mathcal{C}$ can also be considered as a morphism of the category pro^*-$\mathcal{C}$.

Recall that an index set is said to be *cofinite* if its preordering is an ordering and every $\mu \in M$ has finitely many predecessors. Concerning inverse systems indexed by a cofinite index set, we have a very useful lemma which easily follows from [4, Lemma 10].

LEMMA 2.1. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *and* $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ *be inverse systems in* $\mathcal{C}$ *with* M *cofinite. Then every morphism* $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ *of* pro^*-$\mathcal{C}$ *admits a simple representative* $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$. *Moreover, if* $(f', f_\mu'^n)$ *is any simple representative of* $\boldsymbol{f}^*$, *then, for every* $\mu \in M$, *there exists* $n_\mu \in \mathbb{N}$ *such that, for every* $\mu' \leq \mu$ *and every* $n \geq n_\mu$,

$$f_{\mu'}'^n p_{f'(\mu')f'(\mu)} = q_{\mu'\mu} f_\mu'^n$$

In general, a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^*-$\mathcal{C}$ does not admit a level representative. However, the following "reindexing" theorem will help to overcome some technical difficulties concerning this fact.

THEOREM 2.2. *Let* $\boldsymbol{f}^* \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$. *Then there exist inverse systems* $\boldsymbol{X}'$ *and* $\boldsymbol{Y}'$ *in* $\mathcal{C}$ *having the same cofinite index set* $(N, \leq)$, *there exists a morphism* $\boldsymbol{f}'^* : \boldsymbol{X}' \to \boldsymbol{Y}'$ *having a level representative* $(1_N, f_\nu')$ *and there exist isomorphisms* $\boldsymbol{i}^* : \boldsymbol{X} \to \boldsymbol{X}'$ *and* $\boldsymbol{j}^* : \boldsymbol{Y} \to \boldsymbol{Y}'$ *in* pro^*-$\mathcal{C}$, *such that the following diagram in* pro^*-$\mathcal{C}$ *commutes:*

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{f}^*} & \boldsymbol{Y} \\ \boldsymbol{i}^* \downarrow & & \downarrow \boldsymbol{j}^* \\ \boldsymbol{X}' & \xrightarrow{\boldsymbol{f}'^*} & \boldsymbol{Y}' \end{array} .$$

The analogous theorem ([6, Theorem 1.1.3.]) holds in every *pro*-category *pro*-$\mathcal{C}$. Concerning many problems, these "reindexing theorems" allow to assume that each morphism in *pro*-$\mathcal{C}$ or pro^*-$\mathcal{C}$ admits a level representative.

Moreover, we can assume that both inverse systems are indexed by the same cofinite index set.

3. Isomorphisms in a *pro**-category

In this section we are dealing with isomorphisms in pro^*-$\mathcal{C}$ which are induced by morphisms in pro-$\mathcal{C}$. Notice that, if $\mathcal{B}$ is a subcategory of $\mathcal{A}$, and a morphism $f \in \mathcal{B}(X,Y) \subseteq \mathcal{A}(X,Y)$ is an isomorphism in a category $\mathcal{B}$ then f is an isomorphisms in a category $\mathcal{A}$, as well. But the converse is not generally true. Since pro-$\mathcal{C}$ can be viewed as a subcategory of pro^*-$\mathcal{C}$ the following question naturally arises:

Problem 3.1. *If a morphism* $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f}) \in pro^*(\boldsymbol{X},\boldsymbol{Y})$, *induced by* $\boldsymbol{f} \in pro\text{-}\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$, *is an isomorphism of* pro^*-$\mathcal{C}$, *is it true that the morphism* $\boldsymbol{f}$ *is an isomorphism of* pro-$\mathcal{C}$?

Before we answer the above question affirmatively (see Theorem 3.2, below) let us recall an analogue of the well known Morita lemma ([6, Theorem 2.2.5]) which characterizes isomorphisms in a pro^*-category ([4, Theorem 5]).

