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MINIMAL NONABELIAN AND MAXIMAL SUBGROUPS OF A FINITE p-GROUP

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Abstract. The p-groups all of whose nonabelian maximal subgroups are either absolutely regular or of maximal class, are classified (Theorem 2.1). For the main result of [CP] and [ZAX] classifying the p-groups all of whose proper nonabelian subgroups are metacyclic, we offer a proof which is shorter and not so involved. In conclusion we study, in some detail, the p-groups containing an abelian maximal subgroup.

1. Introduction

This note supplements papers [B5] and [BJ2].

Our notation is the same as in [B1-B3] and [BJ1, BJ2]. In what follows, p is a prime and G a finite p-group. A group G is said to be an $\mathcal{A}_n$-group, if all its subgroups of index p^n are abelian but it contains a nonabelian subgroup of index p^{n-1} (so that $\mathcal{A}_1$-groups are minimal nonabelian). The $\mathcal{A}_1$-groups are classified in [R] and $\mathcal{A}_2$-groups are classified by L. Kazarin and V. Sheriev, independently (see [BJ1, Theorem 5.6]). Set $\Omega_1(G) = \langle x \in G \mid x^p = 1\rangle$, $\mho_1(G) = \langle x^p \mid x \in G\rangle$. If $H \leq G$, then $H_G = \cap_{x\in G} H^x$ is the *core* of H in G. A group G is said to be absolutely regular if $|G/\mho_1(G)| < p^p$; by Hall's regularity criterion, such G is regular. Let $\mathrm{cl}(G)$ denote the class of G. A group G of order p^m is of maximal class if $\mathrm{cl}(G) = m - 1 \geq 2$. A group G is said to be an L_s-*group* [B3] (s is a positive integer) if $\Omega_1(G)$ is of order p^s and exponent p and $G/\Omega_1(G)$ is cyclic of order $> p$. By E_{p^n} we denote the elementary abelian group of order p^n. Let $G', \Phi(G)$ and $\mathrm{Z}(G)$ denote the derived subgroup, the Frattini subgroup and the center of G, respectively. We

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write $p^{\mathrm{d}(G)} = |G : \Phi(G)|$; then $\mathrm{d}(G)$ is the minimal number of generators of G.

It is proved in [BJ2, Theorem 2.2] that if all nonabelian maximal subgroups of a nonabelian two-generator 2-group G are two-generator, then G is either minimal nonabelian or metacyclic. The condition $\mathrm{d}(G) = 2$ in that theorem, however, is very restrictive. Indeed, as [BJ1, §4] shows, classification of nonabelian 2-groups G all of whose maximal subgroups are two-generator but $\mathrm{d}(G) = 3$, is one of outstanding open problems of p-group theory. (Note that, for $p > 2$, Blackburn [Bla2] has proved that all such groups are $\mathcal{A}_2$-groups.)

In Theorem 2.1, the main result of this note, the p-groups all of whose nonabelian maximal subgroups are either absolutely regular or of maximal class, are classified. In conclusion of this section we classify (Theorem 1.1) the p-groups all of whose nonabelian maximal subgroups are metacyclic (this is the main result of [ZAX]; in [CP] the case $p = 2$ is considered only). We do not use, in our proof, as in [ZAX], the classification of metacyclic and minimal nonmetacyclic p-groups; note that the proof in [CP] is more elementary. In §3 we treat nonabelian p-groups with abelian subgroup of index p.

The note is self contained modulo the following lemma.

Lemma J. *Let G be a nonabelian p-group.*

(a) *[T]; see also [I, Lemma 12.12]. If $A < G$ is abelian of index p, then $|G| = p|G'||\mathrm{Z}(G)|$.*
(b) *[B2, Lemma 3] The number of abelian maximal subgroups in G equals $0, 1$ or $p + 1$.*
(c) *[B2, Proposition 19(a)] If $B \leq G$ is nonabelian of order p^3 and $\mathrm{C}_G(B) < B$, then G is of maximal class.*
(d) *[Bla1] Let G be of maximal class. If $|G| > p^p$, then G is irregular. If $|G| > p^{p+1}$, then exactly one maximal subgroup of G is not of maximal class (it is absolutely regular).*
(e) *[BJ1, Lemma 3.2(a)] If $G' \leq \mathrm{Z}(G)$, $\exp(G') = p$ and $\mathrm{d}(G) = 2$, then G is an $\mathcal{A}_1$-group.*
(f) *[Bla2]; see also [B1, Theorem 7.6]. If G has no normal subgroup of order p^p and exponent p, it is either absolutely regular or of maximal class. A group G of maximal class and order $> p^{p+1}$ has no normal subgroup of order p^p and exponent p.*
(g) *[Bla2]; see also [B1, Theorem 7.5]. Suppose that G is not absolutely regular. If G contains an absolutely regular maximal subgroup M, then either G is of maximal class or $G = M\Omega_1(G)$ with $|\Omega_1(G)| = p^p$.*
(h) *[B1, Theorem 7.4] Suppose that G is not of maximal class. If G contains a subgroup of maximal class and index p, then $\mathrm{d}(G) = 3$ and the number of subgroups of maximal class and index p in G equals p^2. If, in addition, $|G| > p^{p+1}$, then $|G/\mho_1(G)| = p^{p+1}$ so G has no absolutely regular maximal subgroups.*

