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R GROUPS FOR QUATERNIONIC HERMITIAN GROUPS

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Abstract. In this paper a complete classification of R-groups for hermitian quaternionic groups is given. This completes the work of Goldberg for all classical p–adic groups. As a consequence, multiplicity one result follows.

1. Introduction

Let G be a connected reductive p–adic group defined over a nonarchimedean field F and let M be a Levi subgroup of a F-parabolic subgroup P of G. Let σ be an irreducible admissible representation of M. We are interested in decomposing the representation $\mathrm{Ind}_P^G(\sigma)$ into the irreducible components especially when σ is a discrete series representation, and G a non-split form of the classical p–adic group. We will do this using the concept of the R group.

Let $P = P_\theta = M_\theta N_\theta$ be the standard F-parabolic subgroup corresponding to the subset θ of the basis of the root system $\Phi(G, A_0)$ where A_0 is a maximal split torus in G. Let A_θ be the split component of P_θ with the relative Weyl group $W_\theta = N_G(A_\theta)/M_\theta$. We denote the stability subgroup by $W(\sigma)$, i.e. $W(\sigma) = \{w \in W_\theta : w(\sigma) \cong \sigma\}$.

Let us denote $C(\sigma)$ the intertwining algebra of the representation $\mathrm{Ind}_P^G(\sigma)$. From the work of Casselman [4] we have estimate dim $C(\sigma) \leq |W(\sigma)|$. R group is a subgroup of $W(\sigma)$ which determines the exact dimension and structure of $C(\sigma)$. This approach goes back to the work of Knapp and Stein who used the intertwining operators to calculate the reducibility of $\mathrm{Ind}_P^G(\sigma)$ in the case of the real groups and P_θ minimal parabolic subgroup. We follow the calculation of the R-groups in p–adic case from the work of Keys [9], [10],

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Winarsky [13], Goldberg [6] and Herb [8]. In the preliminaries, we explain the structure of the hermitian quaternionic groups, as non-split forms of the classical p–adic groups. Then, in the next subsection we recall the definitions and the results concerning intertwining operators for the representations induced from the discrete series representations. We recall definition of Plancherel measure, the connection with R-groups and the structure of the algebra of the intertwining operators for representations induced from the discrete series representations of the Levi subgroups.

In the third section, we obtain the structure of the Weyl groups and R-groups related to the Levi subgroups of quaternionic group, analogous to one obtained in [6]. Also, we obtain the formula for R group as product of R-groups of the basic parabolic subgroups. In the fourth section the formula for R-group for the basic parabolic subgroup is obtained. In the fifth section we prove the multiplicity one theorem for our non-split forms, by the methods analogous to those used for the split classical groups by Herb [8].

2. Preliminaries

2.1. Let F be a non-archimedean field of characteristic zero, with the residual field with q elements. Let D be a quaternionic algebra, central over F and let τ be the usual involution, fixing the center of D. Then D is a rank 4 algebra over F with basis $\{1, w_1, w_2, w_3\}$ satisfying

$$w_1^2 = \alpha, w_2^2 = \beta, w_1 w_2 = -w_2 w_1 = w_3 \text{ for some } \alpha, \beta \in F.$$

Then, τ acts on $d = x_0+x_1w_1+x_2w_2+x_3w_3$ as $\tau(d) = x_0-x_1w_1-x_2w_2-x_3w_3$. Division algebra D has well-known matrix representation in $M(2, F(\sqrt{\alpha}))$

$$d = x_0 + x_1w_1 + x_2w_2 + x_3w_3 \mapsto \begin{pmatrix} x_0 + x_1\sqrt{\alpha} & x_2 + x_3\sqrt{\alpha} \\ \beta x_2 - \beta x_3\sqrt{\alpha} & x_0 - x_1\sqrt{\alpha} \end{pmatrix}.$$