Theorem 3.1. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *and* $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ *be inverse systems in* $\mathcal{C}$ *over the same index set. Let a morphism* $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ *in* pro^*-$\mathcal{C}$ *admit a level representative* $(1_\Lambda, f_\lambda^n)$. *Then* $\boldsymbol{f}^*$ *is an isomorphism if and only if, for every* $\lambda \in \Lambda$, *there exist a* $\lambda' \geq \lambda$ *and an* $n \in \mathbb{N}$ *such that, for every* $n' \geq n$, *there exists a morphism* $h_\lambda^{n'} : Y_{\lambda'} \to X_\lambda$ *in* $\mathcal{C}$, *such that the following diagram in* $\mathcal{C}$ *commutes:*

$$\begin{array}{ccc} X_\lambda & \longleftarrow & X_{\lambda'} \\ f_\lambda^{n'} \downarrow & h_\lambda^{n'} \nwarrow & \downarrow f_{\lambda'}^{n'} \\ Y_\lambda & \longleftarrow & Y_{\lambda'} \end{array} . \tag{2}$$

Theorem 3.2. *A morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* pro-$\mathcal{C}$ *is an isomorphism if and only if the induced morphism* $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f}) : \boldsymbol{X} \to \boldsymbol{Y}$ *is an isomorphism of* pro^*-$\mathcal{C}$.

Proof. Since $\underline{J}$ is a functor, the necessity holds trivially. Conversely, suppose that the induced morphism $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f})$ is an isomorphism of pro^*-$\mathcal{C}$. By the "reindexing theorem" ([6, Theorem 1.1.3]) there is no loss of generality in assuming that $\boldsymbol{f}$ is represented by a level morphism $(1_\Lambda, f_\lambda)$. Hence, the induced morphism $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f}) : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^*-$\mathcal{C}$ is represented by the induced level S^*-morphism $(1_\Lambda, f_\lambda^n)$, $f_\lambda^n = f_\lambda$, for all $n \in \mathbb{N}$, $\lambda \in \Lambda$. Since, $\boldsymbol{f}^*$ is an isomorphism, the level representative $(1_\Lambda, f_\lambda^n)$ satisfies the condition of Theorem 3.1. That means that, for every $\lambda \in \Lambda$, there exist a $\lambda' \geq \lambda$ and an $n \in \mathbb{N}$ such that, for every $n' \geq n$, there exists a morphism $h_\lambda^{n'} : Y_{\lambda'} \to X_\lambda$ in $\mathcal{C}$, so that diagram (2) in $\mathcal{C}$ commutes. Now, for every $n' \geq n$, it follows that

$$f_\lambda h_\lambda^{n'} = f_\lambda^{n'} h_\lambda^{n'} = q_{\lambda\lambda'}$$

and

$$h_\lambda^{n'} f_\lambda = h_\lambda^{n'} f_\lambda^{n'} = p_{\lambda\lambda'},$$

which means that the morphism $(1_\Lambda, f_\lambda)$ fulfills the condition of the Morita lemma in *pro*-$\mathcal{C}$ ([6, Theorem 2.2.5]). Therefore, $\boldsymbol{f} = [(1_\Lambda, f_\lambda)]$ is an isomorphism of *pro*-$\mathcal{C}$. □

Notice that, by Morita lemma, it follows that a morphism $f : X \to Y$ is an isomorphism of a category $\mathcal{C}$ if and only if the induced morphism $(f) : (X) \to (Y)$ in *pro*-$\mathcal{C}$ is an isomorphism. Therefore, by Theorem 3.2, the following corollary holds

COROLLARY 3.3. *Let $f : X \to Y$ be a morphism of a category $\mathcal{C}$. Then the following three conditions are equivalent:*

(i) *$f : X \to Y$ is an isomorphism of $\mathcal{C}$.*
(ii) *The induced morphism $(f) : (X) \to (Y)$ is an isomorphism of pro-$\mathcal{C}$.*
(iii) *The induced morphism $\boldsymbol{f}^* = \underline{J}((f)) : (X) \to (Y)$ is an isomorphism of pro*-$\mathcal{C}$.*

4. Bimorphisms in a *pro**-category

We are interested in determining under what conditions the fact that one of the categories in (1) is balanced implies that the two other categories in (1) are balanced.