(i) *[B1, Theorem 5.2] If $p > 2$, G is of maximal class and $H < G$ is such that $\mathrm{d}(H) > p - 1$, then G is isomorphic to a Sylow p-subgroup of the symmetric group of degree p^2.*

(j) *[BJ2, Theorem 2.2] If all nonabelian maximal subgroups of a nonabelian two-generator 2-group G are two-generator, then G is either metacyclic or minimal nonabelian.*

(k) *Let G be an $\mathcal{A}_1$-group. Then G is nonmetacyclic if and only if $\Omega_1(G) \cong \mathrm{E}_{p^3}$. Next, $\mathrm{d(G)} = 2$, $\mathrm{Z(G)} = \Phi(\mathrm{G})$ so, if $N \triangleleft G$ and G/N is noncyclic, then $N \leq \mathrm{Z}(G)$.*

(l) *(Fitting) If $A, B < G$ are normal, then $\mathrm{cl}(AB) \leq \mathrm{cl}(A) + \mathrm{cl}(B)$.*

(m) *[Bla2, Lemma 4.3]; see also [BJ2, Theorem 7.4]. If $p > 2$ and G has no normal subgroup $\cong E_{p^3}$, then G is either metacyclic, or 3-group of maximal class, or $G = \Omega_1(G)C$, where $\Omega_1(G)$ is nonabelian of order p^3 and exponent p and C is cyclic of index p^2 in G.*

(n) *If G has a cyclic subgroup of index p, then either G is a 2-group of maximal class or $G \cong \mathrm{M}_{\mathrm{p^n}}$.*

We use freely basic properties of regular p-groups.

In what follows we use freely the following fact. If G is a nonabelian two-generator p-group, then $\mathrm{Z}(G) \leq \Phi(G)$. Assume that this is false. Then there is in G a maximal subgroup H such that $G = H\mathrm{Z(G)}$; then H is nonabelian. In that case, $H/(H \cap \mathrm{Z(G)}) \cong \mathrm{G/Z(G)}$ is noncyclic so, setting $D = H \cap Z(G)$, we get $G/D = (H/D) \times (\mathrm{Z(G)/D})$ so $\mathrm{d}(G) \geq \mathrm{d(G/D)} = \mathrm{d(H/D)} + \mathrm{d(Z(G)/D)} \geq 2 + 1 = 3$, contrary to the hypothesis.

We offer a new proof of the following

THEOREM 1.1 ([CP] (for $p = 2$), [ZAX]). *Suppose that a nonabelian p-group G is neither minimal nonabelian nor metacyclic nor minimal nonmetacyclic. If all nonabelian maximal subgroups of G are metacyclic, then one and only one of the following holds:*

(a) *$G = M \times C$, where $M \not\cong \mathrm{Q}_8$ is a metacyclic $\mathcal{A}_1$-group and $|C| = p$.*

(b) *$p > 2$, $\mathrm{d}(G) = 2$, $G = \Omega_1(G)C$, where $\Omega_1(G) \cong \mathrm{E}_{p^3}$, C is a cyclic subgroup of index p^2 in G, $C_G = \mho_1(C) = \mathrm{Z}(G)$ is of index p^3 in G (so that, if $|G| > p^4$, then G is an $\mathrm{L_s}$-group and $\mathcal{A}_2$-group).*

PROOF. Let us check that groups of (a) and (b) satisfy the hypothesis. Indeed, let $G = M \times C$ be as in (a) and $U < G$ maximal. If $C < U$, then, by the modular law, $U = C \times (U \cap M)$ so U is abelian since M is an $\mathcal{A}_1$-subgroup. If $C \not\leq U$, then $G = C \times U$ so $U \cong G/C \cong M$ is metacyclic. Now let G be as in (b) and $V < G$ maximal. If $\Omega_1(G) \leq V$, then, by the modular law, $V = \Omega_1(G)\mho_1(C)$ so V is abelian since $\Omega_1(G)$ is abelian and $\mho_1(C) = \mathrm{Z}(G)$. Now assume that $\Omega_1(G) \not\leq V$. Then $\Omega_1(V) = V \cap \Omega_1(G) \cong \mathrm{E}_{p^2}$ so $|V/\mho_1(V)| = |\Omega_1(V)| = p^2$ so V is metacyclic (Lemma J(m)).

Now, assuming that G satisfies the hypothesis, we have to prove that G is either as in (a) or in (b). By hypothesis, there are in G two maximal subgroups

M and A such that M is nonabelian so metacyclic and A is nonmetacyclic so abelian; then $\mathrm{d}(A) > 2$ and $\mathrm{d}(G) \le \mathrm{d}(M) + 1 = 2 + 1 = 3$. Since $M \cap A$ is a noncyclic metacyclic maximal subgroup of A, we get $\mathrm{d}(A) = 3$. Set $E = \Omega_1(A)$; then $\mathrm{E}_{p^3} \cong E \lhd G$. By the product formula, $G = ME$ so $M \cap E \cong \mathrm{E}_{\mathrm{p}^2}$. All maximal subgroups of G containing E, are nonmetacyclic so abelian, hence $\mathrm{d}(G/E) = \mathrm{d}(\mathrm{M}/(\mathrm{M} \cap \mathrm{E})) \le 2$ (Lemma J(b)). In what follows, A, M and E denote the subgroups defined in this paragraph.

