

A REMARK ON THE DIOPHANTINE EQUATION

$$(x^3 - 1)/(x - 1) = (y^n - 1)/(y - 1)$$

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ABSTRACT. In this remark, we use some properties of simple continued fractions of quadratic irrational numbers to prove that the equation

$$\frac{x^3 - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \quad x, y, n \in \mathbb{N}, x > 1, y > 1, n > 3, 2 \nmid n$$

has only the solutions $(x, y, n) = (5, 2, 5)$ and $(90, 2, 13)$.

For any positive integer N with $N > 2$, let $s(N)$ denote the number of solutions (x, m) of the equation

$$(1) \quad N = \frac{x^m - 1}{x - 1}, \quad x, m \in \mathbb{N}, x \geq 2, m > 2.$$

Ratat [17] in 1916 and Goormaghtigh [10] in 1917 found that $s(31) = 2$ and $s(8191) = 2$, respectively. We consider the equation

$$(2) \quad \frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \quad x > 1, y > 1, m > 2, n > 2, x \neq y, \quad \text{for } x, y \in \mathbb{N}.$$

It has been conjectured that the equation (2) has only a finite number of solutions, even that has only two solutions $(x, y, m, n) = (5, 2, 3, 5)$, $(90, 2, 3, 13)$.

This is rather a difficult question. Many authors have proved that if two of the variables x, y, m, n are fixed then the equation (2) has a finite number of solutions. See for examples [1, 3, 4, 5, 12, 13, 19, 20, 21, 16, 22, 23, 24]. Remark that two known solutions of (2) are both satisfying $m = 3$. If $m = 3$, the equation (2) has the form

$$(3) \quad \frac{x^3 - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \quad x, y, n \in \mathbb{N}, x > y > 1, n > 3.$$

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We know that the equation (3) has two solutions $(x, y, n) = (5, 2, 5)$ and $(90, 2, 13)$, and any other possible solution is called an *exceptional solution* [13]. If we prove that (3) has no exceptional solutions, then the conjecture is true under the condition $m = 3$. Le [12] proved that (3) has no exceptional solution with $\omega(y) > 1$, where $\omega(a)$ denote the number of distinct prime divisors of a (the reference [12] contains an error, one can refer to [2] for a correct version). Nesterenko and Shorey [16] proved that any exceptional solution of (3) with $2 \nmid n$ must be $n \geq 25$. Le [14] has given the relative upper bound, namely, $x < 2^{(n^2-4n+6)/2}$ and $y < 2^{(n-3)/2}$.

In [13], Le proved that, for any exceptional solution of (3), we must have $\gcd(x, y) > 1$ and $y \nmid x$. In 2005, Yuan [26] used this result and properties of Pellian equations and proved the following result.

THEOREM 1. *The equation (3) has only the solutions $(x, y, n) = (5, 2, 5)$ and $(90, 2, 13)$ with n is odd.*

In this paper, we prove Theorem 1 using another method. We will use the simple continued fraction expansion to express the solutions of the Pellian equation obtained from (3), and we get a contradiction to the result in [13] by congruence relations.

Now, let us recall some properties of continued fractions. The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{a+\sqrt{d}}{b}$ is periodic. This expansion can be obtained using the following algorithm [11]. Let $s_0 = a$, $t_0 = b$ and

$$(4) \quad a_k = \left\lfloor \frac{s_k + \sqrt{d}}{t_k} \right\rfloor, \quad s_{k+1} = a_k t_k - s_k, \quad t_{k+1} = \frac{d - s_{k+1}^2}{t_k}, \quad k \geq 0.$$

If $(s_c, t_c) = (s_d, t_d)$ for $c < d$, then

$$\alpha = [a_0, \dots, a_{c-1}, \overline{a_c, \dots, a_{d-1}}].$$

Let p_n/q_n denote the n^{th} convergent of α . The following result of Worley [25] and Dujella [6] extends classical results of Legendre and Fatou [9] concerning Diophantine approximations of the form $|\alpha - \frac{a}{b}| < \frac{1}{2b^2}$ and $|\alpha - \frac{a}{b}| < \frac{1}{b^2}$.

LEMMA 2 (Worley [25], Dujella [6]). *Let α be a real number and a and b coprime nonzero integers, satisfying the inequality*

$$\left| \alpha - \frac{a}{b} \right| < \frac{\sigma}{b^2},$$

where σ is a positive real number. Then $(a, b) = (rp_{k+1} \pm up_k, rq_{k+1} \pm uq_k)$, for some $k \geq -1$ and nonnegative integers r and u such that $ru < 2\sigma$.

In fact, by Fatou [9] we have

$$(5) \quad \frac{a}{b} = \frac{p_k}{q_k} \text{ or } \frac{p_{k+1} \pm p_k}{q_{k+1} \pm q_k}$$

for $\sigma = 1$. And explicit versions of above result for $\sigma = 2$, were given by Worley [25, Corollary, p. 206]: $|\alpha - \frac{a}{b}| < \frac{2}{b^2}$, implies

$$(6) \quad \frac{a}{b} = \frac{p_k}{q_k}, \frac{p_{k+1} \pm p_k}{q_{k+1} \pm q_k}, \frac{2p_{k+1} \pm p_k}{2q_{k+1} \pm q_k}, \frac{3p_{k+1} + p_k}{3q_{k+1} + q_k}, \frac{p_{k+1} \pm 2p_k}{q_{k+1} \pm 2q_k} \text{ or } \frac{p_{k+1} - 3p_k}{q_{k+1} - 3q_k}.$$

For the explicit results of the bigger σ , please refer [7].

The next useful result is due to Dujella and Jadrijević [8]. It helps us to simplify our proof.

