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SHORT PROOFS OF SOME BASIC CHARACTERIZATION THEOREMS OF FINITE p-GROUP THEORY

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Abstract. We offer short proofs of such basic results of finite p-group theory as theorems of Blackburn, Huppert, Ito-Ohara, Janko, Taussky. All proofs of those theorems are based on the following result: If G is a nonabelian metacyclic p-group and R is a proper G-invariant subgroup of G', then G/R is not metacyclic. In the second part we use Blackburn's theory of p-groups of maximal class. Here we prove that a p-group G is of maximal class if and only if $\Omega_2^*(G) = \langle x \in G \mid o(x) = p^2 \rangle$ is of maximal class. We also show that a noncyclic p-group G of exponent $> p$ contains two distinct maximal cyclic subgroups A and B of orders $> p$ such that $|A \cap B| = p$, unless $p = 2$ and G is dihedral.

1°. This note is a continuation of the author's previous papers [Ber1, Ber2, Ber4].

Only finite p-groups, where p is a prime, are considered. The same notation as in [Ber1] is used. The nth member of the lower central series of G is denoted by $\mathrm{K}_n(G)$. Given a p-group G and a natural number n, set $\mho_n(G) = \langle x^{p^n} \mid x \in G\rangle$, $\Omega_n(G) = \langle x \in G \mid o(x) \leq p^n\rangle$, $\Omega_n^*(G) = \langle x \in G \mid o(x) = p^n\rangle$, $\mho^2(G) = \mho_1(\mho_1(G))$, $p^{\mathrm{d}(G)} = |G : \Phi(G)|$, where $\Phi(G)$ is the Frattini subgroup of G. Next, G' is the derived subgroup and $\mathrm{Z}(G)$ is the center of G. A group G of order p^m is of *maximal class* if $m > 2$ and $\mathrm{cl}(G) = m - 1$. A group G is *metacyclic* if it contains a normal cyclic subgroup C such that G/C is cyclic. A group G is said to be *minimal nonabelian* if it is nonabelian but all its proper subgroups are abelian. A p-group G is *regular* if, for $x, y \in G$, there is $z \in \langle x, y\rangle'$ such that $(xy)^p = x^p y^p z^p$. A p-group G is *absolutely regular* if $|G/\mho_1(G)| < p^p$. A p-group G is *powerful*

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[LM] provided $G' \le \mho_{\epsilon_p}(G)$, where $\epsilon_2 = 2$ and $\epsilon_p = 1$ for $p > 2$. By $c_n(G)$ we denote the number of cyclic subgroups of order p^n in G.

In Section 2° we show that some basic results of p-group theory are easy consequences of Theorem 2. In Section 3° we use [Ber1, Theorem 5.1] (= Lemma 1(d)), a variant of Blackburn's result [Ber3, Theorem 9.7], characterizing p-groups of maximal class. The following results of this note are new: Supplement 2 to Corollary 11, Corollary 14, theorems 22, 24, 25, 27, 30, Supplements 1 and 2 to Theorem 22 and Remark 18.

In Lemma 1 we gathered some known results. All of them are proved in [Ber3].

LEMMA 1. *Let G be a nonabelian p-group.*

(a) (Tuan) *If G has an abelian subgroup of index p, then $|G| = p|G'||Z(G)|$.*
(b) (Mann)[1] *If M, N are two distinct maximal subgroups of G, then $|G'| \le p|M'N'|$.*
(c) (Blackburn) *If $G/K_{p+1}(G)$ is of maximal class, then G is also of maximal class.*
(d) (Berkovich) *Suppose that G contains the unique subgroup L of index p^{p+1}. If G/L is of maximal class, then G is also of maximal class (obviously, this is also true in the case $|G : L| > p^{p+1}$).*
(e) (Hall) *If $cl(G) < p$ or $\exp(G) = p$, then G is regular.*
(f) (Hall) *If G is regular, then $\exp(\Omega_n(G)) \le p^n$ and $|\Omega_n(G)| = |G/\mho_n(G)|$.*
(g) (Blackburn) *If G is of maximal class and order $\le p^{p+1}$, then $\exp(G) \le p^2$. If $|G| \le p^p$, then $|G : \Omega_1(G)| \le p$. If $|G| = p^{p+1}$, then G is irregular and $|G/\mho_1(G)| = p^p$.*
(h) (Blackburn) *A p-group G of maximal class has an absolutely regular subgroup G_1 of index p, and $\exp(G_1) = \exp(G)$. In particular, if G is of order $> p^{p+1}$, it has no normal subgroup of order p^p and exponent p since, for each $n > 1$, G has at most one normal subgroup of index p^n. If $|G| > p^p$, then $|\Omega_1(G_1)| = p^{p-1}$. Next, all elements of the set $G - G_1$ have orders $\le p^2$.*
(i) (Berkovich) *If G has a nonabelian subgroup B of order p^3 such that $C_G(B) < B$, then G is of maximal class.*
(j) (Lubotzky-Mann) *If G is powerful and X is a maximal cyclic subgroup of G, then $X \not\le \Phi(G)$.*
(k) (Berkovich) *If N is a two-generator G-invariant subgroup of $\Phi(G)$, then N is metacyclic.*
(l) (Blackburn) *If G is of maximal class and order $> p^{p+1}$, then exactly p maximal subgroups of G are of maximal class, the $(p+1)$-th maximal subgroup G_1, the fundamental subgroup of G, is absolutely regular.*

[1]This result is also contained in Kazarin's Ph.D. thesis (unpublished).

(m) (Berkovich) *If $H < G$ and $N_G(H)$ is of maximal class, then G is also of maximal class.*
(n) (Berkovich) *Let G be irregular but not of maximal class. If $U \lhd G$, $|U| < p^p$ and $\exp(U) = p$, then there is in G a normal subgroup V of order p^p and exponent p such that $U < V$.*
(o) (Blackburn) *If G is irregular of maximal class and a normal subgroup V of G is of order p^{p-1}, then $\exp(G/V) = \frac{1}{p} \cdot \exp(G)$.*
(p) (Blackburn) *If an irregular group G has an absolutely regular maximal subgroup H, then either G is of maximal class or $G = H\Omega_1(G)$, where $|\Omega_1(G)| = p^p$.*
(q) (Blackburn) *If an irregular group G has no normal subgroup of order p^p and exponent p, it is of maximal class.*
(r) (Hall's regularity criterion) *Absolutely regular p-groups are regular.*
(s) (Berkovich) *If G is a p-group of maximal class and order $> p^{p+1}$, then $c_2(G) \equiv p^{p-2} \pmod{p^{p-1}}$.*
(t) (Berkovich) *If a p-group G is neither absolutely regular nor of maximal class, then $c_2(G) \equiv 0 \pmod{p^{p-1}}$.*
(u) (Berkovich) *Let G be not of maximal class. Then the number of subgroups of maximal class and index p in G is divisible by p^2.*
(v) (Berkovich) *If G is neither absolutely regular nor of maximal class, then the number of subgroups of order p^p and exponent p is G is $\equiv 1 \pmod{p}$.*
(w) (Blackburn) *If $\Omega_2(G)$ is metacyclic, then G is also metacyclic.*

2°. Blackburn [Bla1, Theorem 2.3] has proved that a p-group G is metacyclic if and only if $G/\mathrm{K}_3(G)\Phi(G')$ is metacyclic. This result is an important source of characterizations of metacyclic p-groups. Here we prove this assertion in slightly another, but equivalent form (Theorem 2). The main point of this section is to deduce from that theorem some basic results of p-group theory. Besides, our proof of Theorem 2 is essentially simpler than the Philip Hall's proof presented in [Bla1].

We prove Blackburn's result in the following form.

THEOREM 2. *The following conditions for a nonabelian p-group G are equivalent:*
(a) *G is metacyclic.*
(b) *The quotient group G/R is metacyclic for some G-invariant subgroup R of index p in G'.*[2]

REMARK 3. If there is a G-invariant subgroup $R < G'$ such that G/R is metacyclic, then G is also metacyclic. Indeed, take $R \le R_1 < G'$, where R_1 is

[2]Since $\mathrm{d}(G) = 2$, R is characteristic in G, by Lemma 7(b).

G-invariant of index p in G'; then G/R_1 is also metacyclic as an epimorphic image of G/R, whence G is metacyclic (Theorem 2).[3]

Theorem 2 and Remark 3, in view of $\mathrm{K}_3(G)\Phi(G') < G'$, imply the original Blackburn's result:

COROLLARY 4 ([Bla1, Theorem 2.3]). *If a p-group G is such that $G/\mathrm{K}_3(G)\Phi(G')$ is metacyclic, then G is also metacyclic.*

The following lemma is a useful criterion for a p-group to be minimal nonabelian.