Let $\mathrm{d}(G/E) = 2$. Then there is a maximal subgroup $B/E < G/E$ with $B \ne A$ so $E \le A \cap B = \mathrm{Z}(G)$ since B, being nonmetacyclic, is abelian. If $x \in E - M$, then $G = M \times X$, where $X = \langle x \rangle$. Let $N < M$ be maximal. Then $N \times X$ is abelian. Indeed, assume that this is false; then $\mathrm{d}(N) = 2$ so $\mathrm{d}(X \times N) = 3$, and $X \times N$ is abelian, by hypothesis. Thus, all maximal subgroups of M are abelian so M is an $\mathcal{A}_1$-group. We conclude that $\Omega_1(G) = E$, unless $M \cong \mathrm{D}_8$.

Now let G/E be cyclic; then $G' < E$.

(i) Let $|G/E| = p$; then $|M| = p^3$. If $\mathrm{C}_G(M) < M$, then G is of maximal class (Lemma J(e)) and $p > 2$ since G is not metacyclic, and $E = A$ is the unique abelian maximal subgroup of G (Lemma J(l)) so $\exp(G) = p^2$ since $M < G$ is metacyclic. If $E = \mathrm{Z}(G) \times L$, then $L_G = \{1\}$ so G is isomorphic to a subgroup of exponent p^2 of a Sylow p-subgroup of the symmetric group S_{p^2}; then G has a nonabelian subgroup of order p^3 and exponent p (Lemma J(i)) which is nonmetacyclic, a contradiction. Thus, $G = M\mathrm{Z}(G)$. If $\mathrm{Z}(G)$ is noncyclic, then $G = M \times L$ is as in (a) (in that case, $M \not\cong \mathrm{Q}_8$ since G is not minimal nonmetacyclic). If $\mathrm{Z}(G)$ is cyclic, then, since $|\mathrm{Z}(G)| = p^2$, we get $G = E\mathrm{Z}(G)$, by the product formula, so G is abelian, a contradiction.

(ii) Now let G/E be cyclic of order $> p$; then $G' < E$ so $|G'| \le p^2$. We have $\Omega_1(G/\Omega_1(G)) < A/\Omega_1(G)$ so $\Omega_1(G) = \Omega_1(A) = E$. Since $M/(M \cap E) = M/\Omega_1(M) \cong G/E$ is cyclic, we get $M \cong \mathrm{M}_{p^n}$, $n > 3$, since M is nonabelian and has a cyclic subgroup of index p (Lemma J(n)). Thus, all nonabelian maximal subgroups of G are $\cong \mathrm{M}_{\mathrm{p}^n}$ (it follows that G is an $\mathcal{A}_2$-group so one can use the classification of $\mathcal{A}_2$-groups [BJ2, Theorem 5.6], however we prefer to present independent, more elementary, proof).

Let $\mathrm{d}(G) = 2$; then $\mathrm{Z}(G) \le \Phi(G)$ and, since G is not an $\mathcal{A}_1$-group, we get $G' \not\le \mathrm{Z(G)}$ so $G' \cong \mathrm{E}_{p^2}$ (Lemma J(e)); then $\mathrm{cl}(G) = 3$ and A is the unique abelian maximal subgroup of G and $|G : \mathrm{Z}(G)| = p|G'| = p^3$ (Lemma J(a)). Since $G' < \Phi(G) < M$, we get $G' = \Omega_1(M)$ so M/G' is a cyclic subgroup of index p in the abelian group G/G'. Since $\mathrm{Z}(G) < M$, then $\mathrm{Z}(G) = \mathrm{Z}(M)$ (compare indices!) so $\mathrm{Z}(G)$ is cyclic. Assume that there is a cyclic $U/\mathrm{Z}(G)$ of index p in $G/\mathrm{Z}(G)$. Then U is abelian and metacyclic so $U \ne A$, a contradiction. Thus, $\exp(G/\mathrm{Z}(G)) = p$ so $G/\mathrm{Z}(G)$ is nonabelian of order p^3 and exponent p (recall that $\mathrm{cl}(G) = 3$); then $p > 2$. We have $\mathrm{Z}(G) < C < M$, where C is cyclic of index p in M. Since $|G : C| = p^2$, we get $G = EC$, and C is not normal in G since G' is noncyclic. Since a Sylow

p-subgroup of $\mathrm{Aut}(E)$ is of exponent p, we conclude that $\mathrm{Z}(G) = \mho_1(C)$ so G is as in (b).

Now we let $\mathrm{d}(G) = 3$. Then G/G' has no cyclic subgroup of index p so $|G'| = p$, and we get $|G : \mathrm{Z}(G)| = p|G'| = p^2$ (Lemma J(a)). Since $|A : \mathrm{Z}(G)| = p$ and $\mathrm{d}(A) = 3$, the subgroup $\mathrm{Z}(G)$ is noncyclic. By what has been proved already, $M \cong \mathrm{M}_{p^n}$. In that case, $\Omega_1(\mathrm{Z}(G)) \not\leq M$ since $\mathrm{Z}(M)$ is cyclic. We have $G = M\mathrm{Z(G)}$ so $M \cap \mathrm{Z(G)} = \mathrm{Z(M)}$ (compare orders!) is cyclic and $|\mathrm{Z(G)}| = \mathrm{p}|\mathrm{Z(M)}$. In that case, $\mathrm{Z}(M)$ is a cyclic subgroup of maximal order in $\mathrm{Z}(G)$ so, by basic theorem on abelian p-groups, $\mathrm{Z}(G) = \mathrm{Z(M)} \times \mathrm{L}$, where $|L| = p$ and $L \not\leq M$. Then $G = M \times L$ so G is as in (a). □

