GLASNIK MATEMATIČKI Vol. 40(60)(2005), 323 – 331

EQUIVARIANT FIBRANT SPACES

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ABSTRACT. In this paper the concept of a G-fibrant space is introduced. It is shown that any compact metrizable group G is a G-fibrant.

1. INTRODUCTION

The general approach to the concept of a fibrant object is the following (c.f.[5]): if in a category \mathcal{C} some class Σ of morphisms is specified then an object Y of \mathcal{C} is called Σ -fibrant if for every morphism $s \in \Sigma$, $s : A \to X$, and every morphism $f : A \to Y$ there is a morphism $F : X \to Y$ such that $F \circ s = f$. The classical fibrant objects appear in [9] for the closed model categories where Σ is the class of trivial cofibrations. A fibrant space in the sense of F. Cathey is a Σ -fibrant object, where Σ is the class of SSDR-maps in the category of metrizable spaces ([6]). In the present paper we provide an equivariant version of a fibrant space.

It is well-known (see [8]) that every compact metrizable group can be represented as an inverse limit of a sequence of Lie groups bonded by fibrations (Proposition 3.3), and therefore it is already a fibrant space in the sense of F. Cathey. On the other hand, due to R. Palais ([7]), every compact Lie group G is a G-ANR (Proposition 3.2) and hence it is a G-fibrant space. These are the basic facts utilized in the proof of our main theorem (Theorem 3.1): every compact metrizable group G is a G-fibrant space. This result justifies the consideration of equivariant fibrant spaces. Also it is clear that equivariant fibrant spaces as well as equivariant SSDR-maps can be used in the construction of the equivariant strong shape category following the way of F. Cathey, which is given in [6] for the "non-equivariant" case.

Supported in part by CONACYT Grant 32132-E.



²⁰⁰⁰ Mathematics Subject Classification. 54C55, 54C56, 54H15, 54B15. Key words and phrases. Fibration, fibrant space, G-ANR.

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2. The basic notions

The basic definitions and facts of the equivariant theory, that is the theory of G-spaces and G-maps, can be found in [4]. Throughout the paper the letter G will denote a compact Hausdorff group. By G-A(N)R, it is denoted the class of G-equivariant absolute (neighborhood) retracts for all G-metrizable spaces (see, for instance, [2] for the equivariant theory of retracts). In this paper all G-spaces are assumed to be metrizable.

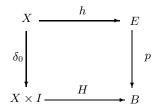
A closed invariant subspace A of a G-space X is called a G-shape strong deformation retract of X if there exists a G-equivariant embedding $i: X \hookrightarrow Y$ for some G-AR space Y such that for any pair of invariant neighborhoods U and V of i(X) and i(A) respectively in Y, there is a G-homotopy $H: X \times I \to U$ rel. A such that H(x, 0) = i(x) and $H(x, 1) \in V$ for any $x \in X$.

Note that if for a *G*-pair (X, A) an embedding $i : X \hookrightarrow M$ satisfies the conditions of the above definition then these conditions hold for any other closed *G*-equivariant embedding $j : X \hookrightarrow Z$ where *Z* is a *G*-AR space.

A closed G-equivariant embedding $s : A \hookrightarrow X$ is called a G-SSDR-map if s embeds A in X as a G-shape strong deformation retract of X.

A G-space Y is called a G-fibrant if for every G-SSDR-map $s : A \hookrightarrow X$ and every G-map $f : A \to Y$, there exists a G-map $F : X \to Y$ such that $F \circ s = f$.

Recall that a map $p: E \to B$ is a *G*-fibration if for every *G*-space X and every commutative diagram of *G*-maps



where $\delta_0(x) = (x, 0)$, there exists $\widetilde{H} : X \times I \to E$ such that $\widetilde{H} \circ \delta_0 = h$ and $p \circ \widetilde{H} = H$.