LEMMA 5 ([BJ1, Lemma 65.2(a)]). *If a p-group G is such that $d(G) = 2$, $G' \le \mathrm{Z}(G)$ and $\exp(G') = p$, then G is minimal nonabelian.*

PROOF. It follows from $\exp(G') = p$ that G is nonabelian. For $x, y \in G$, we have $1 = [x, y]^p = [x, y^p]$ so $y^p \in \mathrm{Z}(G)$ whence $\Phi(G) = G'\mho_1(G) \le \mathrm{Z}(G)$ and $\Phi(G) = \mathrm{Z}(G)$ since $|G : \mathrm{Z}(G)| > p$. If $M < G$ is maximal, then $|M : \mathrm{Z}(G)| = p$ so M is abelian. We are done. ∎

LEMMA 6 ([Red]). *If G is a nonmetacyclic minimal nonabelian p-group, then*

$$(*) \qquad G = \langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1, \ [a, b] = c, \ [a, c] = [b, c] = 1, m \ge n\rangle.$$

Here $G' = \langle c\rangle$ is a maximal cyclic subgroup of G, $\mathrm{Z}(G) = \Phi(G) = \langle a^p, b^p, c\rangle$ has index p^2 in G, $\Omega_1(G) = \langle a^{p^{m-1}}, b^{p^{n-1}}, c\rangle$ is elementary abelian of order p^3, $\mho_1(G) = \langle a^p, b^p\rangle$ and $|G/\mho_1(G)| = p^3$ if and only if $p > 2$.

PROOF. If $a, b \in G$ are not permutable, then $G = \langle a, b\rangle$. If A, B are distinct maximal subgroups of G, then $A \cap B = \mathrm{Z}(G)$ and $G/\mathrm{Z}(G)$ is abelian of type (p, p) so $\Phi(G) = \mathrm{Z}(G)$. We have $|G'| = \frac{1}{p}|G : \mathrm{Z}(G)| = p$ (Lemma 1(a)). Let $G/G' = (U/G') \times (V/G')$, where both factors are cyclic of orders p^m, p^n, respectively, $m \ge n$; then U and V are noncyclic (G is not metacyclic!) so $\Omega_1(G) = \Omega_1(U)\Omega_1(V)$ is elementary abelian of order p^3 (indeed, $\Omega_1(G)/G' = \Omega_1(G/G')$). Assume that $G' < L < G$, where L is cyclic of order p^2. We have $m + n > 2$. Then $G/G' = (C/G') \times (D/G')$, where $L \le C$ and C/G' is cyclic, by [BZ, Theorem 1.16]. It follows from $G' = \Phi(L) \le \Phi(C)$ that $1 = \mathrm{d}(C/G') = \mathrm{d}(C)$ so C is cyclic. Since G/C is cyclic, G is metacyclic, contrary to the hypothesis. All remaining assertions are obvious. ∎

It follows from Lemma 6 that a minimal nonabelian p-group G is not metacyclic if and only if G' is a maximal cyclic subgroup of G.

[3]In our case, R_1 is determined uniquely since $G'/\mathrm{K}_3(G)$ is cyclic and $\mathrm{K}_3(G) \le R_1$; see Lemma 7, below.

LEMMA 7. (a) *If a p-group G is two-generator of class 2, then G' is cyclic.*

(b) [Bla1, Lemma 2.2] *If G is a nonabelian two-generator p-group, then $G'/\mathrm{K}_3(G)$ is cyclic.*

PROOF. (a) Since $\mathrm{cl}(G) = 2$, then $[xy, uv] = [x, u][x, v][y, u][y, v]$ for $x, y, u, v \in G$. Let $G = \langle a, b\rangle$ and $w, z \in G$. Expressing w, z in terms of a and b and using the above identity, we see that the commutator $[w, z]$ is a power of $[a, b]$ so $G' = \langle [a, b]\rangle$.

(b) The statement (b) follows from (a): $G/\mathrm{K}_3(G)$ is two-generator of class 2. □

REMARK 8. Let G be a nonmetacyclic minimal nonabelian 2-group given by $(*)$. We claim that if $G = AB$, where A and B are cyclic, then $n = 1$. Assume that this is false. Set $\bar{G} = G/\langle a^4, b^4\rangle$; then $\bar{G}$ is of order 2^5 and exponent 4 so it is not a product of two cyclic subgroups (of order ≤ 4). This is a contradiction since $\bar{G} = \bar{A}\bar{B}$. Let, in addition, $m > n = 1$. We claim that G is indeed a product of two cyclic subgroups. Set $A = \langle a\rangle$. Then $G/\mho_1(A)$ is dihedral of order 8. Let $U/\mho_1(A) < G/\mho_1(A)$ be cyclic of order 4. If B_0 is a cyclic subgroup which covers $U/\mho_1(A)$, then, by the product formula, $G = AB_0$, as want to be shown.

REMARK 9. Let G be a nonabelian two-generator p-group. It follows from Lemma 6 and Theorem 2 that if R is a G-invariant subgroup of index p in G', then G is metacyclic if and only if G'/R is not a maximal cyclic subgroup of G/R. In particular, we obtain the following theorem from [IO]: The derived subgroup G' of a 2-group $G = AB$ (A and B are cyclic) is contained properly in a cyclic subgroup of G if and only if G is metacyclic.

REMARK 10. If G is a nonmetacyclic p-group, then it contains a characteristic subgroup R such that G/R is one of the following groups: (i) elementary abelian of order $> p^2$, (ii) nonabelian of order p^3 and exponent p, (iii) a 2-group, given in $(*)$, with $m = n = 2$, (iv) a 2-group, given in $(*)$, with $m = 2$, $n = 1$. (Obviously, groups (i)-(iii) are not products of two cyclic subgroups.) Let us prove this. If $\mathrm{d}(G) > 2$, we have case (i) with $R = \Phi(G)$. Next assume that $\mathrm{d}(G) = 2$. If $p > 2$, we have case (ii) with $R = \mathrm{K}_3(G)\Phi(G')\mho_1(G) = \mathrm{K}_3(G)\mho_1(G)$ (Theorem 2 and Lemmas 5-7). If $p = 2$, we have cases (iii) or (iv) with $R = \mathrm{K}_3(G)\Phi(G')\mho_2(G)$ (Corollary 4 and Lemma 6).[4]

It follows from Remark 10 that, if a 2-group G and all its characteristic maximal subgroups are two-generator, then G is either metacyclic or $G/\mathrm{K}_3(G)\Phi(G')\mho_2(G)$ is a group (iii) of Remark 10 (the second group has no

[4]The group (iv) is a product of two cyclic subgroups; see the footnote to the proof of Corollary 17.

characteristic maximal subgroups at all). In particular, a 2-group G is metacyclic if and only if G and all its maximal subgroups are two-generator. This also follows from

COROLLARY 11 ([Bla1]). *Suppose that a nonabelian p-group G and all its maximal subgroups are two-generator. Then G is either metacyclic or $p > 2$ and $K_3(G) = \mho_1(G)$ has index p^3 in G (in the last case, $|G : G'| = p^2$).*

PROOF. Suppose that G is not metacyclic. In cases (iii) and (iv) of Remark 10, G has a maximal subgroup that is not generated by two elements so $p > 2$. By Lemma 6, G has no nonmetacyclic epimorphic image which is minimal nonabelian of order $> p^3$. The group G also has no epimorphic image of order $> p^3$ and exponent p so $|G/\mho_1(G)| = p^3$. Assume that $|G : G'| > p^2$. Let R be a G-invariant subgroup of index p in G'. Then G/R is a nonmetacyclic minimal nonabelian group (Theorem 2 and Lemma 5) of order $> p^3$, contrary to what has just been said. Thus, $|G : G'| = p^2$. Then $G/\mathrm{K}_3(G)$ is minimal nonabelian since its center $G'/\mathrm{K}_3(G)$ has index p^2; moreover, that quotient group is nonmetacyclic (Remark 3). In that case, by the above, $|G/\mathrm{K}_3(G)| = p^3 = |G/\mho_1(G)|$ so $\mathrm{K}_3(G) = \mho_1(G)$ since $\mho_1(G) \leq \mathrm{K}_3(G)$. ∎

COROLLARY 12 (Taussky). *Let G be a nonabelian 2-group. If $|G : G'| = 4$, then G is of maximal class.*