Supplement to Theorem 1.1. Let a nonabelian 2-group G be neither metacyclic nor $\mathcal{A}_i$-group ($i = 1, 2$) nor minimal nonmetacyclic. Suppose that all proper nonabelian subgroups of G are two-generator. Then $\mathrm{d}(G) = 3$ and nonabelian maximal subgroups of G are either metacyclic or minimal nonabelian. Let, in addition, $|G| > 2^5$. Then, if $H < G$ is a nonmetacyclic $\mathcal{A}_1$-subgroup, then $\Omega_1(G) = \Omega_1(H) = E \cong \mathrm{E}_8$. Next, $G/E \in \{\mathrm{Q}_8, \mathrm{M}_{2^n}\}$ and G' is contained in the center of every nonmetacyclic maximal subgroup of G so, if G has two distinct nonmetacyclic maximal subgroups, then $\mathrm{cl}(G) = 2$.

Proof. By Lemma J(j), nonabelian maximal subgroups of G are either metacyclic or minimal nonabelian so $\mathrm{d}(G) = 3$ [B1, Theorem 3.3].

There is a nonabelian maximal $M < G$ which is not an $\mathcal{A}_1$-group so M is metacyclic and, by Lemma J(e), $|M'| > 2$. In view of Theorem 1.1, one may assume that G has a maximal subgroup H which is neither abelian nor metacyclic; then H is an $\mathcal{A}_1$-group with $\Omega_1(H) \cong \mathrm{E}_8$ (Lemma J(k)). If $K < G$ is nonabelian maximal, then K' is cyclic, $K' \leq \Phi(K) \leq \Phi(G) < H$ and H/K' is noncyclic since H is not metacyclic, so $K' \leq \Phi(H) = \mathrm{Z}(H)$. Let $x, y \in G$. Then $\langle x, y\rangle \leq K_1$, where $K_1 < G$ is maximal (recall that $\mathrm{d}(G) = 3$); then $[x, y] \in K_1' \leq \mathrm{Z}(H)$, and we conclude that $G' \leq \mathrm{Z}(H)$. If $H_1 < G$ is another nonmetacyclic maximal subgroup, then $G' \leq \mathrm{Z}(H_1)$ so $\mathrm{C}_G(G') \geq HH_1 = G$ and $\mathrm{cl}(G) = 2$. In what follows, H and E are as defined in this paragraph.

Now we let $|G| > 2^5$. Assume that there is an involution $x \in G - E$ and set $L = E\langle x\rangle$; then $|L| = 2^4$ since $E \lhd G$. However, since M is metacyclic, we get $\exp(M \cap L) > 2$ so L is nonabelian since $\Omega_1(L) = L$. Then L is not an $\mathcal{A}_1$-subgroup (Lemma J(k)), contrary to the hypothesis. Thus $\Omega_1(G) = E$ and $G = ME$ so $G/E \cong M/(M \cap E)$. However, M' is cyclic of order > 2 so $G/E (\cong M/E)$ is nonabelian.

If $E \leq \mathrm{Z}(G)$, then $G = M \times C$ for some $C < E$ of order 2. Since $|M'| > 2$, there is in M a nonabelian maximal subgroup M_1. However, the nonabelian maximal subgroup $M_1 \times C$ of G is neither $\mathcal{A}_1$-subgroup nor metacyclic, a contradiction. Thus, $E \not\leq \mathrm{Z}(G)$.

If a noncyclic subgroup $T/E < G/E$ is maximal, then $E \leq \mathrm{Z}(T)$ (this is obvious if T is abelian, and follows from Lemma J(k) if T is an $\mathcal{A}_1$-subgroup;

note that T is nonmetacyclic). Since $E \not\leq \mathrm{Z}(G)$, the nonabelian group G/E has at most one noncyclic maximal subgroup. If all maximal subgroups of G/E are cyclic, then $G/E \cong \mathrm{Q}_8$. If G/E has exactly one noncyclic maximal subgroup, then, by Lemma J(n), $G/E \cong \mathrm{M}_{2^k}$. ☐

2. p-groups, all of whose nonabelian maximal subgroups are either absolutely regular or of maximal class

Let G be a nonabelian 2-group all of whose nonabelian maximal subgroups are of maximal class. Suppose that G is neither minimal nonabelian nor a group of maximal class. Then G contains a subgroup of maximal class and index 2 so, by Lemma J(h), $\mathrm{d}(G) = 3$ and G contains exactly 4 subgroups of maximal class and index 2. It follows that G contains exactly 3 abelian maximal subgroups so $\mathrm{cl}(G) = 2$, and we conclude that $|G| = 2^4$. By Lemma J(c), $G = M\mathrm{Z}(\mathrm{G})$, where M is nonabelian of order 8. Therefore, since absolutely regular 2-groups are cyclic, we confine, in the following theorem, to case $p > 2$.