LEMMA 3. *Let ab be a positive integer which is not a perfect square, and let $\frac{p_k}{q_k}$ denotes the k^{th} convergent of continued fraction expansion of $\sqrt{\frac{a}{b}}$. Let the sequences (s_k) and (t_k) be defined by (4) for the quadratic irrational $\frac{\sqrt{ab}}{b}$. Then*

$$a(rq_{k+1} + uq_k)^2 - b(rp_{k+1} + up_k)^2 = (-1)^k (u^2 t_{k+1} + 2rus_{k+2} - r^2 t_{k+2}).$$

The following lemma is due to Le [13].

LEMMA 4. *If (x, y, n) is a exceptional solution of equation (3), then $\gcd(x, y) > 1$ and $y \nmid x$.*

PROOF OF THEOREM 1. Let (x, y, n) be a solution of (3) with n odd. Let us rewrite (3) into

$$(7) \quad (y - 1)(2x + 1)^2 - 4y(y^{(n-1)/2})^2 = -3y - 1, \quad n > 3.$$

Let $\gcd(2x + 1, y) = d$. Then d is a divisor of $-3y - 1$. This implies $d = 1$, since $\gcd(-3y - 1, y) = 1$. Now, assume that $y \geq 2$. Let us put $X = 2x + 1$ and $Y = y^{(n-1)/2}$ with $\gcd(X, Y) = 1$. Then we have

$$\begin{aligned} \left| \sqrt{\frac{y-1}{4y}} - \frac{Y}{X} \right| &= \left| \frac{y-1}{4y} - \frac{Y^2}{X^2} \right| \cdot \left| \sqrt{\frac{y-1}{4y}} + \frac{Y}{X} \right|^{-1} \\ &< \frac{3y+1}{4yX^2} \cdot \left| 2\sqrt{\frac{y-1}{4y}} \right|^{-1} = \frac{3y+1}{4\sqrt{y(y-1)}} \cdot X^{-2}. \end{aligned}$$

It follows that

$$(8) \quad \left| \sqrt{\frac{y-1}{4y}} - \frac{Y}{X} \right| < \frac{\sigma}{X^2},$$

where $\sigma = 1$ if $y \geq 4$ and $\sigma = 2$ if $y = 2$ or 3 .

On the other hand, let $\alpha = \sqrt{\frac{y-1}{4y}} = \frac{\sqrt{4y(y-1)}}{4y}$, one can see that

$$\alpha = [0, 2, \overline{y-1}, 4],$$

$$(s_0, t_0) = (0, 4y), (s_1, t_1) = (0, y-1),$$

$$(s_2, t_2) = (2y-2, 4), (s_3, t_3) = (2y-2, y-1), (s_4, t_4) = (2y-2, 4).$$

Since the period of continued fraction expansion of α is equal to 2, according to Lemma 2, we only need to consider $(X, Y) = (rq_{k+1} \pm uq_k, rp_{k+1} \pm up_k)$

for $k = 0, 1, 2$. We use Lemma 3 to check all possibilities $(k, r, \pm u)$ such that the equation

$$(9) \quad (y-1)X^2 - 4yY^2 = \gamma$$

satisfies the inequality (8). Thus we have $\gamma \in \{-4, y-1, -3y-1, 5y-9\}$ for $y \geq 4$ and $\gamma \in \{-4, y-1, -3y-1, -4y, 5y-9, -7y-9, 9y-25, -11y-25, 12y-16, 13y-49\}$ for $2 \leq y \leq 3$. Moreover, the result $\gamma = -3y-1$ comes from

$$(k, r, \pm u) = \begin{cases} (2t, 1, -1), (2t-1, 1, 1), & \text{if } y \geq 4, \\ (2t, 1, -1), (2t-1, 1, 1), (2t, 1, -3), (2t-1, 3, 1), & \text{if } 2 \leq y \leq 3. \end{cases}$$

- The cases $(r, \pm u) = (1, 1)$ or $(1, -1)$ imply

$$(10) \quad (2x+1, y^{(n-1)/2}) = (q_{2t+1} - q_{2t}, p_{2t+1} - p_{2t}),$$

or

$$(11) \quad (2x+1, y^{(n-1)/2}) = (q_{2t} + q_{2t-1}, p_{2t} + p_{2t-1}).$$

By simple computations, we get

$$\begin{aligned} q_0 &= 1, & q_2 &= 2y-1, & q_{2t+4} &= (4y-2)q_{2t+2} - q_{2t}, \\ q_1 &= 2, & q_3 &= 8y-2, & q_{2t+3} &= (4y-2)q_{2t+1} - q_{2t-1}. \end{aligned}$$

Then by induction one can easily prove the following property:

$$(12) \quad q_{2t} \equiv (-1)^t \pmod{2y} \text{ and } q_{2t+1} \equiv 2(-1)^t \pmod{2y}.$$

From (10), (11) and (12), we get

$$x \equiv 0 \text{ or } -1 \pmod{y}.$$

But this and Lemma 4 give a contradiction.

- The additional cases $(r, \pm u) = (3, 1)$ or $(1, -3)$ (for $y = 2, 3$) gives

$$(13) \quad (2x+1, y^{(n-1)/2}) = (q_{2t+1} - 3q_{2t}, p_{2t+1} - 3p_{2t}),$$

or

$$(14) \quad (2x+1, y^{(n-1)/2}) = (3q_{2t} + q_{2t-1}, 3p_{2t} + p_{2t-1}).$$

We use a similar argument to get

$$x \equiv 0 \text{ or } -1 \pmod{y}.$$

We get the contradiction as in the above case.

This completes the proof. \square

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