PROOF. Let R be a G-invariant subgroup of index 2 in G'. Then G/R is nonabelian of order 8 so metacyclic; then G is metacyclic (Theorem 2) so G has a normal cyclic subgroup $U < G$ such that G/U is cyclic. Since $G' < U$, we get $|G : U| = 2$, and the result follows from description of 2-groups with cyclic subgroup of index 2. ∎

COROLLARY 13 (Huppert [Hup]). *Let G be a p-group, $p > 2$, and let $|G/\mho_1(G)| \leq p^2$. Then G is metacyclic.*

PROOF. Assuming that G is not metacyclic, we must consider cases (i) and (ii) of Remark 10. We have there $|G/\mho_1(G)| > p^2$, a contradiction. ∎

SUPPLEMENT 1 TO COROLLARY 11. *Let G be a p-group.*

(a) *G is metacyclic if and only if $G/\mho^2(G)$ is metacyclic.*
(b) [Ber1, Theorem 3.4] *G is metacyclic if and only if $G/\mho_2(G)$ is metacyclic.*

PROOF. (b) ⇒ (a) since $\mho_2(G) \leq \mho^2(G)$ (indeed, $\exp(G/\mho^2(G)) \leq p^2$). If G is not metacyclic, then $G/\mho^2(G)$ is not metacyclic (Remark 10), proving (b). ∎

SUPPLEMENT 2 TO COROLLARY 11. *Suppose that a nonabelian p-group G and all its characteristic subgroups of index $\frac{1}{p^2}|G : G'|$ are two-generator.*

Then either G is metacyclic or $p > 2$ and $G/\mathrm{K}_3(G)$ is of order p^3 and exponent p. If, in addition, a nonmetacyclic p-group G and all its characteristic subgroups are two-generator, then $K_3(G) = \mho_1(G)$.

PROOF. By Lemma 7(b), a G-invariant subgroup R of index p in G' is characteristic in G. Suppose that G is nonmetacyclic; then G/R is also nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5). Assume that $|G/R| > p^3$. Then $H/R = \Omega_1(G/R)$ is elementary abelian of order p^3 (Lemma 6), $\mathrm{d}(H) > 2$, $|G/H| = \frac{1}{p^2}|G/G'|$ and H is characteristic in G, contrary to the hypothesis. Thus, $|G/R| = p^3$ so $|G/G'| = \frac{1}{p}|G/R| = p^2$; then $p > 2$ since G/R is nonmetacyclic (Corollary 12). It follows that $G/\mathrm{K}_3(G)$ is minimal nonabelian so $|G'/\mathrm{K}_3(G)| = p$ (Lemma 6); then $R = \mathrm{K}_3(G)$ and $\exp(G/R) = p$ since G/R is not metacyclic (Corollary 11).

Now suppose, in addition, that all characteristic subgroups of a nonmetacyclic p-group G are two-generator. Set $\bar{G} = G/\mho_1(G)$. Assume that $|\bar{G}| > p^3$. Let $\bar{G}$ be of order p^4; then it contains an abelian subgroup $\bar{A}$ of index p and $\mathrm{d}(A) \geq \mathrm{d}(\bar{A}) = 3$ so, by hypothesis, $\bar{A}$ is not characteristic in $\bar{G}$. Then $\bar{G}$ has another abelian maximal subgroup $\bar{B}$. We have $\bar{A} \cap \bar{B} = \mathrm{Z}(\bar{G})$ so $\bar{G}$ is minimal nonabelian since $\mathrm{d}(G) = 2$. But a minimal nonabelian group of exponent p has order p^3 (Lemma 6), a contradiction. Now let $|\bar{G}| > p^4$. Then $\mathrm{d}(\bar{G}') = 2$, by hypothesis, so $|\bar{G}'| = p^2$ since $\exp(\bar{G}') = p$ (Lemma 1(k)). In that case, $|\bar{G}| = |\bar{G} : \bar{G}'||\bar{G}'| = p^4$, contrary to the assumption. Thus, $|G/\mho_1(G)| = p^3$ so $\mathrm{K}_3(G) = \mho_1(G)$ since $\mathrm{K}_3(G) \leq \mho_1(G)$ and $|G/\mathrm{K}_3(G)| = p^3$. ∎

In particular, if a 2-group G and all its characteristic subgroups of index $\frac{1}{4}|G : G'|$ are two-generator, then G is metacyclic, and this implies Corollary 12.

In the proof of Theorem 2 we use only Lemma 7(b) which is independent of all other previously proved results.

PROOF OF THEOREM 2. It suffices to show that (b) $\Rightarrow$ (a). Since G/R is metacyclic, it has a normal cyclic subgroup U/R such that G/U is cyclic. Assume that U is noncyclic. Then U has a G-invariant subgroup T such that U/T is abelian of type (p, p). Set $\bar{G} = G/T$. In that case, $R \not\leq T$ since $\bar{U} = U/T$ cannot be an epimorphic image of the cyclic group U/R; then $G' \not\leq T$ so $\bar{G}$ is nonabelian. Next, $\bar{G}/\bar{G}'$ is noncyclic so $\bar{G}' < \bar{U}$ and $|\bar{G}'| = p$ since $|\bar{U}| = p^2$. It follows from $\bar{G}' = G'T/T \cong G'/(G' \cap T)$ that $G' \cap T = R$, by Lemma 7(b). Then $R = G' \cap T < T$, a contradiction.[5] ∎

If a p-group G is nonmetacyclic but all its proper epimorphic images are metacyclic, then either G is of order p^3 and exponent p or G is as given in $(*)$

[5]Isaacs proved the following equivalent of Theorem 2. Let G be a p-group and let $Z < G'$ be G-invariant of order p. If G/Z is metacyclic, then G is metacyclic; see [Ber5, Lemma 11].

with $m = 2$ and $n = 1$. Indeed, the result is trivial for abelian G. Now let G be nonabelian. Let R be a G-invariant subgroup of index p in G'; then G/R is not metacyclic (Theorem 2) so $R = \{1\}$, and we get $|G'| = p$. By Lemma 5, G is minimal nonabelian. Now the assertion follows from Lemma 6.

Corollary 14. *Suppose that a nonabelian and nonmetacyclic p-group G and all its maximal subgroups are two-generator, $p > 2$ and $|G| = p^m$, $m > 3$; then $\mathrm{cl}(G) > 2$. Set $K = \mathrm{K}_4(G)$ and $\bar{G} = G/K$. Then one of the following holds:*

(a) *$\bar{G}$ is of order p^4. In particular, if $p = 3$, then G is of maximal class.*
(b) *$|\bar{G}| = p^5$, all maximal subgroups of $\bar{G}$ are minimal nonabelian (see* [BJ2, Theorem 5.5] *for defining relations of $\bar{G}$).*

Proof. By Corollary 11, $\mathrm{K}_3(G) = \mho_1(G)$ has index p^3 in G so that $\mathrm{cl}(G) > 2$ since $m > 3$ and $|G : G'| = p^2$, $\mathrm{d}(G) = 2$. Then $\mathrm{Z}(\bar{G}) = \mathrm{K}_3(G)/K$ has index p^3 in $\bar{G}$ since $\mathrm{cl}(\bar{G}) = 3$. Let $\bar{M} < \bar{G}$ be maximal; then $|\bar{M} : \mathrm{Z}(\bar{G})| = \frac{1}{p}|\bar{G} : \mathrm{Z}(\bar{G})| = p^2$ and, since $\mathrm{d}(\bar{M}) = 2$, it follows that $\bar{M}$ is either abelian or minimal nonabelian. In view of Lemma 6, $\bar{G}$ has a nonabelian maximal subgroup, say $\bar{M}$. By Lemma 1(a), $\bar{G}$ has at most one abelian maximal subgroup.

Suppose that $\bar{G}$ has an abelian maximal subgroup, say $\bar{A}$. Then $|\bar{G}'| \leq p|\bar{M}'\bar{A}'| = p^2$ (Lemma 1(b)) so $|\bar{G}| = |\bar{G}'||\bar{G} : \bar{G}'| = p^4$, and we get $\mathrm{cl}(\bar{G}) = 3$. In particular, if $p = 3$, then G is of maximal class (Lemma 1(c)). Thus, G is as stated in part (a).

Now suppose that all maximal subgroups of $\bar{G}$ are minimal nonabelian; then $|\bar{G}| > p^4$. If $\bar{U}, \bar{V}$ are distinct maximal subgroups of $\bar{G}$, then $|\bar{G}'| \leq p|\bar{U}'\bar{V}'| = p^3$ so $|\bar{G}'| = p^3$ since $p^5 \leq |\bar{G}| = |\bar{G} : \bar{G}'||\bar{G}'| \leq p^5$. □

Blackburn found indices of the lower central series of groups of Corollary 14 for $p > 3$ (the case $p = 3$ is open); see [Bla2].