Theorem 2.1. *Let a nonabelian p-group G be neither minimal nonabelian nor absolutely regular, $p > 2$ and $|G| > p^p$. If all nonabelian maximal subgroups of G are either absolutely regular or of maximal class, then one of the following holds:*

(i) *G is of maximal class and order $> p^{p+1}$,*
(ii) *G is of maximal class and order p^{p+1} with $|\Omega_1(G)| = p^{p-1}$,*
(iii) *G is of maximal class and order p^{p+1} with abelian maximal subgroup,*
(iv) *G is of maximal class and order p^{p+1}, $\Omega_1(G) = G$ and all maximal subgroups of G of exponent p are of maximal class,*
(v) *$p = 3$, $|G| = 3^4$, $G = M\mathrm{Z}(G)$, where $|\mathrm{Z}(G)| = 3^2$, M is nonabelian of order 3^3,*
(vi) *$G = B \times C$ where B is absolutely regular, $|C| = p$, $|\Omega_1(G)| = p^p$, $\Omega_1(G) \leq \mathrm{Z}(G)$, $\mathrm{d}(\mathrm{G}/\Omega_1(\mathrm{G})) = 2$. All maximal subgroups of B containing $\Omega_1(B)$, are abelian,*
(vii) *G is regular of order p^{p+1}, $\Omega_1(G)$ of order p^p is either abelian or of maximal class,*
(viii) *G is an L_p-group, $|G : \mathrm{C}_G(\Omega_1(G))| = p$.*

Groups (i)–(viii) satisfy the hypothesis.

Proof. The last assertion is checked easily as will be clear from the proof. It remains to show that if G satisfies the hypothesis, it is one of groups (i)–(viii).

(a) Suppose that G is of maximal class. If $|G| > p^{p+1}$, then G satisfies the hypothesis (Lemma J(d)). Now let $|G| = p^{p+1}$. If G has an abelian subgroup of index p, then all its nonabelian maximal subgroups are of maximal class (Lemma J(l)) so G satisfies the hypothesis. Next assume that G has no abelian subgroup of index p. If all maximal subgroups of G are absolutely regular, then $|\Omega_1(G)| = p^{p-1}$ so G is as in (ii). If $M < G$ is maximal and of exponent p, it is of maximal class and G is as in (iv). In what follows we assume that G is not of maximal class.

(b) Suppose that $|G| = p^{p+1}$. Then G is regular, by assumption in (a).

Suppose that $\exp(G) = p$. Then G has no absolutely regular maximal subgroup. Since not all maximal subgroups of G are of maximal class, there is in G a subgroup $A \cong \mathrm{E}_{p^p}$. By hypothesis, G has a nonabelian maximal subgroup M; then M is of maximal class. By Lemma J(h), there are in G exactly p^2 subgroups of maximal class and index p so it has exactly $p+1 > 1$ abelian maximal subgroups; then $|G : \mathrm{Z}(G)| = p^2$, $|G'| = \frac{1}{p}|G : \mathrm{Z}(G)| = p$ (Lemma J(a)). Then $|M| = p^3$ so $|G| = p^4$. Since $|G| = p^{p+1}$, we get $p = 3$. Since $\mathrm{Z}(G) \not\leq M$, we get $G = M \times \langle x \rangle$ for $x \in \mathrm{Z}(G) - M$, and G is as in (v).

Now let $\exp(G) > p$. Then $|\Omega_1(G)| = p^p$ since G is not absolutely regular so $\Omega_1(G)$ is either abelian or of maximal class; then G is as (vii). Next we assume that $|G| > p^{p+1}$. By Lemma J(f), there is in G a normal subgroup R of order p^p and exponent p.

(c) Suppose that $|G| > p^{p+2}$. Then all maximal subgroups of G containing R are neither absolutely regular nor of maximal class (Lemma J(f)). Therefore, if $R < A$, where A is maximal in G, then A is abelian. Assume that $R < \Omega_1(G)$. Let $x \in G - R$ be of order p; then $L = \langle x, R \rangle$ is elementary abelian of order p^{p+1}. Consideration of intersection of a maximal subgroup, say H, with L shows that H is neither of maximal class (Lemma J(i) or J(f) since $|H| > p^{p+1}$) nor absolutely regular. Then all maximal subgroups of G are abelian, a contradiction since G is not minimal nonabelian. Thus, $R = \Omega_1(G)$. Therefore, if G/R is cyclic, then G is an L_p-group so it is as in (viii).

Suppose that G/R is noncyclic. Since all maximal subgroups of G, containing R, are abelian, it follows that $R \leq \mathrm{Z}(G)$ and $|G : \mathrm{Z}(G)| = p^2$ so $\mathrm{cl}(G) = 2$, and $\mathrm{d}(G/R) = 2$ (Lemma J(b,k)). Since G is not minimal nonabelian, it contains a nonabelian maximal subgroup B. Since $|B \cap R| > p$, B is not of maximal class so it is absolutely regular. Then $R \not\leq B$ so $G = B \times C$ for some $C < R$ of order p, and G is as in (vi).

(d) Suppose that $|G| = p^{p+2}$. If G/R is cyclic, then G is an L_p-group. Indeed, let $R < M < G$. Then M is either abelian or of maximal class. If M is abelian, then, as in (c), $R = \Omega_1(G)$ so G is an L_p-group. Assume that M is of maximal class. Let $D < R$ be G-invariant of index p^2. Then $M \leq \mathrm{C}_G(R/D)$ so M/D is abelian of order p^3 and M is not of maximal class, a contradiction.

Let $G/R \cong \mathrm{E}_{p^2}$.