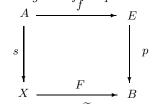
For example, the *G*-fibrations naturally appear in the following situation. Let *U* be a *G*-space. The space U^I of paths $\omega : I \to U$, provided with the compact-open topology, can be treated as a *G*-space with the action: $(g \star \omega)(t) = g\omega(t)$. Then the projection $p : U^I \to U \times U$, given by $p(\omega) = (\omega(0), \omega(1))$, is a *G*-fibration.

The following theorem is an equivariant version of Theorem 1.2 of [6].

THEOREM 2.1. Let $s : A \hookrightarrow X$ be a closed G-embedding. Then the following conditions are equivalent:

(a) s is a G-SSDR-map;

- (b) for any G-equivariant map f: A → Y, where Y is G-ANR, there is a G-equivariant extension f̃: X → Y such that f̃ ∘ s = f, and if f̃₁, f̃₂: X → Y are any two such extensions, then f̃₁ ≃_G f̃₂ rel. s(A);
- (c) For any G-fibration $p: E \to B$, where E and B are G-ANR-spaces and any commutative diagram of G-equivariant maps



there exists a G-equivariant map $\widetilde{F}: X \to E$ such that $\widetilde{F} \circ s = f$ and $p \circ \widetilde{F} = F$.

We shall give the proof of the theorem though it is quite analogous to the proof of its "non-equivariant" case.

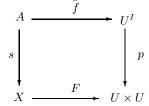
PROOF. (a) \Rightarrow (b). Clearly, we can assume that $A \subset X$ and s(a) = a. Let $X \hookrightarrow M$ be an equivariant closed embedding of X in some G-AR space M (See [3], Proposition 1). Since each G-AR space for metrizable spaces is a G-AE ([3], Proposition 2), there is a G-extension $\hat{f}: V \to Y$ of f on some invariant open neighborhood V of A in M. By the definition of a G-SSDR-map, we can find a G-homotopy $H: X \times I \to M$ such that H(x,0) = x, $H(x,1) \in V$ and H(a,t) = a for $x \in X$, $a \in A$, $t \in I$. The required extension $\tilde{f}: X \to Y$ can be given by $\tilde{f}(x) = \hat{f}(H(x,1))$.

Let $\tilde{f}_1, \tilde{f}_2 : X \to Y$ be two *G*-extensions of *f*. Define a *G*-equivariant map $F : X \times 0 \cup A \times I \cup X \times 1 \to Y$ by $F(x,0) = \tilde{f}_1(x), F(x,1) = \tilde{f}_2(x),$ F(a,t) = f(a) for $x \in X, a \in A, t \in I$. Considering *X* as a closed invariant subset of a *G*-AR space *M*, and therefore $X \times I$ as a closed invariant subset of the *G*-AR space $M \times I$, we find a *G*-extension $\overline{F} : W \to Y$ of *F* on some invariant neighborhood *W* of $X \times 0 \cup A \times I \cup X \times 1$ in $M \times I$. Clearly, one can choose an invariant neighborhood *U* of *X* in *M* such that $U \times \{0\} \subset W$ and $U \times \{0\} \subset W$. Besides, a standard compactness argument guarantees the existence of an invariant neighborhood *V* of *A* in *M* such that $V \times I \subset W$. Taking a *G*-homotopy $D : X \times I \to U$ such that $D(x,0) = x, D(x,1) \in V$ and D(a,t) = a for $x \in X, a \in A, t \in I$, we can establish *G*-homotopies $F' : \tilde{f}_1 \simeq_G h_1$ rel. $A, F'' : \tilde{f}_2 \simeq_G h_2$ rel. *A* and $H : h_1 \simeq_G h_2$ rel. *A* by $F'(x,t) = \overline{F}(D(x,t),0), F''(x,t) = \overline{F}(D(x,t),1)$ and $H(x,t) = \overline{F}(D(x,1),t)$. Thus $\tilde{f}_1 \simeq_G \tilde{f}_2$ rel. *A*.