Our arguments in Corollary 15 and Remark 16 are based on [Jan2].

Corollary 15 (Janko [Jan2]). *If every maximal cyclic subgroup of a noncyclic p-group G is contained in a unique maximal subgroup of G, then G is metacyclic.*

Proof. Let N be a proper normal subgroup of G and let $U/N \leq G/N$ be maximal cyclic. Then $U = AN$ for a cyclic A. Let $B \geq A$ be a maximal cyclic subgroup of G; then $B \cap N = A \cap N$ and $U/N = BN/N$ so $|A| = |B|$ and $A = B$, i.e., A is a maximal cyclic subgroup of G. Assume that $K/N, M/N$ are distinct maximal subgroups of G/N containing U/N. Then $A \leq U \leq K \cap M$, contrary to the hypothesis. Thus, the hypothesis is inherited by epimorphic images.

Let $A < G$ be maximal cyclic. Then $A\Phi(G)/\Phi(G)$ is contained in a unique maximal subgroup of $G/\Phi(G)$ so $A\Phi(G)$ is maximal in G, and we conclude that $\mathrm{d}(G) = 2$. Assume that G is nonmetacyclic. Let R be a G-invariant subgroup of index p in G'. Then $\bar{G} = G/R$ is nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5) so $\bar{G}'$ is maximal cyclic in $\bar{G}$ (Lemma 6). Since $\bar{G}/\bar{G}'$ is abelian of rank 2, $\bar{G}'$ is contained in $1 + p > 1$ maximal subgroups of $\bar{G}$, contrary to the previous paragraph. □

Remark 16. Obviously, metacyclic p-groups are powerful for $p > 2$. Let us show (this is Janko's result as well) that G of Corollary 15 is also powerful for $p = 2$, unless G is of maximal class. Assume that G is not of maximal class. Then $|G/G'| > 4$ (Corollary 12) so $W = G/\mho_2(G)$ cannot be nonabelian of order 8. It suffices to show that W is abelian. Assume that this is false. Then $W = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1}\rangle$ is the unique nonabelian metacyclic group of order 2^4 and exponent 4 (Corollary 15). In that case, $W/\langle a^2b^2\rangle$ is ordinary quaternion so has two distinct maximal subgroups $U/\langle a^2b^2\rangle$ and $V/\langle a^2b^2\rangle$. Since $\langle a^2b^2\rangle$ is a maximal cyclic subgroup of W, we get a contradiction. Thus, G is powerful. Then, by Lemma 1(j), if $X < G$ is maximal cyclic, then X is not contained in $\Phi(G)$ (Lemma 1(j)) so $X\Phi(G)$ is the unique maximal subgroup of G containing X since $\mathrm{d}(G) = 2$. Thus, G satisfies the hypothesis of Corollary 15 if and only if it is powerful and metacyclic.

It follows from Corollary 13 that a p-group $G = AB$, where A and B are cyclic, is metacyclic if $p > 2$. This is not true for $p = 2$, however, we have

Corollary 17 (Ito-Ohara [IO]). *If a nonmetacyclic 2-group $G = AB$ is a product of two cyclic subgroups A and B, then G/G' is of type $(2^m, 2)$, $m > 1$.*

Proof. Let R be a G-invariant subgroup of index 2 in G'. Then $\bar{G} = G/R$ is nonmetacyclic (Theorem 2) and minimal nonabelian (Lemma 5) as in $(*)$. Since $\bar{G} = \bar{A}\bar{B}$, we get $n = 1$ (Remark 8). Next, $m > 1$ (Corollary 12). □

Remark 18. Suppose that a nonmetacyclic 2-group $G = AB$ is a product of two cyclic subgroups A and B. Since $A \cap B = \Phi(A) \cap \Phi(B)$, we get $\Phi(G) = \Phi(A)\Phi(B)$, by the product formula, so $\Phi(G)$ is metacyclic (Lemma 1(k)). It follows that all subgroups of G are three-generator. By Corollary 11, G has a maximal subgroup M with $\mathrm{d}(M) = 3$. We claim that M is the unique maximal subgroup of G which is not generated by two elements. Indeed, let U, V be maximal subgroups of G, containing A, B, respectively; then $U \neq V$. By the modular law, $U = A(U \cap B)$ and $V = B(V \cap A)$ so $\mathrm{d}(U) = 2 = \mathrm{d}(V)$ since G in nonmetacyclic. Since the set of maximal subgroups of G is $\{M, U, V\}$, our claim follows. In particular, M is characteristic in G. Set $\bar{G} = G/\mho_2(G)$; then $\bar{G} = \bar{A}\bar{B}$ so $|\bar{A}| = 4 = |\bar{B}|$ since $\bar{G}$ is of exponent 4 (in fact, $\bar{G}$ is a group (iv) of Remark 10).[6]

[6]The author and Janko [J5] have proved independently that subgroups U and V are metacyclic; see the proof of Supplement to Corollary 17 due to the author.

Suppose that X is a 2-group such that $\mathrm{d}(X) = 2$, $\exp(X) > 2$ and $\Phi(X)$ is metacyclic. We claim that $|X/\mho_2(X)| \leq 2^4$. Assume that this is false. Clearly, $\mho_2(X) \leq \mho_1(\Phi(X)) < \Phi(X)$ and $|\Phi(X)/\mho_2(X)| \leq |\Phi(X)/\mho_2(\Phi(X))| \leq 2^4$. To obtain a contradiction, one may assume that $\mho_2(X) = \{1\}$, i.e., $\exp(X) = 4$. Then $2^3 \leq |\Phi(X)| \leq 2^4$ since $\Phi(X)$ is metacyclic of exponent ≤ 4. By Burnside, $\Phi(X)$ cannot be nonabelian of order 8 so it is either abelian of type $(4, 2)$, or abelian of type $(4, 4)$, or $\Phi(X) = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1}\rangle$. In any case, every generating system of $\Phi(X)$ must contain an element of order 4. It follows from $\Phi(X) = \mho_1(X)$ that X has an element of order 8, a contradiction since $\exp(X) = 4$.

Supplement to Corollary 17. *Let $G = AB$ be a nonmetacyclic 2-group, where A and B are cyclic and let G/G' be abelian of type $(2^m, 2)$, $m > 1$ (see Corollary 17). Then the set $\Gamma_1 = \{U, V, M\}$ is the set of maximal subgroups of G, where $A < U$, $B < V$, the subgroups U, V are metacyclic but not of maximal class and $d(M) = 3$.*

Proof. By Remark 18, $\Phi(G)(= \mho_1(G))$ is metacyclic but not cyclic since G has no cyclic subgroup of index 2.

Since $\mathrm{d}(G) = 2$ and G is not minimal nonabelian, we get $\mathrm{Z}(G) < \Phi(G)$.

Assume that U is of maximal class. Since G is nonmetacyclic, it is not of maximal class. Then, by [Ber1, Theorem 7.4(a)], we get $\mathrm{d}(G) = 3$, a contradiction. Similarly, V is also not of maximal class.

Let us prove, for example, that U is metacyclic. Assume that this is false. Then $U/\mho_2(U)$ is nonmetacyclic, by Blackburn's result [Ber1, Theorem 3.4]; in particular, $|U/\mho_2(U)| \geq 2^4$ and $G/\mho_2(U)$ is nonmetacyclic. Since $\mathrm{d}(U) = 2$ and $\Phi(U)$ is metacyclic, we get $|U/\mho_2(U)| = 2^4$ (see the paragraph preceding the supplement). We have $\mho_2(U) \lhd G$ and $\mho_2(U) < \Phi(M)$ (otherwise, all maximal subgroups of two-generator nonmetacyclic group $G/\mho_2(U)$ are two-generator, contrary to [Ber1, Theorem 3.3]). We conclude that $\mathrm{d}(M/\mho_2(U)) = 3$. Next, $G/\mho_2(U) = (A\mho_2(U)/\mho_2(U))(B\mho_2(U)/\mho_2(U))$, where both factors are cyclic. Therefore, to get a contradiction, one may assume that $\mho_2(U) = \{1\}$. In that case, $|G| = 2^5$, $U = \langle x, y \mid x^4 = b^2 = z^2 = 1, z = [x, y], [x, z] = [y, z] = 1\rangle$ is minimal nonabelian. Since U is not metacyclic and two-generator, it has no normal cyclic subgroup of order 4. Since $G = AB$ is of order 2^5 and exponent ≤ 8, one of the factors A, B, namely B (since $|A| \leq \exp(U) = 4$) has order 8, by the product formula. Then $\exp(V) = 8$ and $|V : B| = 2$. It follows from $\Phi(V) = \mho_1(B)$ that $\mho_1(B) \lhd G$. But $\mho_1(B) = \Phi(B) < \Phi(G) < U$, and the cyclic subgroup $\mho_1(B)$ of order 4 is normal in G so in U, contrary to what has been said already. Thus, U is metacyclic. Similarly, V is metacyclic. ∎

Remark 19. Let G be a metacyclic 2-group with $\mathrm{c}_1(G) > 3$. Assume that G is not of maximal class. Then G has a normal abelian subgroup R of type $(2,2)$. Let $x \in G - R$ be an involution. Then $D = \langle x, R\rangle \cong \mathrm{D}_8$. By Lemma 1(i), $D\mathrm{C}_G(D)$ is nonmetacyclic, a contradiction. It follows that then G is either dihedral or semidihedral[7]. If, in addition, G is nonabelian and satisfies $\Omega_1(G) = G$, then it is dihedral.