(d1) Suppose that all $M < G$ such that $R < M$, are abelian. Then $R = \mathrm{Z}(\mathrm{G})$ and $\mathrm{cl}(G) = 2$ so G has no subgroups of maximal class and index p. By hypothesis, G has a nonabelian absolutely regular maximal subgroup

B. Then $R = \Omega_1(G) \not\leq B$ so $G = B \times C$, where $C < R$ is of order p so G is as in (vi).

(d2) Now suppose that there is nonabelian $M < G$ such that $R < M$. Then M is of maximal class so the number of subgroups of maximal class and index p in G is exactly p^2 (Lemma J(h)). Since $\mathrm{d}(G) = 3$ and G has no absolutely regular maximal subgroup (Lemma J(h)), the number of abelian subgroups of index p in G is exactly $p+1$. In that case, as in (b), $|G| = p^4 < p^{p+2}$, a final contradiction. □

3. Nonabelian p-groups containing an abelian maximal subgroup

Let a nonabelian p-group contains an abelian maximal subgroup. Such groups, playing important role in finite p-group theory, were classified in two long papers [NR] and [NRSB], however, it is fairly difficult to extract from these papers the results about their subgroup structure. A nonabelian two-generator p-group G containing an abelian subgroup A of index p is considered in [XZA, Lemma 3.1]. In Proposition 3.1 we consider more general situation.

To facilitate future considerations, we prove using induction on $|G|$ that, if a nonabelian p-group G contains an abelian maximal subgroup A and $|G : G'| = p^2$, then G is of maximal class. If $|G| = p^3$, the assertion is obvious so we let $|G| > p^3$. By Lemma J(a), $|\mathrm{Z}(G)| = \frac{1}{p}|G : G'| = p$ so $\mathrm{Z}(G) < G'$. In that case, $|(G/\mathrm{Z}(G)) : (G/\mathrm{Z}(G))'| = |G : G'| = p^2$ so, by induction, $G/\mathrm{Z}(G)$ is of maximal class, and we are done since $|\mathrm{Z}(G)| = p$.

Proposition 3.1. *Let A be a maximal subgroup of a nonabelian two-generator p-group G. Suppose that $R = \langle x^p \mid x \in G - A\rangle \leq \mathrm{Z}(G)$ and A/R is abelian. Then $\Omega_1(G/R) = G/R$ and G/R is of maximal class, unless G is minimal nonabelian.*

Proof. Write $\bar{G} = G/R$; then $\bar{G}$ is noncyclic since $R \leq \mathrm{Z}(G)$. Since all elements of the set $\bar{G} - \bar{A}$ have the same order p, it follows that $\Omega_1(\bar{G}) \geq \langle \bar{G} - \bar{A}\rangle = \bar{G}$ so $\bar{G}' = \Phi(\bar{G})$, and hence $\bar{G}/\bar{G}' \cong \mathrm{E}_{p^2}$ since $\mathrm{d}(\bar{G}) = \mathrm{d}(G) = 2$ in view of $R \leq \mho_1(G) \leq \Phi(G)$. If $\bar{G}' = \{\bar{1}\}$, then $\mathrm{Z}(G) = R = \Phi(G)$ so G is minimal nonabelian. If $\bar{G}' > \{\bar{1}\}$, then $\bar{G}$ is nonabelian so it is of maximal class, by the paragraph preceding the proposition. □

Suppose that A is an abelian maximal subgroup of a nonabelian p-group G; then $\mathrm{Z}(G) < A$. Write $\bar{G} = G/\mathrm{Z}(G)$. Then all elements of the set $\bar{G} - \bar{A}$ have the same order p so $\Omega_1(\bar{G}) = \bar{G}$ and $\bar{G}' = \Phi(\bar{G})$. Indeed, if $x \in G - A$, then $\mathrm{C}_G(x^p) \geq \langle x, A\rangle = G$ so $x^p \in \mathrm{Z}(G)$, and all claims in the previous sentence follow. If, in addition, $\mathrm{d}(G) = 2$, then either G/R is of maximal class or G is minimal nonabelian (here R is as in Proposition 3.6). Thus,

Corollary 3.2. *Let G be a nonabelian two-generator p-group and $A < G$ abelian of index p. Then $R = \langle x^p \mid x \in G - A\rangle \leq \mathrm{Z}(G)$, $\Omega_1(G/R) = G/R$ and either G/R is of maximal class or G is minimal nonabelian.*

PROPOSITION 3.3. *Let A be an abelian maximal subgroup of a nonabelian p-group G and let $x \in G - A$ be fixed. Then the following conditions are equivalent:*

(a) $\mathrm{cl}(G) = 2$.
(b) *For every $a \in A - \mathrm{Z}(G)$, the subgroup $H_a = \langle x, a\rangle$ is minimal nonabelian.*

PROOF. (a) $\Rightarrow$ (b): Since $G = A\langle x\rangle$, we get $\mathrm{C}_A(x) = \mathrm{Z}(G)$. Therefore, if $a \in A - \mathrm{Z}(G)$, then $xa \neq ax$ so $\mathrm{cl}(H_a) = 2$, where $H_a = \langle a, x\rangle$. Then $H_a/\mathrm{Z}(H_a)$ is abelian and its exponent equals p (Corollary 3.2) since $A \cap H_a$ is maximal abelian in H_a. Since $\mathrm{d}(H_a) = 2$, we get $H_a/\mathrm{Z}(H_a) \cong \mathrm{E}_{p^2}$ so H_a is minimal nonabelian, and (b) is proved.