THEOREM 4.1. *If pro*-$\mathcal{C}$ or pro-$\mathcal{C}$ is a balanced category, then $\mathcal{C}$ is also a balanced category.*

PROOF. Assuming on the contrary, i.e. if $\mathcal{C}$ is not a balanced category, then there exists a bimorphism $f : X \to Y$ of $\mathcal{C}$ which is not an isomorphism. According to [5, Corollary 3 and Corollary 6], every bimorphism of $\mathcal{C}$ is also a bimorphism of *pro*-$\mathcal{C}$ and *pro**-$\mathcal{C}$. Thus, the induced morphisms (f) in *pro*-$\mathcal{C}$ and $\underline{J}((f))$ in *pro**-$\mathcal{C}$ are bimorphisms. This bimorphisms cannot be isomorphisms of *pro*-$\mathcal{C}$ nor *pro**-$\mathcal{C}$, respectively, because, by Corollary 3.3, it would imply that f is a $\mathcal{C}$-isomorphism. Hence, *pro*-$\mathcal{C}$ and *pro**-$\mathcal{C}$ are not balanced, which is a contradiction. □

THEOREM 4.2. *Let $\mathcal{C}$ be a category admitting sums and products. If pro*-$\mathcal{C}$ is balanced, then pro-$\mathcal{C}$ is also a balanced category.*

PROOF. Assuming on the contrary, i.e. if *pro*-$\mathcal{C}$ is not a balanced category, then there exists a bimorphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *pro*-$\mathcal{C}$ which is not an isomorphism. According to [5, Theorem 3 and Theorem 5], every bimorphism of *pro*-$\mathcal{C}$ is also a bimorphism of *pro**-$\mathcal{C}$. Thus, the induced morphism $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f})$ is a bimorphism of *pro**-$\mathcal{C}$. Now, $\boldsymbol{f}^*$ cannot be an isomorphism, because, by Theorem 3.2, that would imply that $\boldsymbol{f}$ is an isomorphism. That means, the morphism $\boldsymbol{f}^*$ is a bimorphism, but not an isomorphism of *pro**-$\mathcal{C}$. Hence, *pro**-$\mathcal{C}$ is not balanced, which is a contradiction. □

The question: *"Is the category pro-$\mathcal{C}$ balanced when $\mathcal{C}$ is balanced?"* posed in [1] was recently answered negatively in [2]. It has been proved ([2], Proposition 3.12.) that the category whose objects are compact connected spaces and morphisms are maps is a balanced category, but the corresponding *pro*-category is not balanced. Since the category of compact connected spaces is balanced but it does not admit sums, we cannot apply Theorem 4.2. Consequently, the following problem remains open.

PROBLEM 4.1. *If $\mathcal{C}$ be a balanced category, is the category pro^*-$\mathcal{C}$ balanced?*

We will prove that in two important cases of balanced categories, $\mathcal{C} = Set$ and $\mathcal{C} = Grp$, the category pro^*-$\mathcal{C}$ is balanced.

DEFINITION 4.3. *We say that a category $\mathcal{C}$ $*$-additive if it has zero-objects, admits sums and products, every morphism of $\mathcal{C}$ has the kernel and cokernel and every morphism set $\mathcal{C}(X,Y)$ is a group such that the composition $\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ is bilinear.*

A zero-object of a category $\mathcal{C}$ is denoted by 0. For any two objects, a unique zero-morphism which factorizes through 0 is denoted by $o_{XY} : X \to Y$ (briefly o). Notice that a zero-morphism is the unit element of the group $\mathcal{C}(X,Y)$ (see [3]), therefore we denote the group operation additively, although $\mathcal{C}(X,Y)$ is not, in general, an abelian group. The kernel of a morphism $f : X \to Y$ we denote by $\ker f : N \to X$, while $\operatorname{coker} f : Y \to C$ denotes the cokernel of f.

PROPOSITION 4.4. *Let $\mathcal{C}$ be a $*$-additive category. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be inverse systems in $\mathcal{C}$ over the same cofinite index set. Let a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ in pro^*-$\mathcal{C}$ admit a level representative $(1_\Lambda, f_\lambda^n)$. A morphism $\boldsymbol{f}^*$ is a monomorphism if and only if $(1_\Lambda, f_\lambda^n)$ satisfies the following condition:*

(M-Add) *For every $\lambda \in \Lambda$, there exist a $\lambda' \geqslant \lambda$ and an $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, $p_{\lambda\lambda'}(\ker f_{\lambda'}^n) = o$.*

Further, $\boldsymbol{f}^*$ is an epimorphism if and only if $(1_\Lambda, f_\lambda^n)$ satisfies the following condition:

(E-Add) *For every $\lambda \in \Lambda$, there exists a $\lambda' \geqslant \lambda$ and an $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, $(\operatorname{coker} f_\lambda^n)\, q_{\lambda\lambda'} = o$.*

PROOF. Assume that $\boldsymbol{f}^*$ is a monomorphism. Then, according to [5, Theorem 4], a level representative $(1_\Lambda, f_\lambda^n)$ satisfies condition (M) of the same theorem. For an arbitrary λ, let $\lambda' \geq \lambda$ be as in condition (M). Let us put $u^n = \ker f_{\lambda'}^n : N_{\lambda'}^n \to X_{\lambda'}$ and $v^n : N_{\lambda'}^n \to X_{\lambda'}$, $v^n = o$, for every $n \in \mathbb{N}$. Obviously, $f_{\lambda'}^n u^n = f_{\lambda'}^n v^n = o$ holds for every $n \in \mathbb{N}$. Now by (M), there exist an $n_0 \in \mathbb{N}$ such that, $p_{\lambda\lambda'} \ker f_{\lambda'}^n = p_{\lambda\lambda'} u^n = p_{\lambda\lambda'} v^n = o$, for every $n \geq n_0$, which establishes (M-Add). Conversely, suppose that $(1_\Lambda, f_\lambda^n)$

satisfies `(M-Add)`. Then, for an arbitrary $\lambda \in \Lambda$, there exist a $\lambda' \geqslant \lambda$ and an $n_0 \in \mathbb{N}$ as in condition `(M-Add)`. Let (u^n), (v^n) be a pair of sequences of $\mathcal{C}$-morphism $u^n, v^n : P^n \to X_{\lambda'}$, $P^n \in Ob(\mathcal{C})$, such that $f^n_{\lambda'} u^n = f^n_{\lambda'} v^n$, for every $n \in \mathbb{N}$. It implies $f^n_{\lambda'}(u^n - v^n) = o$, for every $n \in \mathbb{N}$. Consider the kernel $\ker f^n_{\lambda'} : N^n_{\lambda'} \to X_{\lambda'}$ of $f^n_{\lambda'}$, for every $n \in \mathbb{N}$. According to the universal property of a kernel, for every $n \in \mathbb{N}$, there exists a morphism $h^n : P^n \to N^n_{\lambda'}$ such that $(\ker f^n_{\lambda'}) h^n = (u^n - v^n)$. Now, by `(M-Add)`, it follows that $p_{\lambda\lambda'}(u^n - v^n) = p_{\lambda\lambda'}(\ker f^n_{\lambda'}) h^n = o$, for every $n \geq n_0$, which implies that $p_{\lambda\lambda'} u^n = p_{\lambda\lambda'} v^n$, for every $n \geq n_0$. Thus, $(1_\Lambda, f^n_\lambda)$ satisfies condition `(M)` of [5, Theorem 4], and, by the same theorem, $\boldsymbol{f}^*$ is a monomorphism.

Analogously, one can prove a statement concerning an epimorphism, i.e. the equivalence, for a $*$-additive category, between `(E-Add)` and the condition `(E)` of [5, Theorem 2]. ☐

EXAMPLE 4.5. Clearly, the category *Grp* is a $*$-additive category. Thus, we can apply Proposition 4.4 to characterize epimorphisms and monomorphisms in *pro**-*Grp*. We point out that a group operation is denoted additively and the unit element is denoted by 0, for every group, not necessarily abelian. Recall that a kernel of a homomorphism $f : X \to Y$ is the inclusion $i : f^{-1}(0) \hookrightarrow X$ (the group $f^{-1}(0)$ is usually denoted by $\ker f$) and a cokernel of f is the quotient homomorphism $p : Y \to Y/\operatorname{Im} f$. One can easily verify that, for $\mathcal{C} = Grp$, `(E-Add)` is equivalent to the following condition:

`(E-Grp)` *For every* $\lambda \in \Lambda$, *there exist a* $\lambda' \in \Lambda$, $\lambda' \geqslant \lambda$, *and an* $n_0 \in \mathbb{N}$ *such that* $\operatorname{Im} q_{\lambda\lambda'} \subseteq \operatorname{Im} f^n_\lambda$, *for all* $n \geqslant n_0$.