Remark 20. Suppose that a metacyclic 2-group G of exponent $\geq 2^3$ satisfies $\Omega_2^*(G) = G$. Then G is either generalized quaternion or $G/\Omega_1(G)$ is dihedral with $\Omega_1(G) \leq \mathrm{Z}(G)$. Obviously, G is nonabelian. If G is of maximal class, it is generalized quaternion. Next assume that G is not of maximal class. Then G has a normal four-subgroup R (Lemma 1(q)) and $R = \Omega_1(G)$ (Remark 19). If $U < G$ is cyclic of order 4, then $U \cap R = \Omega_1(U)$ so $|RU/R| = 2$. It follows that $\Omega_1(G/R) = G/R$ so G/R is dihedral, by Remark 19. We claim that if G is metacyclic and G/R is dihedral ($R = \Omega_1(G)$ is a four-subgroup), then $R \leq \mathrm{Z}(G)$. Indeed, let $U/\Omega_1(G) < G/\Omega_1(G)$ be of order 2; then U is abelian (Remark 19). Since all such U centralize $\Omega_1(G)$ and generate G, $\Omega_1(R) \leq \mathrm{Z}(G)$.[8]

Remark 21. Let G be a 2-group. Suppose that $H = \Omega_2(G)$ is metacyclic of exponent $\geq 2^3$. Then one of the following holds: (a) G is of maximal class (in that case, $H = G$), (b) G is metacyclic with dihedral $G/\Omega_1(G)$ (then $H = G$ and $\Omega_1(G) \leq \mathrm{Z}(G)$) or semidihedral (then $|G/H| = 2$). Indeed, by Lemma 1(w), G is metacyclic. By Remark 20, H is one of groups (a), (b). If H is of maximal class, then $\mathrm{c}_2(G) = \mathrm{c}_2(H) \equiv 1 \pmod 4$ so G is of maximal class, by Lemma 1(p) and 1(q). Now let H be not of maximal class and let $R < H$ be G-invariant of type $(2,2)$. We have $\Omega_1(H/R) = H/R$ so H/R is dihedral and $R \leq \mathrm{Z}(H)$ (remarks 19, 20).

If G is a nonmetacyclic 2-group of order 2^m and $m > n \geq 4$, then the number of normal subgroups D of G such that G/D is metacyclic of order 2^n, is even [Ber5].

3°. In this section, most proofs are based on properties of p-groups of maximal class and counting theorems.

Let G be a p-group of exponent $p^e > p^2$, $p > 2$, and let $1 < k < e$. Suppose that $H < G$ is metacyclic of exponent p^k such that whenever $H < L$, then $\exp(L) > p^k$. Then G is also metacyclic. This is a consequence of Corollary 13 and the following

Theorem 22. *Let G be a p-group of exponent $p^e > p^2$ and let $1 < k < e$. Suppose that U is a maximal member of the set of subgroups of G having exponent p^k.*

(a) *If U is absolutely regular then G is also absolutely regular, $U = \Omega_k(G)$ and the subgroup U is not of maximal class.*

[7]The above argument also shows that if G has a nonabelian subgroup of order 8, it is of maximal class.

[8]Janko (see [BJ1, §86]) has classified the 2-groups G with metacyclic $\Omega_2^*(G)$.

(b) *If U is irregular of maximal class, then G is also of maximal class.*

Proof. If G is absolutely regular, then U is also absolutely regular. If G is a 2-group of maximal class, then U is also of maximal class (and order 2^{k+1}).

Let G be of maximal class, $p > 2$ and let U be absolutely regular. Then G is irregular since $e > 2$ (Lemma 1(g)). Denote by G_1 *the* absolutely regular subgroup of index p in G; then $\exp(G_1) = \exp(G) = p^e > p^k$ (Lemma 1(h)). Assume that $U < G_1$. Then $U = \Omega_k(G_1) < G_1$ since $k < e$, hence $U \triangleleft G$. Since $|G : U| > p$, then all elements of the set $(G/U) - (G_1/U)$ have the same order p [Ber3, Theorem 13.19], so there exists $H/U < G/U$ such that $H \not\leq G_1$ and $|H : U| = p$. Then H is of maximal class [Ber3, Theorem 13.19] so $\exp(H) = \exp(U)$ (Lemma 1(h)), contrary to the choice of U. Now suppose that $U \not\leq G_1$. We get $k = 2$ (otherwise, $U = \Omega_k^*(U) \leq \Omega_k^*(G) \leq G_1$, by Lemma 1(h)). Assume that $\Omega_1(G_1) \not\leq U$. Let $R \leq \Omega_1(G_1)$ be a minimal G-invariant subgroup such that $R \not\leq U$. In that case, $|UR : R| = p$. By Lemma 1(f), 1(h) and 1(p), $\exp(UR) = \exp(U)$, contrary to the choice of U. Thus, $\Omega_1(G_1) < U$ so $|U| \geq p^p$ (Lemma 1(h)); moreover, by [Ber3, Theorem 13.19], $|U| = p^p$. Let $U < H \leq G$, where $|H : U| = p$. Then H is of maximal class [Ber3, Theorem 13.19] and order p^{p+1} so $\exp(H) = p^2 = \exp(U)$ and $U < H$, contrary to the choice of U. Thus, if G is irregular of maximal class, then U must be also irregular of maximal class and $\Omega_k(G_1)$ has index p in U.

In what follows we may assume that G is not of maximal class.

Next we proceed by induction on $|G|$.

(i) Let G be noncyclic and regular; then U is absolutely regular. Then $U = \Omega_k(G)$ (Lemma 1(f)) so $\Omega_1(G) = \Omega_1(U)$ and $p^p > |U/\mho_1(U)| = |\Omega_1(U)| = |\Omega_1(G)| = |G/\mho_1(G)|$, whence G is absolutely regular; in that case, $p > 2$. Assume that, in addition, U is of maximal class. Then $|U : \Omega_1(U)| = p$ (Lemma 1(g)) so $|\Omega_1(G/\Omega_1(G))| = p$. It follows that $G/\Omega_1(G)$ is cyclic (of order $> p$). Let D be a G-invariant subgroup of index p^2 in $\Omega_1(U) = \Omega_1(G)$, and set $C = \mathrm{C}_G(\Omega_1(U)/D)$; then C/D is abelian and $U \leq C$ so U/D is abelian of order p^3, and we conclude that U is not of maximal class, contrary to the assumption. Thus, U is not of maximal class.

In what follows we assume that G is irregular.

(ii) Let U be absolutely regular; then $|\Omega_1(U)| = |U/\mho_1(U)| < p^p$. We write $R = \Omega_1(U)$ and $N = \mathrm{N}_G(R)$; then $U < N$.

Assume that $N = G$. Then, by Lemma 1(n), there is in G a normal subgroup S of order $p|\Omega_1(U)|$ and exponent p such that $R < S$. Set $H = US$. Then $H/S \cong U/R$ is of exponent p^{k-1} so, since $U < H$, we get $\exp(H) = p^k$, contrary to the choice of U.

Now let $N < G$. Then N is absolutely regular, by induction and Lemma 1(m). In that case, $U = \Omega_k(N)$ so $R = \Omega_1(N)$ is characteristic in N whence $N = G$, contrary to the assumption.