(b) \Rightarrow (c). Since *E* is a *G*-ANR there exists a *G*-extension $\overline{F} : X \to E$ such that $\overline{F} \circ s = f$. We have $F \circ s = p \circ \overline{F} \circ s$ and by the second part of (b) there is a *G*-homotopy $H : F \simeq_G p\overline{F}$ rel. s(A). Applying the covering

homotopy property we get a G-homotopy $\widetilde{H} : X \times I \to E, \ \widetilde{H} : \widetilde{F} \simeq_G \widetilde{F}$ rel. s(A), such that $p \circ \widetilde{H} = H$. So $\widetilde{F} \circ s = f$ and $p \circ \widetilde{F} = F$ as required.

 $(c) \Rightarrow (a)$. As above, we can assume that X is an invariant closed subset of some G-AR space M and that A is an invariant closed subset of of X, so s(a) = a for $a \in A$. Let U and V be invariant open neighborhoods of X and A respectively in M. First applying (c) to the G-fibration $V \rightarrow *$ and the inclusion $i: A \rightarrow V$ we get a G-map $r: X \rightarrow V$ such that $r \circ s = i$. Afterwards applying (c) to the commutative diagram



where $p(\omega) = (\omega(0), \omega(1)), f(a)(t) = a, F(x) = (x, r(x))$, we obtain a *G*-map $\overline{F} : X \to U^I$ such that $\overline{F} \circ s = f, p \circ \overline{F} = F$. Now observe that the map $D : X \times I \to U$ defined by $D(x,t) = \overline{F}(x)(t)$ satisfies the conditions of the definition of a *G-SSDR*-map.

COROLLARY 2.2. Every G-ANR is a G-fibrant space.

3. Main result

The main result of this paper is the following

THEOREM 3.1. Every compact metrizable group G is a G-fibrant space.

In the proof of this theorem, we shall use the propositions given below.

PROPOSITION 3.2. ([7], Proposition 1.6.6) Let G be a compact Lie group and H be its closed subgroup. Then G/H is a G-ANR space.

The following result is actually proved in the classical book of Pontrjagin [8]. Note that it can be easily obtained from Corollary 4.4 of [4]: for every neighborhood U of the unit e of a compact group G, there exists a group morphism $\varphi: G \to \mathbf{O}(n)$ such that ker $\varphi \subseteq U$.

PROPOSITION 3.3. Let G be a compact metrizable group. Then there exists a decreasing sequence $\{N_i\}_{i\in\mathbb{N}}$ of its normal closed subgroups such that the quotient groups G/N_i are Lie groups, $\bigcap_i N_i = \{e\}$ and

$$\lim\{G/N_i, q_i^j\} = G$$

where $q_i^j : G/N_j \to G/N_i, j \ge i$, are the natural projections.

We omit a routine proof of the following statement.

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PROPOSITION 3.4. Let G be a compact metrizable group and $\{N_i\}_{i\in\mathbb{N}}$ be a sequence of its closed normal subgroups satisfying Proposition 3.3.

(a) If X is a G-space, then

$$X = \lim \{X/N_i, p_i^j\}$$

where $p_i^j : X/N_j \to X/N_i, j \ge i$, are the natural projections. (b) Let X and Y be G-spaces represented according to (a) as

$$X = \lim\{X/N_i, p_i^j\} and Y = \lim\{Y/N_i, q_i^j\}$$

 $\begin{aligned} & If \ the \ G/N_i \text{-maps } f_i : X/N_i \to Y/N_i, \ i \in \mathbb{N}, \ are \ such \ that \ q_i^{i+1}f_{i+1} = \\ & f_i p_i^{i+1}, \ i.e. \ the \ diagram \\ & X/N_{i+1} \longrightarrow Y/N_{i+1} \\ & & \downarrow \\ & & f_i & \downarrow \\ & & & f_i & \downarrow \\ & & & & f_i & \downarrow \\ & & & & & & Y/N_i \end{aligned}$

commutes for each $i \in \mathbb{N}$, then there exists a unique G-map $f: X \to Y$ such that $q_i f = f_i p_i$ for each i, where $p_i: X \to X/N_i$, $q_i: Y \to Y/N_i$ are N_i -orbit projections.