Further, for $\mathcal{C} = Grp$, `(M-Add)` is equivalent to the condition

`(M-Grp)` *For every* $\lambda \in \Lambda$ *there exist a* $\lambda' \in \Lambda$, $\lambda' \geqslant \lambda$, *and an* $n_0 \in \mathbb{N}$ *such that* $\ker f^n_{\lambda'} \subseteq \ker p_{\lambda\lambda'}$, *for every* $n \geqslant n_0$.

REMARK 4.6. Let us consider a morphism $\boldsymbol{f} = [(f, f_\mu)]$ in *pro-Grp*. For the induced morphism $\boldsymbol{f}^* = [(f, f^n_\mu)] = \underline{J}(\boldsymbol{f})$, $f^n_\mu = f_\mu$, for every $n \in \mathbb{N}$, Proposition 4.4 allows to put $n_0 = 1$, for every λ. Thus, for the induced morphism, the dependence on the indices n in the conditions `(E-Grp)` and `(M-Grp)` vanishes. Consequently, the conditions `(E-Grp)` and `(M-Grp)` in the subcategory *pro-Grp* become the well-known conditions for $\boldsymbol{f}$ being a monomorphism and an epimorphism of *pro-Grp* respectively (see [6, Theorem 2.2.1 and Theorem 2.2.3]).

THEOREM 4.7. *The categories pro***-Set and pro***-Grp are balanced.*

PROOF. We need to prove that every bimorphism in *pro**-*Grp* (*pro**-*Set*) is an isomorphism. Suppose $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ is a bimorphism of *pro**-*Grp* (*pro**-*Set*). By the "reindexing theorem" (Theorem 2.2), there is no loss of generality in assuming that $\boldsymbol{f}^*$ is represented by a level morphism $(1_\Lambda, f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ where $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are inverse systems in *pro**-*Grp* (*pro**-*Set*) over the same cofinite index set. Therefore, by Lemma 2.1, for every $\lambda \in \Lambda$, there exist an $n_\lambda \in \mathbb{N}$ such that, for every $\lambda' \leq \lambda$ and every $n \geq n_\lambda$, $f^n_{\lambda'} p_{\lambda'\lambda} = q_{\lambda'\lambda} f^n_\lambda$.

First, let us show that a bimorphism $\boldsymbol{f}^*$ of *pro**-*Grp* is an isomorphism. By Example 4.5, $(1_\Lambda, f_\lambda)$ fulfills conditions (M-Grp) and (E-Grp). Hence, for an arbitrary $\lambda \in \Lambda$, there exist a $\lambda' \geqslant \lambda$ and an $n_0 \in \mathbb{N}$ such that

$$\ker p_{\lambda\lambda'} \supseteq \ker f^{n'}_{\lambda'}, \text{ for every } n' \geq n_0. \tag{3}$$

Further, for this λ', there exist a $\lambda'' \geq \lambda'$ and an n'_0 such that

$$\operatorname{Im} f^{n'}_{\lambda'} \supseteq \operatorname{Im} q_{\lambda'\lambda''},\ n' \geqslant n'_0. \tag{4}$$

Let us put $n = \max\{n_0, n'_0, n_{\lambda'}, n_{\lambda''}\}$. For every $n' \geqslant n$, we will now define a homomorphism $h^{n'}_\lambda : Y_{\lambda''} \to X_\lambda$ as follows. For $y \in Y_{\lambda''}$, we put

$$h^{n'}_\lambda(y) = p_{\lambda\lambda'}(x),$$

where $x \in X_{\lambda'}$ is chosen such that

$$f^{n'}_{\lambda'}(x) = q_{\lambda'\lambda''}(y). \tag{5}$$

By (4), such an element exists. Notice that the value of the function $h^{n'}_\lambda$ at y does not depend on a particular choice of x for which (5) holds. Indeed, for any $x' \in X_{\lambda'}$, $f^{n'}_{\lambda'}(x') = q_{\lambda'\lambda''}(y)$, by (5), it follows $f^{n'}_{\lambda'}(x) - f^{n'}_{\lambda'}(x') = 0$, i.e. $f^{n'}_{\lambda'}(x - x') = 0$, which, by (3), implies $p_{\lambda\lambda'}(x - x') = 0$, i.e. $p_{\lambda\lambda'}(x) = p_{\lambda\lambda'}(x')$. Thus, $h^{n'}_\lambda$ is well defined. Let us show that it is a homomorphism. Let $y_1, y_2 \in Y_{\lambda''}$. Then there exist $x_1, x_2 \in X_{\lambda'}$ such that $f^{n'}_{\lambda'}(x_1) = q_{\lambda'\lambda''}(y_1)$, $f^{n'}_{\lambda'}(x_2) = q_{\lambda'\lambda''}(y_2)$. Then,