It is straightforward that

$$D \otimes_F \overline{F} \cong M(2, \overline{F}), \tag{2.1}$$

so we can naturally extend the involution τ to $M(2, \overline{F})$. By Skolem-Noether theorem there exists a matrix h_0 such that

$$\tau(x) = h_0 x^t h_0^{-1} \text{ for all } x \in M(2, \overline{F}).$$

Actually, with the above matrix representation of D we can take $h_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We define an involution on the space $M(k, D)$ by $g \mapsto g^* = \tau(g)^t$. Again, we can extend the involution $*$ from $M(k, D)$ on

$$M(k, D) \otimes_F \overline{F} \cong M(2k, \overline{F}), \tag{2.2}$$

and by Skolem-Noether theorem realize it through transposition and conjugation by matrix

$$h = \begin{pmatrix} h_0 & & & \\ & h_0 & & \\ & & \ddots & \\ & & & h_0 \end{pmatrix}.$$

Also, we can assume by (2.2) that $M(k, D)$ is embedded in $M(2k, \overline{F})$. Division algebra D defines a reductive group G over F as follows. Let

$$V_n = e_1 D \oplus \cdots \oplus e_n D \oplus e_{n+1} D \oplus \cdots \oplus e_{2n} D$$

be a right vector space over D. Relations $(e_i, e_{2n-j+1}) = \delta_{ij}$ for $i = 1, 2, \ldots, n$ define the hermitian form on V_n:

$$\begin{gathered} (v, v') = \varepsilon\tau((v', v)), v, v' \in V_n,\ \varepsilon \in \{-1, 1\} \\ (vx, v'x') = \tau(x)(v, v')x', x, x' \in D. \end{gathered}$$

Let $G(F, \varepsilon) = G_n(D, \varepsilon)$ be the group of isometries of the form $(\cdot, \cdot)$. In the sequel, we will do explicit calculations for the case $\varepsilon = -1$, so we drop ε from the notation and, unless otherwise specified, assume $\varepsilon = -1$. The remaining case is quite similar and we will turn our attention to that case in the end. We can describe group $G(F, \varepsilon) = G(F)$ also as

$$G(F) = \left\{ g \in GL(2n, D) : g^* \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

where

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

So, if we consider the group

$$G_n(\overline{F}) = \left\{ g \in GL(4n, \overline{F}) : g^* \begin{pmatrix} 0 & J'_{2n} \\ -J'_{2n} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J'_{2n} \\ -J'_{2n} & 0 \end{pmatrix} \right\}$$

with J'_{2n} obtained from J_n by embedding which follows from (2.2) we see that $G_n(D)$ is the group of F-rational points of the group $G_n(\overline{F})$. Of course, the action of the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ is given by isomorphism (2.2).

2.2. The group $G_n(\overline{F})$ is conjugated to $SO(4n,\overline{F})$ in $GL(4n,\overline{F})$, so we conclude that

$$T=\left\{\begin{pmatrix}\lambda_1 &&&&&&&\\ &\lambda_2&&&&&&\\ &&\ddots&&&&&\\ &&&\lambda_{2n}&&&&\\ &&&&\lambda_{2n}^{-1}&&&\\ &&&&&\ddots&&\\ &&&&&&\lambda_2^{-1}&\\ &&&&&&&\lambda_1^{-1}\end{pmatrix}:\lambda_i\in\overline{F}\right\}$$

is a maximal torus in $G_n(\overline{F})$. To determine whether T is defined over F we observe the following. Let χ_{ij} denote a regular function (in algebraic sense) on the group $G_n(\overline{F})$ given by $g=(g_{ij})\mapsto g_{ij}$. Then $\sigma\in\Gamma$ acts in the following way

$$\chi_{ij}^{\sigma}=\begin{cases}\chi_{ij} & \text{if } \sigma(\sqrt{\alpha})=\sqrt{\alpha},\\ \begin{cases}\chi_{i+1j+1} & \text{if } i,j \text{ are odd}\\ \frac{1}{\beta}\chi_{i+1j-1} & \text{if } i \text{ odd and } j \text{ even,}\\ \beta\chi_{i-1j+1} & \text{if } i \text{ is even and } j \text{ is odd,}\\ \chi_{i-1j-1} & \text{if } i,j \text{ are even}\end{cases} & \text{if } \sigma(\sqrt{\alpha})=-\sqrt{\alpha}.\end{cases}$$