PROPOSITION 3.5. Let G be a compact group and N be a closed normal subgroup of G. If $s : A \to X$ is a G-SSDR-map, then the induced map $s/N : A/N \to X/N$ is a G/N-SSDR-map.

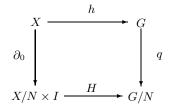
PROOF. Let $j_0 : X/N \hookrightarrow Y$ be a closed G/N-embedding of X/N in a G/N-ANR space Y. By Lemma 1 of [1] there exist a G-space Z and a closed G-embedding $\hat{j}_0 : X \hookrightarrow Z$ such that Z/N = Y and $j_0 \circ p = q_1 \circ \hat{j}_0$, where $p : X \to X/N$, $q_1 : Z \to Y$ are the N-orbit maps. Let $\hat{j}_1 : Z \hookrightarrow M$ be a closed G-embedding of Z in a G-AR space M (See [3], Proposition 1). Then by Theorem 1 of [2] M/N is a G/N-ANR space and we get a closed G/N-embedding $j = j_1 \circ j_0 : X/N \hookrightarrow M/N$, where the embedding $j_1 : Z/N \hookrightarrow M/N$ is induced by \hat{j}_1 . Moreover, for the closed G-embedding $\hat{j} = \hat{j}_1 \circ \hat{j}_0 : X \hookrightarrow M$ and the N-orbit map $q : M \to M/N$ we have $q \circ \hat{j} = j \circ p$.

Now let U and V be invariant neighborhoods of X/N and A/N respectively in M/N. Then $\hat{U} = q^{-1}(U)$ and $\hat{V} = q^{-1}(V)$ are invariant neighborhoods of X and A respectively in M. Since $s : A \hookrightarrow X$ is a G-SSDR-map there exists a G-homotopy $\hat{H} : X \times I \to \hat{U}$ rel. A such that $\hat{H}(x,0) = x$ and $\hat{H}(x,1) \in \hat{V}$. It is clear that the induced G/N-homotopy $H : X/N \times I \to U$, defined by $H(N(x),t) = N(\hat{H}(x,t))$, satisfies the analogous properties and it means that $s/N : A/N \hookrightarrow X/N$ is a G/N-SSDR-map.

We need the following version of the covering homotopy theorem (compare [4], Ch. II, Theorem 7.3).

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PROPOSITION 3.6. Let G be a compact Lie group and N be a normal closed subgroup of G. Suppose that for a G-space X, a G/N-homotopy $H: X/N \times I \to G/N$ and a G-map $h: X \to G$ are such that the following diagram commutes



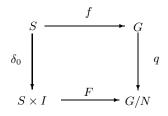
where $\partial_0(x) = (p(x), 0)$ and $p : X \to X/N$, $q : G \to G/N$ are the Norbit maps. Then there exists a G-homotopy $\widetilde{H} : X \times I \to G$ such that $\widetilde{H}(x, 0) = h(x)$ and $q \circ \widetilde{H} = H \circ (p \times 1_I)$.

Moreover, if A is an invariant closed subset of X such that H(p(a),t) = H(p(a),0) for any $a \in A$, $t \in I$, then the covering homotopy \widetilde{H} can be chosen so that $\widetilde{H}(a,t) = h(a)$ for any $a \in A$, $t \in I$.

PROOF. Note that the existence of the *G*-map $h: X \to G$ implies that the action of the group *G* on *X* is of a quite simple structure. Indeed, let $S = h^{-1}(e)$, where *e* is the unit element of the group *G*, and let $\rho: X \to S$ be a map defined by $\rho(x) = (h(x))^{-1}x$. Then ρ is a retraction such that $\rho(gx) = \rho(x)$ for any $g \in G$, $x \in X$. Now consider the product $G \times S$ as a *G*-space endowed with the action g(g', s) = (gg', s). It can be easily verified that the map $\varphi: X \to G \times S$, given by $\varphi(x) = (h(x), \rho(x))$, is a *G*-map and, moreover, it is a *G*-equivalence because the *G*-map $\psi: G \times S \to X$, where $\psi(g, s) = gs$, is the inverse map for φ . In fact, we shall use this *G*-equivalence in the construction of the covering homotopy \widetilde{H} .