$$f^{n'}_{\lambda'}(x_1 + x_2) = f^{n'}_{\lambda'}(x_1) + f^{n'}_{\lambda'}(x_2) = q_{\lambda'\lambda''}(y_1) + q_{\lambda\lambda''}(y_2) = q_{\lambda\lambda''}(y_1 + y_2).$$

Now, by the definition of $h^{n'}_\lambda$,

$$h^{n'}_\lambda(y_1 + y_2) = p_{\lambda\lambda'}(x_1 + x_2) = p_{\lambda\lambda'}(x_1) + p_{\lambda\lambda'}(x_2) = h^{n'}_\lambda(y_1) + h^{n'}_\lambda(y_2),$$

which proves that $h^{n'}_\lambda$ is a homomorphism. In order to show that $\boldsymbol{f}^*$ is an isomorphism, it suffices, by Theorem 3.1, to check that, for every $n' \geqslant n$,

$$h^{n'}_\lambda f^{n'}_{\lambda''} = p_{\lambda\lambda''} \tag{6}$$

and

$$f^{n'}_\lambda h^{n'}_\lambda = q_{\lambda\lambda''} \tag{7}$$

hold. By definition of $h^{n'}_\lambda$, for an $x \in X_{\lambda''}$,

$$h^{n'}_\lambda f^{n'}_{\lambda''}(x) = p_{\lambda\lambda'}(x'), \tag{8}$$

holds, where x' is an element of $X_{\lambda'}$ such that

$$f^{n'}_{\lambda'}(x') = q_{\lambda'\lambda''} f^{n'}_{\lambda''}(x). \tag{9}$$

Since $n' \geqslant n_{\lambda''}$, for the pair $\lambda' \leq \lambda''$, the appropriate commutativity condition holds, i.e.

$$q_{\lambda'\lambda''} f_{\lambda''}^{n'} = f_{\lambda'}^{n'} p_{\lambda'\lambda''}. \tag{10}$$

Now, (9) and (10) imply $f_{\lambda'}^{n'} (x' - p_{\lambda'\lambda''} (x)) = 0$, and therefore,

$$x' - p_{\lambda'\lambda''} (x) \in \ker f_{\lambda'}^{n'}.$$

Using (3) one obtains $p_{\lambda\lambda'} (x' - p_{\lambda'\lambda''} (x)) = 0$, which implies

$$p_{\lambda\lambda'} (x') = p_{\lambda\lambda'} p_{\lambda'\lambda''} (x) = p_{\lambda\lambda''} (x). \tag{11}$$

Now, (8) and (11) establish (6). Further, by definition of $h_{\lambda}^{n'}$, for an arbitrary $y \in Y_{\lambda''}$, one gets

$$f_{\lambda}^{n'} h_{\lambda}^{n'} (y) = f_{\lambda}^{n'} p_{\lambda\lambda'} (x'), \tag{12}$$

where x' is an element of $X_{\lambda'}$ such that

$$f_{\lambda'}^{n'} (x') = q_{\lambda'\lambda''} (y). \tag{13}$$

Since $n' \geqslant n_{\lambda'}$, for the pair $\lambda \leq \lambda'$, the appropriate commutativity condition holds, i.e.

$$f_{\lambda}^{n'} p_{\lambda\lambda'} = q_{\lambda\lambda'} f_{\lambda'}^{n'}. \tag{14}$$

Now, by combining (12) and (14), one obtains

$$f_{\lambda}^{n'} h_{\lambda}^{n'} (y) = q_{\lambda\lambda'} f_{\lambda'}^{n'} (x'). \tag{15}$$

Finally, (15) and (13) imply (7). Thus, we have proved that $(1_\Lambda, f_\lambda)$ fulfills the conditions of Theorem 3.1, and therefore, $\boldsymbol{f}^*$ is an isomorphism of *pro**-*Grp*.