(iii) In what follows we assume that U is irregular of maximal class. Set $V = \Omega_1(\Phi(U))$ and $N = \mathrm{N}_G(V)$. If $N < G$, then, by induction, N is of maximal class so G is also of maximal class (Lemma 1(m)), contrary to the assumption. Now let $N = G$. Then, as in (ii), G has a normal subgroup R of order p^p and exponent p such that $V < R$. Set $H = UR$; then $H/R \cong U/V$ is of exponent p^{k-1}. This is a contradiction since $\exp(H) = p^k = \exp(U)$ and $U < H$. □

Supplement 1 to Theorem 22. *Let G be a p-group of exponent $p^e > p$, $1 < k \le e$. Set $H = \Omega_k^*(G)$.*

(a) *If H is absolutely regular, then G is either absolutely regular or irregular of maximal class.*
(b) *If H is of maximal class, then G is also of maximal class.*

Proof. We proceed by induction on $|G|$. One may assume that $H < G$.

(a) Suppose that H is absolutely regular. Set $R = \Omega_1(H)$; then $R \triangleleft G$.

Assume that G is neither absolutely regular nor of maximal class. Then G contains a normal subgroup S of order $p|R|$ and exponent p such that $R < S$ (Lemma 1(n)). Set $U = HS$. Assume that U is of maximal class. Then $|S| = |H| = \frac{1}{p}|U|$ (Lemma 1(h)), $|HS| = p^{p+1}$, $\exp(HS) = p^2$ so $k = 2$ and $H(= \Omega_2^*(G))$ is the unique maximal subgroup of HS of exponent p^2. In that case, $\mathrm{c}_2(G) = \mathrm{c}_2(H) \not\equiv 0 \pmod{p^{p-1}}$ so G is either absolutely regular or of maximal class (Lemma 1(s) and 1(t)), contrary to the assumption. The proof of (a) is complete.

(b) Suppose that H is irregular of maximal class.

Assume that $|H| > p^{p+1}$. Then $\mathrm{c}_2(G) = \mathrm{c}_2(H) \equiv p^{p-2} \pmod{p^{p-1}}$, so G is of maximal class (Lemma 1(s) and 1(t)), a contradiction.

It remains to consider the possibility $|H| = p^{p+1}$; then $\exp(H) = p^2$ (Lemma 1(g)) so $k = 2$. In that case, $\mathrm{c}_2(H) = \mathrm{c}_2(G) \equiv 0 \pmod{p^{p-1}}$ (Lemma 1(t)) so $\Omega_1(H)$ is of order p^p and exponent p. Let $H < A \le G$ and $|A : H| = p$. By [Ber3, Theorem 13.21], one may assume that A is not of maximal class. By [Ber1, Theorem 7.4(c)], A contains exactly $p + 1$ regular subgroups $T_1, \dots, T_{p+1}$ of index p which are not absolutely regular. It follows that $\exp(T_i) = p$ for all i (otherwise, $T_i = \Omega_2^*(T_i) \le \Omega_2^*(G) = H$, which is not the case). Then $T_i \cap H = \Omega_1(H)$. It follows that $\Omega_1(H)$ is contained in $p + 2$ pairwise distinct subgroups $H, T_1, \dots, T_{p+1}$ of index p in A, a contradiction since $A/\Omega_1(H)$ is of order p^2. □

Supplement 2 to Theorem 22. *Let H be a metacyclic subgroup of exponent 2^k of a 2-group G. Suppose that H is maximal among subgroups of exponent 2^k in G. Then G has no H-invariant elementary abelian subgroup of order 8 (see* [Jan1]*).*

PROOF. Assume that G has an H-invariant elementary abelian subgroup E of order 8. To get a contradiction, one may assume, without loss of generality, that $G = HE$; then $E \triangleleft G$. Set $L = H \cap E$; then $|L| \leq 4$ and L is normal in G.

Let $L = \{1\}$. If $L_0 \leq E \cap \mathrm{Z}(G)$ is of order 2, then $H < H \times L_0$ and $\exp(H \times L_0) = 2^k$, contrary to the choice of H.

Let L be of order 4. Then $G/L = (E/L) \times (H/L)$ is of exponent 2^{k-1} so $\exp(G) = 2^k$, contrary to the choice of H.

Now let $|L| = 2$. In view of Theorem 22, one may assume that H is not of maximal class. Then H contains a normal abelian subgroup R of type $(2,2)$. By the product formula, $|ER| = 16$. Note that ER is H-invariant. We also have $R < \mathrm{C}_E(R)$ and $\mathrm{C}_E(R)$ is H-invariant. Let $R < F < R\mathrm{C}_E(R)$, where F is an H-invariant subgroup of order 8; then F is elementary abelian, the quotient group $HF/R = (H/R)\times(F/R)$ has exponent 2^{k-1} so $\exp(HF) = 2^k$, contrary to the choice of H since $H < HF$. □

For related results, see [Ber4].

Let s be a positive integer. A p-group G is said to be an L_s-*group*, if $\Omega_1(G)$ is of order p^s and exponent p and $G/\Omega_1(G)$ is cyclic of order $> p$ ($\Omega_1(G)$ is said to be the *kernel* of G).

Below we use the following

LEMMA 23 ([Ber1, Lemma 2.1]). *Let G be a p-group with $|\Omega_2(G)| = p^{p+1} < |G|$. Then one of the following holds:*

(a) *G is absolutely regular.*

(b) *G is an L_p-group.*

(c) *$p = 2$ and $G = \langle a, b \mid a^{2^n} = 1, a^{2^{n-1}} = b^4, a^b = a^{-1+2^{n-2}} \rangle$.*

It is known that an irregular p-group G has a maximal regular subgroup R of order p^p if and only if G is of maximal class [Ber3, §10].[9] The following theorem supplements this result.

THEOREM 24. *Let G be a p-group and let $H < G$ be a maximal member of the set of subgroups of G of exponent p^2. Suppose that $|H| = p^{p+1}$. Then one of the following holds:*

(a) *$p = 2$ and G is of maximal class.*

(b) *$H = \Omega_2(G)$ (see Lemma 23).*

PROOF. If G is regular, then $H = \Omega_2(G)$ so G is a group of Lemma 23. Next let G be irregular. By hypothesis, $\exp(H) < \exp(G)$.

[9] This is an easy consequence of Lemma 1(m). Indeed, write $N = \mathrm{N}_G(R)$. If $N < G$, then N is of maximal class, by induction, and we are done (Lemma 1(m)). Now let $N = G$. Take D, a G-invariant subgroup of index p^2 in N, and set $C = \mathrm{C}_G(R/D)$. If $B/R \leq C/R$ is of order p, then B is regular since B/D is abelian of order p^3 (Lemma 1(e)), a contradiction since $R < B$.

Suppose that G is irregular of maximal class. It follows from [Ber3, theorems 9.5 and 9.6] that then $p = 2$, and we get case (a). Indeed, assume that $p > 2$. If $H \leq G_1$, then $H = \Omega_2(G_1)$. If $H = G_1$, then $\exp(H) = \exp(G)$, contrary to the choice of H. Thus, $H < G_1$. Let U/H be a subgroup of G/H of order p not contained in G_1/H. Then U is of maximal class and exponent p^2 [Ber3, Theorem 13.19], contrary to the choice of H. Now let $H \not\leq G_1$; then $\Omega_1(G_1) \leq H$ and H is of maximal class. Let $H < F \leq G$ with $|F : H| = p$. Then $\exp(F) = \exp(H)$, contrary to the choice of H. The 2-groups of maximal class satisfy the hypothesis.

In what follows we assume that G is not of maximal class. Then, in view of Theorem 22, one may assume that H is neither absolutely regular nor of maximal class so $\mathrm{cl}(H) < p$. It follows that H is regular (Lemma 1(e)) and $\Omega_1(H)$ is of order p^p and exponent p. Set $N = \mathrm{N}_G(\Omega_1(H))$; then $H < N$ since $\Omega_1(H)$ is characteristic in $H < G$. We use induction on $|G|$.

Assume that $N < G$. Then, by induction, N is one of groups (a,b). However, in case (b), $\Omega_1(H)$ is characteristic in N (Lemma 23) so $N = G$, contrary to the assumption. On the other hand, N cannot be a 2-group of maximal class since H is abelian of type $(4, 2)$, by the previous paragraph.

Thus, $N = G$ so $\Omega_1(H) \triangleleft G$. By hypothesis, $G/\Omega_1(H)$ has no abelian subgroup $K/\Omega_1(H)$ of type (p, p) such that $H < K$, so $G/\Omega_1(H)$ is either cyclic or generalized quaternion (then $p = 2$). In that case, $\Omega_1(G) = \Omega_1(H)$ so that $\Omega_2(G) = H$. □

Let a natural number $n \geq p - 1$. A p-group G is said to be a U_n^p-*group* provided it has a normal subgroup R of order p^n and exponent p such that G/R is irregular of maximal class and, if T/R is absolutely regular of index p in G/R, then $\Omega_1(T) = R$.[10] Let us prove that if a normal subgroup R_1 of G is of exponent p, then $R_1 \leq R$. Assume that this is false and that every proper G-invariant subgroup of R_1 is contained in R; then $|RR_1 : R| = p$ so $RR_1/R < T/R$ since G/R has only one minimal normal subgroup. This is a contradiction: $RR_1 \leq \Omega_1(T) = R < RR_1$. It follows that R is characteristic in G. We call R the *kernel* of the U_n^p-group G. It follows from Lemma 1(p) that U_{p-1}^p-groups are of maximal class. Note that $\exp(G) = p{\cdot}\exp(G/R) = \exp(T)$.