Denote by J the ideal of functions in $\overline{F}[G_n(\overline{F})]$ vanishing on T, and by $J_F=J\cap F[G_n(\overline{F})]$, where $F[G_n(\overline{F})]=\overline{F}[G_n(\overline{F})]^{\Gamma}$. For T to be defined over F it is enough to show (e.g. from [3]) that

$$J=\overline{F}[G_n(\overline{F})]J_F.$$

Consider, for example, for i odd and j even, function $\sqrt{\alpha}(\chi_{ij}-\frac{1}{\beta}\chi_{i+1j-1})$. This function as well as function $\chi_{ij}+\frac{1}{\beta}\chi_{i+1j-1}$ belongs to J_F, so χ_{ij} belongs to $\overline{F}[G_n(\overline{F})]J_F$ and, in general, χ_{ij}, for $i\neq j$ belongs to $\overline{F}[G_n(\overline{F})]J_F$. The same reasoning implies also that functions $\chi_{ii}\chi_{4n+1-i4n+1-i}-1$, $i=1,\ldots,2n$ belong to $\overline{F}[G_n(\overline{F})]J_F$. So, T is defined over F. Now, it is easy to see that the maximal F-split subtorus of T is

$$A_0 = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_n & & & & & & \\ & & & & \lambda_n & & & & & \\ & & & & & \lambda_n^{-1} & & & & \\ & & & & & & \lambda_n^{-1} & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \lambda_1^{-1} & \\ & & & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in \overline{F}^* \right\}.$$

From (2.2) follows

$$A_0(F) = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in F^* \right\}.$$

Since the absolute root system $\Phi(G_n(\overline{F}), T)$ is of the type D_{2n}, we can take

$$\Phi(G_n(\overline{F}), T) = \{\pm(\varepsilon_i \pm \varepsilon_j); i \neq j, i, j = 1, 2, \ldots 2n\}.$$

Here, $\{\varepsilon_i\}$'s have obvious meaning. It follows that the relative root system obtained from the absolute one restricted on A_0 is

$$\Phi(G_n(\overline{F}), A_0) = \{\pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i; i \neq j, i, j = 1, 2, \ldots n\},$$

i.e. the relative root system is of the type C_n. We can choose the basis

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} : i = 1, 2, \ldots n\} \cup \{2\varepsilon_n\}.$$

We abbreviate $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_n = 2\varepsilon_n$. Let us denote

$$\tilde{J}_{2k} = \begin{pmatrix} & & & & & -1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & \iddots & & & \\ & -1 & & & & \\ 1 & & & & & \end{pmatrix}.$$

Now, the (standard) Levi subgroup $M_0 = Z_{G_n(\overline{F})}(A_0)$ of the minimal F-parabolic subgroup $P_0 = M_0 N_0$ corresponding to the above basis is

$$M_0 = \left\{ \begin{pmatrix} A_1 & & & & & & & \\ & A_2 & & & & & & \\ & & \ddots & & & & & \\ & & & A_n & & & & \\ & & & & \tilde{A}_n & & & \\ & & & & & \ddots & & \\ & & & & & & \tilde{A}_2 & \\ & & & & & & & \tilde{A}_1 \end{pmatrix} : A_i \in GL(2, \overline{F}) \right\}.$$

The matrices $\tilde{A}_i$ are obtained as $\tilde{A}_i = -\tilde{J}_2 {A_i}^{-t} \tilde{J}_2$. Hence:

$$M_0(F) = \left\{ \begin{pmatrix} \xi_1 & & & & & & & \\ & \xi_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \xi_n & & & & \\ & & & & \tau(\xi_n)^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \tau(\xi_2)^{-1} & \\ & & & & & & & \tau(\xi_1)^{-1} \end{pmatrix} : \xi_i \in D^* \right\}.$$