(b) $\Rightarrow$ (a): As in (a), $\mathrm{C}_A(x) = \mathrm{Z}(G)$, and $H_a \not\leq A$ so $|H_a : (H_a \cap A)| = p$ hence $H_a \cap A$ is a maximal abelian subgroup of H_a. Therefore, $\mathrm{Z}(H_a) = \Phi(H_a) < H_a \cap A$ since H_a is an $\mathcal{A}_1$-subgroup. Since $\mathrm{C}_G(\mathrm{Z}(H_a)) \geq AH_a = G$, we get $\mathrm{Z}(H_a) \leq \mathrm{Z}(G)$. Set $R = \langle \mathrm{Z}(H_b) \mid b \in A - \mathrm{Z}(G)\rangle$; then $R \leq \mathrm{Z}(G)$ and $H_a \cap R = \mathrm{Z}(H_a)$ for all $a \in A - \mathrm{Z}(G)$. Write $\bar{G} = G/R$. Assume that $\bar{G}$ is not abelian. Then there is $\bar{b} \in \bar{A} - \mathrm{Z}(\bar{G})$ such that $\bar{K} = \langle \bar{x}, \bar{b}\rangle$ is nonabelian. In that case, $H_b = \langle x, b\rangle$ is an $\mathcal{A}_1$-group since $b \in A - \mathrm{Z}(G)$. However, $\bar{K} = \bar{H}_b \cong H_b/(H_b \cap R) \cong H_b/\mathrm{Z}(H_b) \cong \mathrm{E}_{p^2}$, a contradiction. Thus, $\bar{G} = G/R$ is abelian so $\mathrm{cl}(G) = 2$ since $R \leq \mathrm{Z}(G)$, and (a) is proved. □

REMARK 3.4. Let A be a normal abelian subgroup of a nonabelian p-group G, $A \not\leq \mathrm{Z}(G)$ and $x \in G - \mathrm{C}_G(A)$; then $A\langle x\rangle$ is nonabelian. To prove that there exists $a \in A - \mathrm{C}_A(x)$ such that the subgroup $H_a = \langle x, a\rangle$ is minimal nonabelian, we suppose that G is a counterexample of minimal order; then $G = A\langle x\rangle$. Write $C = \mathrm{C}_G(x)$; then $C = \langle x\rangle\mathrm{Z}(G)$ is a maximal abelian subgroup of $G = CA$. Let $C < B \leq G$ be such that $|B : C| = p$. Then B is nonabelian, $B = C(A \cap B)$ and $A \cap B$ is an abelian normal subgroup of B. It follows from $G = CA = BA$ that $B/(A \cap B) \cong G/A$ is cyclic, and so $A \cap B \not\leq \mathrm{Z}(B)$; therefore, $G = B$ so $|G : C| = p$. Write $\bar{G} = G/\mathrm{Z}(G)$; then $\bar{C} \cong \langle x\rangle/(\langle x\rangle \cap \mathrm{Z}(G))$ is cyclic of index p in $\bar{G}$. Since $G = CA$, where C and A are G-invariant abelian subgroups, we get $\mathrm{cl}(G) = 2$ (Lemma J(l)) so $\bar{G}$ is noncyclic abelian, and so it has a cyclic subgroup $\bar{C}_1$ of index p which is $\neq \bar{C}$. Then C and C_1 are different abelian subgroups of G of index p so $C \cap C_1 = \mathrm{Z}(G)$ and $|G'| = \frac{1}{p}|G : \mathrm{Z}(G)| = p$ (Lemma J(a)). If $a \in A - \mathrm{C}_G(x)$, then $H_a = \langle x, a\rangle$ is minimal nonabelian (Lemma J(e)) since $H_a' = G'$ is of order p. In particular (Janko), if $A < G$ is a maximal abelian normal subgroup, then for every $x \in G - A$ there exists $a \in A$ such that $\langle x, a\rangle$ is minimal nonabelian [BJ2, Lemma 4.1].

It is trivial that a p-group G is not covered by p proper subgroups. I am indebted to Moshe Roitman (University of Haifa) for the following, probably, known

Remark 3.5. Let $P_1, \dots, P_{p+1}$ be pairwise distinct subgroups of a p-group G of order p^n. First assume that these subgroups are maximal. We prove by induction on k, $1 < k \le p+1$, that

$$|\bigcup_{i=1}^{k} P_i| \le p^{n-2} + k(p^{n-1} - p^{n-2}),$$

and that we have equality just if $|\bigcap_{i=1}^{k} P_i| = p^{n-2}$. This is clear for $k = 2$. Since the intersection of two distinct maximal subgroups of G has order p^{n-2}, we get, by induction on k, that

$$\begin{aligned} |\bigcup_{i=1}^{k} P_i| &= |\bigcup_{i=1}^{k-1} P_i| + |P_k| - |(\bigcup_{i=1}^{k-1} P_i) \cap P_k| \\ &\le p^{n-2} + (k-1)(p^{n-1} - p^{n-2}) + p^{n-1} - p^{n-2} \\ &= p^{n-2} + k(p^{n-1} - p^{n-2}). \end{aligned}$$

Moreover, we have equality if and only if

$$|(\bigcup_{i=1}^{k-1} P_i)| = p^{n-2} + (k-1)(p^{n-1} - p^{n-2}) \quad \text{and} \quad |(\bigcup_{i=1}^{k-1} P_i) \cap P_k| = p^{n-2};$$

this is equivalent to the condition $|\bigcap_{i=1}^{k} P_i| = p^{n-2}$. Now let $k = p + 1$ and maximal subgroups $P_1, \dots, P_{p+1}$ cover G. Then, since

$$|(\bigcup_{i=1}^{p+1} P_i)| = p^n = p^{n-2} + (p+1)(p^{n-1} - p^{n-2}),$$

we obtain that $|\bigcap_{i=1}^{p+1} P_i| = p^{n-2}$. In the general case, we have to show that all the subgroups P_i are maximal in G if they cover G. Assume, for example, that P_1 is not maximal in G. For each i, let Q_i be a maximal subgroup containing P_i, and let $H = \bigcap_{i=1}^{p+1} Q_i$. There exists an element $x \in Q_1 - (P_1 \cup H)$. Since H is equal to the intersection of any two distinct subgroups among the Q_i's by what has been proved above, we see that x belongs to a unique subgroup Q_i, namely to Q_1. Hence $x \notin \bigcup_{i=1}^{p+1} P_i$, a contradiction.