Let $F: S \times I \to G/N$ be a homotopy defined by F(s,t) = H(p(s),t). For the given invariant closed subset $A \subseteq X$, let $U = A \cap S$, that is $U = \rho(A)$. Then, if $u \in U$, we have F(u,t) = H(p(u),t) = H(p(u),0) = F(u,0) for any $t \in I$.

Since N is a Lie group, the projection $q: G \to G/N$ is a locally trivial fibration (see [4], Ch. II, Theorem 5.8) and hence it has the regular homotopy lifting property. In particular, considering the commutative diagram



where f(s) = e and $\delta_0(s) = (s, 0)$ for $s \in S$, one can find a homotopy \widetilde{F} : $S \times I \to G$ which preserves the commutativity of the diagram, that is to say, $q \circ \widetilde{F} = F$, $\widetilde{F} \circ \delta_0 = f$ and, moreover, $\widetilde{F}(u, t) = e$ for any $u \in U$, $t \in I$.

Finally, the required covering homotopy $\widetilde{H} : X \times I \to G$ can be defined by $\widetilde{H}(x,t) = h(x)\widetilde{F}(\rho(x),t)$. The verification of this fact is straightforward.

Indeed, we have $\tilde{H}(x,0) = h(x)\tilde{F}(\rho(x),0)$, but

$$\widetilde{F}(\rho(x),0) = \widetilde{F} \circ \delta_0(\rho(x)) = f(\rho(x)) = e,$$

and hence $\widetilde{H}(x,0) = h(x)$.

Since $q: G \to G/N$ is a group morphism and $q \circ \tilde{F} = F$, we get

$$q(\tilde{H}(x,t)) = q(h(x))q(\tilde{F}(\rho(x),t)) = q(h(x))F(\rho(x),t).$$

By the definition of the homotopy F, we have $F(\rho(x), t) = H(p(\rho(x)), t)$, but H is a G/N-map, and therefore, $q(h(x))H(p(\rho(x)), t) = H(q(h(x))p(\rho(x)), t)$. Reminding that $q: G \to G/N$ and $p: X \to X/N$ are N-orbit maps, we obtain

$$q(h(x))p(\rho(x)) = p(h(x)\rho(x)) = p(x)$$

Thus $q(\tilde{H}(x,t)) = H(p(x),t)$, that is $q \circ \tilde{H} = H \circ (p \times 1_I)$. Besides, if $a \in A$, we have $\rho(a) \in U$, and therefore, $\tilde{H}(a,t) = h(a)\tilde{F}(\rho(a),t) = h(a)e = h(a)$ for any $t \in I$.

PROOF OF THEOREM 3.1. Let $j : A \hookrightarrow X$ be a *G*-SSDR-map and $f : A \to G$ be a *G*-map. In order to show that *G* is a *G*-fibrant, we must find a *G*-map $F : X \to G$ such that $F \circ j = f$.

According to Proposition 3.3, we represent the group G as an inverse limit of Lie groups $G = \lim \{G/N_i, q_i^j\}$.

The *G*-maps j and f induce for each k the G/N_k -maps $j_k = j/N_k$: $A/N_k \hookrightarrow X/N_k, f_k = f/N_k : A/N_k \to G/N_k$. Then for each k the following diagram commutes

where p_k^{k+1} , q_k^{k+1} , r_k^{k+1} are the natural projections. It is clear that these projections can be treated as orbit projections with respect to the action of the closed subgroup N_i/N_{i+1} of the Lie group G/N_{i+1} on X/N_{i+1} , A/N_{i+1} and G/N_{i+1} respectively. For each k the map j_k is a G/N_k -SSDR-map by Proposition 3.5, and G/N_k is a G/N_k -ANR by Proposition 3.2, and therefore there exists a G/N_k -equivariant extension $F_k: X/N_k \to G/N_k$ of f_k such that

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 $F_k j_k = f_k$. Using these extensions we shall construct by induction G/N_k maps $T_k : X/N_k \to G/N_k$ satisfying the conditions $T_k p_k^{k+1} = q_k^{k+1} T_{k+1}$ and $T_k j_k = f_k$ for every $k \in \mathbb{N}$. Let $T_1 = F_1$ and suppose that T_k is already found. We have to construct the map T_{k+1} .