Analogously, the standard Levi F-subgroups correspond to the subsets θ of the basis Δ. They have the following form:
(i) If $\alpha_n \notin \theta$ there exist positive integers $n_1, n_2, \ldots, n_k$ with $\Sigma n_i = n$, such that
(2.3)

$$M_\theta = \left\{ \begin{pmatrix} A_{n_1} & & & & & & & \\ & A_{n_2} & & & & & & \\ & & \ddots & & & & & \\ & & & A_{n_k} & & & & \\ & & & & \tilde{A}_{n_k} & & & \\ & & & & & \ddots & & \\ & & & & & & \tilde{A}_{n_2} & \\ & & & & & & & \tilde{A}_{n_1} \end{pmatrix} : A_i \in GL(2n_i, \overline{F}) \right\}.$$

(ii) If $\alpha_n \in \theta$ there exist positive integers $n_1, n_2, \ldots, n_k, r$ with $\Sigma n_i + r = n$, such that

$$M_\theta = \left\{ \begin{pmatrix} A_{n_1} & & & & & & & & \\ & A_{n_2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & A_{n_k} & & & & & \\ & & & & G_r(\overline{F}) & & & & \\ & & & & & \tilde{A}_{n_k} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \tilde{A}_{n_2} & \\ & & & & & & & & \tilde{A}_{n_1} \end{pmatrix} : A_i \in GL(2n_i, \overline{F}) \right\},$$

where the matrices $\tilde{A}_{n_i}$ are obtained as $\tilde{A}_{n_i} = -\tilde{J}_{2n_i} {A_{n_i}}^{-t} \tilde{J}_{2n_i}$. In the same way, their groups of F-rational points are

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & \\ & \ddots & & & & \\ & & A_{n_k} & & & \\ & & & J_{n_k}(A_{n_k}^{-1})^* J_{n_k} & & \\ & & & & \ddots & \\ & & & & & J_{n_1}(A_{n_1}^{-1})^* J_{n_1} \end{pmatrix} : A_i \in GL(n_i, D) \right\},$$

with $\Sigma n_i = n$ in the first case, and

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & \\ & \ddots & & & \\ & & G_r(D) & & \\ & & & \ddots & \\ & & & & J_{n_1}(A_{n_1}^{-1})^* J_{n_1} \end{pmatrix} : A_i \in GL(n_i, D) \right\},$$

with $\Sigma n_i + r = n$ in the second case.

2.3. Let us denote the complexified dual of the real Lie algebra of A_θ by $(\mathfrak{a}^*_{M_\theta})_{\mathbb{C}}$ and by $\mathcal{E}_2(M_\theta)$ the set of equivalence classes of the irreducible square integrable modulo center representations of M_θ. If θ is fixed we write M for M_θ and P for P_θ. Let K be a maximal, good compact subgroup of G and let H_P be the homomorphism from subgroup P to $\mathfrak{a}_M$ trivial on the subgroup N such that the following holds

$$q^{<H_P(a),\alpha>} = |\alpha(a)|_F \quad \forall a \in A, \quad \forall \alpha \in \mathrm{Rat}(A).$$

We extend it from P, trivially on K, to G. For $\nu \in (\mathfrak{a}^*_{M_\theta})_{\mathbb{C}}$ and σ discrete series representation of M we denote by $J_{P'|P}(\nu, \sigma)$ standard (integral) intertwining operator between $\mathrm{Ind}_P^G(\sigma \otimes q^{\langle \nu, H_P()\rangle})$ and $\mathrm{Ind}_{P'}^G(\sigma \otimes q^{\langle \nu, H_{P'}()\rangle})$, defined in the following way

$$J_{P'|P}(\nu,\sigma)f(g) = \int_{N\cap N'\backslash N'} f(ng)\mathrm{d}n.$$

Here, P' is another parabolic subgroup with Levi subgroup M. This integral (weakly) converges in a cone in $(\mathfrak{a}^*_{M_\theta})_{\mathbb{C}}$ and has meromorphic continuation on the whole space. Let $R_{P'|P}(\sigma,\nu)$ denote the normalization of the operator $J_{P'|P}(\sigma,\nu)$ which satisfies number of properties, e.g. see [1]. When ν is 0, we exclude it from the notation. Let us denote by $l(w)$ an operator of the left translation by element w^{-1} of the Weyl group. This operator acts on the spaces of the induced representations. If σ acts on the space V, then for $w \in W(\sigma)$ there exist the isomorphisms T_w on V such that