THEOREM 25. *Let G be a p-group and let $H < G$ be a maximal member of the set of subgroups of G of exponent* $\exp(H)$. *If H is a U_n^p-group, then G is also a U_n^p-group.*

PROOF. We use induction on $|G|$. In view of Theorem 22(b), one may assume that H is not of maximal class so that $n > p - 1$. Let R be the kernel of H and set $N = \mathrm{N}_G(R)$. If $N < G$, then N is a U_n^p-group, by induction. In that case, R is also kernel of N so characteristic in N. It follows that $N = G$,

[10]It follows from Lemma 1(p) and 1(q), that U_n^p-groups do no exist for $n < p - 1$. The U_2^2-groups are classified by Janko; see [Jan3] or [BJ1, §67].

contrary to the assumption. Thus, $N = G$. Then H/R is a maximal member of the set of subgroups of exponent $\frac{1}{p} \cdot \exp(H)$ in G/R and H/R is irregular of maximal class. Then G/R is of maximal class, by Theorem 22. Let us show that G is a U_n^p-group. Let T/R be *the*[11] absolutely regular subgroup of index p in G/R (Lemma 1(h)) and set $U/R = (H/R) \cap (T/R)$. Then U/R is an absolutely regular subgroup of index p in H/R so $\Omega_1(U) = R$ since H is a U_n^p-group. Let $F/R < T/R$ be G-invariant of order p. It follows from the subgroup structure of G/R (see [Ber3, §9 and Theorem 13.19]) that $F/R \leq \Phi(G/R) < H/R$ so $F/R \leq \Phi(H/R) < U/R$, and we get $\exp(F) = p^2$ since F is not contained in $R = \Omega_1(U)$. In that case, $R = \Omega_1(T)$ so G is a U_n^p-group. ∎

Remark 26. Let G be a p-group and let $H < G$ be a maximal member of the set of subgroups of G of exponent $\exp(H)$. If H is an L_n-group, then G is also an L_n-group. To prove this, it suffices to repeat, with small modifications, the proof of Theorem 25 and use the following easy fact: If $C < G$ is a cyclic subgroup of order $p^k > p$ which is not contained properly in a subgroup of exponent p^k, then G is cyclic.

The following theorem is an analogue of Supplement 1 to Corollary 11 and dual, in some sense, to Theorem 22.

Theorem 27. *Suppose that a p-group G is such that $G/\mho^2(G)$ is of maximal class. Then G is also of maximal class.*

Proof. (a) Suppose that G is regular. Then $|G/\mho_1(G)| = p^k$, where $k < p$, and $|G/\mho^2(G)| = p^{k+1}$ (Lemma 1(g)) so $|\mho_1(G) : \mho_1(\mho_1(G))| = p$, and we conclude that $\mho_1(G)$ is cyclic. Let $|\mho_1(G)| = p^e$; then $\exp(G) = p^{e+1}$. By Lemma 1(f), $|\Omega_1(G)| = p^k$. Since $|G| = p^{k+e}$, it follows that $G/\Omega_1(G)$ is cyclic of order p^e. By hypothesis, $|G : G'| = p^2$ so $e = 1$. In that case, $\mho^2(G) = \{1\}$ so G is of maximal class, by hypothesis.

(b) Now let G be irregular. One may assume that $|G| > p^{p+1}$ (otherwise, in view of Lemma 1(e), it is nothing to prove). By Lemma 1(r), $|G/\mho_1(G)| \geq p^p$ so $|G/\mho^2(G)| \geq p^{p+1}$ and we conclude that $G/\mho^2(G)$ is irregular (Lemma 1(g)). By hypothesis and Lemma 1(g), we get $|G/\mho_1(G)| = p^p$ and $|G/G'| = p^2$.

(i) Let L be a normal subgroup of index p^{p+1} in G. By the previous paragraph, $\exp(G/L) > p$. Let $R/L = \mho^2(G/L)$; then $\mho^2(G) \leq R$. It follows from properties of irregular p-groups of maximal class[12] that $|G/R| \geq p^{p+1} = |G/L|$ so $R = L$, and we conclude that $\exp(G/L) = p^2$ and $\mho^2(G) \leq L$.

(ii) Assume that L and L_1 are distinct normal subgroups of the same index p^{p+1} in G. Then $\mho^2(G) \leq L \cap L_1$, by (i). In that case, $L/\mho^2(G)$ and

[11] 'the' since $|G/R| > |H/R| \geq p^{p+1}$.

[12] If X is irregular p-group of maximal class, then every its epimorphic image of order p^p has exponent p.

$L_1/\mho^2(G)$ are different normal subgroups of index $p^{p+1} > p$ in a p-group of maximal class $G/\mho^2(G)$, which is impossible (Lemma 1(h)). Thus, G has the unique normal subgroup, say L, of index p^{p+1}. By the above, G/L, as a nonabelian epimorphic image of $G/\mho^2(G)$, is of maximal class. Then, by Lemma 1(d), G is also of maximal class. □

The case $p = 2$ of Theorem 27 follows immediately from Corollary 12.

A p-group G is of maximal class if and only if $G/\mho_2(G)$ is of maximal class. Indeed, $\mho_2(G) \le \mho^2(G)$ so $G/\mho^2(G)$ is of maximal class as a nonabelian epimorphic image of $G/\mho_2(G)$, and the result follows from Theorem 27.

Remark 28. Now we offer another argument for part (b) of the proof of Theorem 27. Let $H/\mho^2(G)$ be an absolutely regular subgroup of index p in $G/\mho^2(G)$, existing, by Lemma 1(h). Assume that H is not absolutely regular. Then, by Lemma 1(r), we have $|H/\mho_1(H)| \ge p^p$. Clearly, $\mho^2(G) \le \mho_1(H)$ so $H/\mho_1(H)$ of order $\ge p^p$ and exponent p is an epimorphic image of the absolutely regular group $H/\mho^2(G)$, a contradiction.[13] Thus, H is absolutely regular. Assume that G is not of maximal class. Then $G = H\Omega_1(G)$, where $\Omega_1(G)$ is of order p^p and exponent p (Lemma 1(p)). By hypothesis, $|G/G'| = p^2$. We have $G/(H \cap \Omega_1(G)) = G/\Omega_1(H) \cong (H/\Omega_1(H)) \times (\Omega_1(G)/\Omega_1(H))$ so $|H/\Omega_1(H)| = p$, $|H| = p|\Omega_1(H)| = p^p$ and $|G| = p^{p+1}$. In that case, $\mho^2(G) = \{1\}$ so G is of maximal class, contrary to the assumption.

In Remark 29 we use the following fact. If G is neither absolutely regular nor maximal class and $E_1, \dots, E_r$ are all its subgroups of order p^p and exponent p, then $\bigcup_{i=1}^r E_i = \{x \in G \mid x^p = 1\}$. Indeed, if D is a normal subgroup of G of order p^{p-1} and exponent p and $x \in G - D$ is of order p, then the subgroup $\langle x, D\rangle$ is of order p^p and exponent p so coincides with some E_i.

Remark 29. If G is a p-group such that $H = \Omega_1(G)$ is of maximal class, then one of the following holds: (a) H is of order $\le p^p$ and exponent p, (b) G is of maximal class. Indeed, this is the case if G is regular, by Lemma 1(g). Now assume that G is not of maximal class and $|H| > p^p$. Let $E_1, \dots, E_r$ be all subgroups of order p^p and exponent p in G; then $r > 1$ and, by Lemma 1(v), $r \equiv 1 \pmod p$. We have $E_i < H$ for all i so H has a G-invariant subgroup, say E_1, of order p^p and exponent p. It follows that $|H| = p^{p+1}$ (Lemma 1(h)). Since $r \ge p+1$ and $\mathrm{d}(H) = 2$, we get $\exp(H) = p$ so H is regular (Lemma 1(e)), a contradiction.