Now, let us show that a bimorphism $\boldsymbol{f}^*$ of *pro**-*Set* is an isomorphism. Since $\boldsymbol{f}^*$ is a monomorphism, by [5, Theorem 4], $(1_\Lambda, f_\lambda)$ fulfills condition `(M″)` of the same theorem. Therefore, for an arbitrary $\lambda \in \Lambda$, there exist a $\lambda' \geqslant \lambda$, such that the appropriate condition `(M″)` holds. Since $\boldsymbol{f}^*$ is an epimorphism, $(1_\Lambda, f_\lambda^n)$ satisfies condition `(E-Set)` from [5, Example 1]. Hence, for this λ', there exist a $\lambda'' \geqslant \lambda'$ and an $n_0 \in \mathbb{N}$ such that

$$q_{\lambda'\lambda''} (Y_{\lambda''}) \subseteq f_{\lambda'}^{n'} (X_{\lambda'}), \tag{16}$$

for every $n \geqslant n_0$. Now, for every $n \geq n_0$, let $\widetilde{h}_\lambda^n : Y_{\lambda''} \to X_{\lambda'}$ be any function such that

$$\widetilde{h}_\lambda^n (y) \in (f_{\lambda'}^n)^{-1} (q_{\lambda'\lambda''} (y)),$$

for every $y \in Y_{\lambda''}$. Because of (16), such a function exists. Clearly, for every $n \geq n_0$,

$$f_{\lambda'}^n \widetilde{h}_\lambda^n = q_{\lambda'\lambda''}. \tag{17}$$

Let us define functions $u^n, v^n : X_{\lambda''} \to X_{\lambda'}$ by putting $u^n = \widetilde{h}_\lambda^n f_{\lambda''}^n$ and $v^n = p_{\lambda'\lambda''}$, for every $n \geq \max\{n_0, n_{\lambda''}\}$, and $u^n = v^n = p_{\lambda'\lambda''}$, for every

$n < \max\{n_0, n_{\lambda''}\}$. For every $n \geq \max\{n_0, n_{\lambda''}\}$, using (10) and (17), one obtains

$$f^n_{\lambda'} u^n = f^n_{\lambda'} \widetilde{h}^n_\lambda f^n_{\lambda''} \overset{(17)}{=} q_{\lambda'\lambda''} f^n_{\lambda''} \overset{(10)}{=} f^n_{\lambda'} p_{\lambda'\lambda''} = f^n_{\lambda'} v^n.$$

If $n < \max\{n_0, n_{\lambda''}\}$, it trivially follows that

$$f^n_{\lambda'} u^n = f^n_{\lambda'} v^n$$

Now, by condition (M $''$) from [5, Theorem 4], there exists an $n_1 \in \mathbb{N}$ such that,

$$p_{\lambda\lambda'} u^{n'} = p_{\lambda\lambda'} v^{n'},$$

for every $n \geq n_1$. Therefore, for every $n \geq \max\{n_0, n_1, n_{\lambda''}\}$, one has

$$p_{\lambda\lambda'} \widetilde{h}^n_\lambda f^n_{\lambda''} = p_{\lambda\lambda''}. \tag{18}$$

Further, for $n \geqslant n_{\lambda'}$, for the pair $\lambda \leq \lambda'$, the appropriate commutativity condition (14) holds. Therefore, for $n \geq \max\{n_0, n_1, n_{\lambda''}, n_{\lambda'}\}$, one has

$$f^n_\lambda p_{\lambda\lambda'} \widetilde{h}^n_\lambda \overset{(14)}{=} q_{\lambda\lambda'} f^n_{\lambda'} \widetilde{h}^n_\lambda \overset{(17)}{=} q_{\lambda\lambda''}. \tag{19}$$

Finally, for every $n' \geq n = \max\{n_0, n_1, n_{\lambda''}, n_{\lambda'}\}$, we define

$$h^{n'}_\lambda = p_{\lambda\lambda'} \widetilde{h}^{n'}_\lambda.$$

Now, by (18) and (19), a straightforward argument shows that (6) and (7) hold. Thus, we have proved that $(1_\Lambda, f^n_\lambda)$ fulfills the conditions of Theorem 3.1 and therefore, $\boldsymbol{f}^*$ is an isomorphism of *pro*-Set*. ∎

An immediate consequence of Theorem 4.2 and Theorem 4.7 is the following:

COROLLARY 4.8. *The category pro-Set is balanced.*

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