One has

$$T_k j_k r_k^{k+1} = f_k r_k^{k+1} = q_k^{k+1} f_{k+1} = q_k^{k+1} F_{k+1} j_{k+1} = F_k^* p_k^{k+1} j_{k+1} = F_k^* j_k r_k^{k+1},$$

where $F_k^* : X/N_k \to G/N_k$ is the G/N_k -map induced by F_{k+1} . Hence $T_k j_k = F_k^* j_k$ because r_k^{k+1} is surjective. According to Theorem 2.1, there is a G/N_k -homotopy $H : F_k^* \simeq T_k \ rel.j_k(A/N_k), \ H : X/N_k \times I \to G/N_k$.

Now consider the following commutative diagram

$$\begin{array}{c|c} X/N_{k+1} & \xrightarrow{F_{k+1}} & G/N_{k+1} \\ \hline \\ \partial_0 & & & & \\ \partial_0 & & & & \\ & & & & \\ X/N_k \times I & \xrightarrow{H} & G/N_k \end{array}$$

where $\partial_0([x]) = (p_k^{k+1}[x], 0)$ for $[x] = N_{k+1}(x) \in X/N_{k+1}$.

Taking into account this diagram, we are going to apply Proposition 3.6 to the Lie group G/N_{k+1} acting on X/N_{k+1} and to its closed normal subgroup N_k/N_{k+1} . Note that one can consider the G/N_{k+1} -space X/N_k as the orbit space $X/N_{k+1}/N_k/N_{k+1}$.

By Proposition 3.6, we get a G/N_{k+1} -homotopy

$$H: X/N_{k+1} \times I \to G/N_{k+1}$$

such that $\widetilde{H}([x], 0) = F_{k+1}([x])$, $q_k^{k+1}(\widetilde{H}([x], t) = H(p_k^{k+1}([x]), t)$ and, for any $t \in I$, $\widetilde{H}([a], t) = F_{k+1}([a])$ if $[a] \in j_{k+1}(A/N_{k+1})$. Putting $T_{k+1}([x]) = \widetilde{H}([x], 1)$ we get the required G/N_{k+1} -map $T_{k+1} : X/N_{k+1} \to G/N_{k+1}$. This completes the inductive step.

The sequence $\{T_k\}_{k\in\mathbb{N}}$, according to Proposition 3.4, determines a unique G-map $F: X \to G$ such that $q_k F = T_k p_k$ for each k. Since $T_k j_k = f_k$ for each k, we can state that $F \circ j = f$.

ACKNOWLEDGEMENTS.

The authors wish to thank the referee for many helpful comments and suggestions.

References

 S.A. Antonyan, Preservation of k-connectedness by a symmetric n-th power functor, Moscow Univ.Math. Bull. 49 (1994) 22-25.

- [2] S.A. Antonyan, Extensorial properties of orbit spaces of proper group actions, Topology and Appl. 98 (1999) 35-46.
- [3] S.A. Antonyan, S.Mardešić, Equivariant Shape, Fund. Math. 127 (1987), 213-224.
- [4] G.E.Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
- [5] A. I. Bykov and L. G. Zerkalov, Cotelescopes and approximate lifting properties in shape theory, Topology and Appl. 73 (1996), 197-212.
- [6] F. Cathey, Strong shape theory, in: Shape Theory and Geometric Topology, Lecture Notes in Math. 870, Springer, Berlin, 1981, 216-239.
- [7] R.S.Palais, The classification of G-spaces, Memoirs AMS, 36, 1960.
- [8] L.S. Pontrjagin, Topological groups, Princeton Univ. Press, 1939.
- [9] D.G.Quillen, Homotopical algebra, Lecture Notes in Math. 43, Springer, 1967.

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Received: 21.1.2004. Revised: 2.3.2005.