4°. In this section we prove the following

Theorem 30. *Let A be a maximal cyclic subgroup of order $> p$ of a noncyclic p-group G. Then there exists in G a maximal cyclic subgroup B of order $> p$ such that $|A \cap B| = p$, unless $p = 2$ and G is dihedral.*

[13] This argument is similar to one from the proof of Theorem 2.

Proof. If A is the unique cyclic subgroup of its order in G, then $p = 2$ and G is of maximal class [Ber2, Remark 6.2], and the theorem is true. In what follows we assume that there is in G another cyclic subgroup of order $|A|$.

Suppose that $|G : A| = p$ and G is either abelian $\langle a\rangle \times \langle b\rangle$ of type (p^n, p) or $G = \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}}\rangle$, $A = \langle a\rangle$, $n > 1$ and $n > 2$ if G is nonabelian 2-group. In both cases G has exactly p cyclic subgroups of order p^i, $i = 2, \dots, n$. If $n = 2$ and B is a cyclic subgroup of index p in G, $B \neq A$, then $|A \cap B| = p$. Now let $n > 2$; then $\Phi(G) = \langle a^p\rangle$. Let $B < G$ be a cyclic subgroup of order p^2 not contained in $\Phi(G)$. Then B is a maximal cyclic subgroup of G (indeed, if $B < C \leq G$ and C is cyclic of order $p|B|$, then $B = \Phi(C) \leq \Phi(G)$, contrary to the choice of B). We have $|A \cap B| = p$ again.

If G is a 2-group of maximal class and G is not dihedral, it has a maximal cyclic subgroup B of order 4 with $B \not\leq A$; then $|A \cap B| = 2$.

In what follows we assume that $|G : A| > p$. Let $A < H < G$, where $|H : A| = p$.

Suppose that H is not dihedral. Then, by the above, there is in H a maximal cyclic subgroup B_1 of order p^2 such that $|A \cap B_1| = p$. Let $B_1 \leq B < G$, where B is a maximal cyclic subgroup of G. Then $A \cap B = A \cap B_1$, completing this case.

Now suppose that H is dihedral. Let $H < F \leq G$, where $|F : H| = 2$. Then $A \triangleleft F$ since A is characteristic in H. Let A_1 be a subgroup of order 4 in A; then $A_1 \triangleleft F$. In that case, $C_F(A_1)$ is maximal in F and contains A as a subgroup of index 2. Since A is maximal cyclic subgroup of G, the subgroup $C_F(A_1)$ is noncyclic. Since $C_F(A_1)$ is not dihedral, it has a maximal cyclic subgroup B_1 of order > 2 such that $|A \cap B_1| = 2$, by induction. If $B_1 \leq B < G$, where B is a maximal cyclic subgroup of G, then $A \cap B = A \cap B_1$, completing the proof. ∎

Suppose that a p-group G is neither abelian nor minimal nonabelian. We claim that then G contains p pairwise distinct minimal nonabelian subgroups, say $B_1, \dots, B_p$, of the same order, say p^n, such that $B_1 \cap \dots \cap B_p \geq \Phi(B_i)$ for $i = 1, \dots, p$ (in particular, $|B_1 \cap \dots \cap B_p| \geq p^{n-2}$). Indeed, let B_1 be a minimal nonabelian subgroup of G of minimal order, and set $|B_1| = p^n$. Let $B_1 < U \leq G$, where $|U : B_1| = p$. It follows from the choice of B_1 that each maximal subgroup of U is either abelian or minimal nonabelian (of order p^n). By [Ber6, Remark 1], U contains at least p distinct minimal nonabelian subgroups, say $B_1, \dots, B_p$. If $i \neq j$, then $|B_i \cap B_j| = p^{n-1}$ so $B_i \cap B_j$ is maximal in B_i. It follows that $\Phi(B_i) < B_i \cap B_j$ for all $i \neq j$, and our claim follows.

Problems

1. Classify the p-groups G in which every maximal cyclic subgroup of composite order is contained in a unique maximal subgroup of G.
2. Study the p-groups G, all of whose maximal cyclic subgroups are not contained in $\Phi(G)$.
3. Study the p-groups G, $p > 2$, such that $\mathrm{K}_p(G) = \mho_1(G)$ has index p^p in G.
4. Let H be a maximal member of the set of subgroups of exponent $p > 2$ in a p-group G. Study the structure of G provided H is of maximal class.
5. Study the p-groups G such that $G/\Omega_1(G)$ is irregular of maximal class and $\Omega_1(G)$ is irregular.
6. Let H be a metacyclic subgroup of exponent $2^k > 2$ of a 2-group G. Study the structure of G provided every subgroup of G containing H properly, has exponent $> 2^k$.
7. Let H be a subgroup of exponent 4 in a 2-group G such that every subgroup of G properly containing H, has exponent > 4. Study the structure of G provided $|H| \leq 2^5$.
8. Classify the nonmetacyclic p-groups G containing a normal subgroup R of order p such that G/R is metacyclic.
9. Let H be a maximal member of the set of subgroups of exponent $\exp(H)$ in a p-group G. Study the structure of G provided H is extraspecial.
10. Let a nonmetacyclic 2-group $G = BC$, where B and C are cyclic. (i) Describe the maximal subgroup of G that is not generated by two elements (see Remark 18). (ii) Find all possible numbers of involutions in G. (iii) Does there exist $A < G$ such that $|A/\mho_2(A)| = p^5$? If so, study its structure and embedding in G. (iv) Is it true that $\mho_2(A) = \mho^2(A)$ for all $A < G$?
11. Classify the 2-groups G such that $\Omega_k^*(G)$ is metacyclic, for $k > 2$.[14]

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References

[Ber1] Y. Berkovich, *On subgroups and epimorphic images of finite p-groups*, J. Algebra **248** (2002), 472-553.

[Ber2] Y. Berkovich, *Alternate proofs of some basic theorems of finite group theory*, Glas. Mat. Ser. III **40(60)** (2005), 207-233.

[Ber3] Y. Berkovich, *Groups of Prime Power Order, Part I*, in preparation.

[Ber4] Y. Berkovich, *Alternate proofs of two theorems of Philip Hall on finite p-groups, and some related results*, J. Algebra **294** (2005), 463-477.

[14]This question was solved by Janko [Jan4] for $k = 2$.

[Ber5] Y. Berkovich, *On the metacyclic epimorphic images of finite p-groups*, Glas. Mat. Ser. III **41(61)** (2006), 259-269.

[Ber6] Y. Berkovich, *Finite p-groups with few minimal nonabelian subgroups*. With an appendix by Z. Janko, J. Algebra **297** (2006), 62-100.

[BJ1] Y. Berkovich and Z. Janko, *Groups of Prime Power Order, Part II*, in preparation.

[BJ2] Y. Berkovich and Z. Janko, Structure of finite p-groups with given subgroups, in: Contemp. Math. 402, Amer. Math. Soc., Providence, 2006, 13-93.

[BZ] Ya. G. Berkovich and E. M. Zhmud, Characters of Finite Groups, Part 1, American Mathematical Society, Providence, 1998.

[Bla1] N. Blackburn, *On prime-power groups with two generators*, Proc. Cambridge Philos. Soc. **54** (1958), 327-337.

[Bla2] N. Blackburn, *Generalizations of certain elementary theorems on p-groups*, Proc. London Math. Soc. (3) **11** (1961), 1-22.

[Bla3] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45-92.

[Hup] B. Huppert, *Über das Produkt von paarweise vertauschbaren zyklischen Gruppen*, Math. Z. **58** (1953), 243-264.

[Isa] I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York-London, 1976.

[IO] N. Ito and A. Ohara, *Sur les groupes factorisables par deux 2-groupes cycliques I, II*, Proc. Japan Acad. **32** (1956), 736-743.

[Jan1] Z. Janko, *Finite 2-groups with no normal elementary abelian subgroups of order* 8, J. Algebra **246** (2001), 951-961.

[Jan2] Z. Janko, *On maximal cyclic subgroups in finite p-groups*, Math. Z. **254** (2006), 29-31.

[Jan3] Z. Janko, *Finite 2-groups with exactly four cyclic subgroups of order* 2^n, J. Reine Angew. Math. **566** (2004), 135-181.

[Jan4] Z. Janko, *Finite 2-groups G with $\Omega_2^*(G)$ metacyclic*, Glas. Mat. Ser. III **41(61)** (2006), 71-76.

[Jan5] Z. Janko, *Finite 2-groups with exactly one nonmetacyclic maximal subgroup*, submitted.

[LM] A. Lubotzky and A. Mann, *Powerful p-groups. I. Finite groups*, J. Algebra **105** (1987), 484-505.

[Red] L. Redei, *Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören*, Comment. Math. Helv. **20** (1947), 225-264.

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