$$T_w w(\sigma)(m) = \sigma(m)T_w, \text{ for all } m \in M.$$

We define the operators $R(w,\sigma) = T_w l(w) R_{w^{-1}Pw|P}(\sigma)$ which are the intertwining operators for the representation $\mathrm{Ind}_P^G(\sigma)$. By the results of Harish-Chandra, family $\{R(w,\sigma) : w \in W(\sigma)\}$ spans (as a vector space) algebra $C(\sigma)$. Now, we could introduce R group without referring to Plancherel measures: we could say that R group is the subgroup of elements in $W(\sigma)$ which act on $(\mathfrak{a}^*_{M_\theta})_{\mathbb{C}}$ in such a way that they fix positive chamber, but for our calculations it is easier to use Plancherel measures. We also denote $A(\sigma,\nu,w) = l(w)J_{w^{-1}Pw|P}(\sigma,\nu)$ and to avoid misinterpretation once more we note that

$$A(\sigma,\nu,w)f(g) = \int_{N_w} f(w^{-1}ng)\mathrm{d}n,$$

where $N_w = N_0 \cap w^{-1}\overline{N}w$.

Definition 2.1. *Plancherel measures $\mu(\sigma,\nu,w)$ are defined as follows*

$$A(\sigma,\nu,w)A(w(\sigma),w(\nu),w^{-1}) = \mu(\sigma,\nu,w)^{-1}\gamma_w^2(G/P)$$

where

$$\gamma_w^2(G/P) = \int_{\overline{N_w}} q^{\langle 2\rho_P, H_P(\overline{n})\rangle}\mathrm{d}\overline{n}.$$

Here ρ_P is half the sum of the positive roots in N_θ.

When w is the longest element in the corresponding Weyl group we exclude it from the notation. We shall denote by $\mu_\alpha(\sigma,\nu)$ the Plancherel measure corresponding to the representation $\mathrm{Ind}_{P_\theta^*}^{M_\alpha}(\sigma \otimes q^{\langle \nu, H_{P_\theta^*}()\rangle})$. Root α is in the set of roots in P_θ corresponding to A_θ. Group M_α is the Levi subgroup of G centralizing torus $A_{\theta\cup\{\alpha\}}$ and $P_\theta^* = M_\alpha \cap P_\theta$ is the maximal parabolic subgroup of M_α. Let us denote $\Delta' = \{\alpha \in \Phi(P_\theta, A_\theta) : \mu_\alpha(\sigma) = 0\}$.

Definition 2.2. *R-group of the representation $\sigma \in \mathcal{E}_2(M_\theta)$ is*

$$R_\sigma = \{w \in W(\sigma) : w(\Delta') > 0\}.$$

We observe that w is an element of R_σ if and only if $w(\Delta') = \Delta'$. In the sequel, if the representation σ is fixed, we write $R_\sigma = R$.

THEOREM 2.3 (Harish-Chandra). *We have*

$$\dim C(\sigma) = |R_\sigma|.$$

The following holds

$$R(r_1 r_2, \sigma) = \eta_\sigma(r_1, r_2) R(r_1, \sigma) R(r_2, \sigma),\ \forall r_1, r_2 \in R_\sigma,$$

where $\eta_\sigma(r_1, r_2) = T_{r_1 r_2} T_{r_2}^{-1} T_{r_1}^{-1}$ is a non-zero complex number, and the above operators have appropriate domains. It is easy to see that η_σ is a 2-cocycle for R_σ with values in $\mathbb{C}^*$, so we can form the finite central extension ([2]) where η_σ splits, i.e. we have short exact sequence

$$1 \to Z_\sigma \to \tilde{R}_\sigma \to R_\sigma \to 1,$$

for certain group $\tilde{R}_\sigma$. Then, we have bijective correspondence between certain class of the irreducible representations of $\tilde{R}_\sigma$ and the irreducible components of $\mathrm{Ind}_{P_\theta}^{G}(\sigma)$. During the calculations, one can assume that in the Levi subgroups, all the GL- blocks of the same size are grouped together. This is possible because each parabolic subgroup has associate parabolic subgroup with such structure, and associate parabolic subgroups have equal corresponding Plancherel measures.