Let $\alpha_1(G)$ denote the number of $\mathcal{A}_1$-subgroups in p-group G. Recall that a nonabelian p-group G is generated by $\mathcal{A}_1$-subgroups (see [B7] and [B4]).

Remark 3.6. The result of Remark 3.4 allows us to produce in a p-group G, which is neither abelian nor minimal nonabelian, a lot of $\mathcal{A}_1$-subgroups. Indeed, let $A < G$ be a maximal abelian normal subgroup; then $C_G(A) = A$. By Remark 3.4, the set-theoretic union U of all $\mathcal{A}_1$-subgroups of G contains the set $G - A$ so $G = U \cup A$ (this coincides with [BJ2, Lemma 4.1]). Thus, G is the set-theoretic union of $\alpha_1(G) + 1$ proper subgroups, one of which is A and other $\alpha_1(G)$ are $\mathcal{A}_1$-subgroups, so, by Remark 3.5, $\alpha_1(G) \ge p$. Thus, if $\alpha_1(G) = p$, then all $\mathcal{A}_1$-subgroups and A are maximal in G (Remark 3.5) so G is an $\mathcal{A}_2$-group. Next we prove that if $\alpha_1(G) = p+1$, then G is an $\mathcal{A}_2$-group again. Let A be as above and $M \leq G$ be an $\mathcal{A}_2$-subgroup; then $\mathrm{d}(M) \leq 3$. Assume that $M < G$. Then there is an $\mathcal{A}_1$-subgroup $L < G$ such that $L \not\leq M$ so $p+1 = \alpha_1(G) \geq \alpha_1(M)+1$. It follows that $\alpha_1(M) = p$, by the above, and $G = A \cup M \cup L$ is the set-theoretic union of three proper subgroups (Remark 3.4) which is impossible for $p > 2$ (Remark 3.5). Now we let $p = 2$. By Remark 3.5, A, M and L are maximal in G and their intersection has index 4 in G. Next, $L \cap A$ is maximal abelian in L so $\mathrm{Z}(L) = \Phi(L) < L \cap A$, and we get $\mathrm{C}_G(\mathrm{Z}(L)) \geq AL = G$. Thus, $\mathrm{Z}(L) \leq \mathrm{Z}(G)$, $|G : \mathrm{Z}(L)| = 8$. We have $\mathrm{Z}(L) = \Phi(L) \leq \Phi(G) < M$ so $|M : \mathrm{Z}(L)| = 4$. It follows that $\mathrm{d}(M) = 3$ (otherwise, M is minimal nonabelian) so M has exactly 7 maximal subgroups. Then, by Lemma J(b), $\alpha_1(M) \geq 7 - 3 = 4 > 3 = \alpha_1(G)$, a contradiction. Thus, $G = M$ so G is an $\mathcal{A}_2$-group.[1]

4. Problems

Below we formulate some related problems.

1. Classify the irregular p-groups, $p < 5$, all of whose nonabelian maximal subgroups are either minimal nonabelian or of maximal class. (If $p \geq 5$, then our group has no minimal nonabelian subgroup of index p, by Lemma J(k,f,g), so Theorem 2.1 solves the problem.)

2. Classify the p-groups all of whose nonabelian maximal subgroups are either minimal nonabelian or metacyclic.

3. Let $M < G$ be maximal and $\mathrm{Z}(G) < M$. Study the structure of M if, whenever $x \in G - M$ and $a \in M$, then either $xa = ax$ or $\langle x, a \rangle$ is (i) an $\mathcal{A}_1$-group, (ii) a group of maximal class, (iii) a metacyclic group, (iv) a group of class 2 (four different problems).

4. Classify the p-groups all of whose maximal subgroups are of the form $M \times E$, where M is metacyclic and E is abelian.

5. Classify the p-groups G such that $|M'| \leq p$ for all maximal subgroups of G.

6. Classify the 2-groups all of whose two-generator subgroups are metacyclic.

7. Study the p-groups all of whose $\mathcal{A}_1$-subgroups are metacyclic.

8. Classify the p-groups allowing irredundant covering by $p+2$ subgroups.

9. (i) Classify the p-groups all of whose maximal subgroups (nonabelian maximal subgroups) are special. (ii) Does there exist a special p-group all of whose maximal subgroups are special? If it exists, classify such groups.

[1] The remark yields a new proof of [B4, Lemma 6]. Moreover, it is proved in [B4, Theorem 9] that, if $\alpha_1(G) < p^2$, then G is also an $\mathcal{A}_2$-group.

10. Study the p-groups all of whose nonabelian maximal subgroups have cyclic centers.

11. Classify the p-groups all of whose nonabelian maximal subgroups, but one, are minimal nonabelian.

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