3. WEYL GROUPS AND R GROUPS

In this section we compute relative Weyl groups which we need for the calculation of R-groups. We adopt the settings from [6]. Let $M = M_\theta$ be the standard F-Levi subgroup of $G_n(D)$. We assume that M is composed from the GL-blocks which are arranged in the following way. Let $\theta = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_k$ be the decomposition of $\theta \subset \Delta$ into disjoint union of the connected components of the Dynkin diagram. First consider the case when $\alpha_n \notin \theta$. We assume that there are no gaps between the components of θ and that $\alpha_1 \in \theta$. Further, let

$$\begin{aligned} X_1 &= \{\theta_{11}, \theta_{12}, \ldots, \theta_{1n_1}\}, \\ X_2 &= \{\theta_{21}, \theta_{22}, \ldots, \theta_{2n_2}\}, \\ &\vdots \\ X_r &= \{\theta_{r1}, \theta_{r2}, \ldots, \theta_{rn_r}\}, \end{aligned}$$

where each X_i assembles components of the same length. If we put $m_i = |\theta_{i1} + 1|$, then let $b = \sum_{i=1}^{r} m_i n_i$ and $n_{r+1} = n - b$. With this notation we have

$$M_\theta(F) \cong GL(m_1, D)^{n_1} \times\ GL(m_2, D)^{n_2}\ \times \cdots \times \\ \times GL(m_r, D)^{n_r}\ \times GL(1, D)^{n_{r+1}}, \tag{3.1}$$

where all the blocks are arranged in above strict order.

In the case when $\alpha_n \in \theta$ we assume $\alpha_n \in \theta_k$. Let $m_i = |\theta_{i1} + 1|$ for $i = 1, 2, \ldots, r-1$ and $m = |\theta_k|$. We set $b' = \sum_{i=1}^{r-1} m_i n_i$ and $n_{r+1} = n - m - b'$. Then

$$\begin{aligned} (3.2) \quad M_\theta(F) \cong GL(m_1, D)^{n_1} \times \ GL(m_2, D)^{n_2} \ \times \cdots \times \\ \times GL(m_{r-1}, D)^{n_{r-1}} \ \times GL(1, D)^{n_{r+1}} \times \ G_m(D). \end{aligned}$$

We can make the above assumptions because of the remark given in the preliminaries: in the calculation of R-groups we can reduce the case of the arbitrary standard Levi subgroup to the case of one given above. Now, with $G = G_n(D)$ let us compute relative Weyl groups $W(G/A_\theta) = W_\theta = N_G(A_\theta)/M_\theta$. Because $N_G(A_\theta)/M_\theta$ is a subgroup of $N_G(A_0)/M_\theta \cap N_G(A_0) \simeq W_0/(M_\theta \cap N_G(A_0))/M_0$ we can take for the representatives of W_θ elements of Weyl group W_0 modulo certain subgroup (in fact, modulo Weyl group of system $\Phi(M_\theta, A_0)$).

We know that the Weyl group of the root system C_n is $W_0 \cong S_n \ltimes \mathbb{Z}_2^n$. We can interpret S_n as the group of permutations on the set of diagonal entries $\{\lambda_i\}$ of the torus $A_0(F)$. $\mathbb{Z}_2^n$ acts in such a way that if we denote by c_i the nontrivial element in the i–th copy of $\mathbb{Z}_2$, then c_i sends $\lambda_i \mapsto \lambda_i^{-1}$, hence it is called the sign change. Indeed, it is of no importance that the group $G(\overline{F})$ is not even quasi-split: it easy to see that every member of the Weyl group of $Sp(2n, F) \subset G_n(D)$, appropriately embedded in $G(\overline{F})$ normalizes A_0 modulo M_0, and, vice versa, that every element of $N_G(A_0)$ comes from Weyl group of $Sp(2n, F)$ modulo M_0.

Analogously, we have the same situation with the relative Weyl groups.

Proposition 3.1. (a) *If $\alpha_n \notin \theta$, we have*

$$W_\theta \cong (\prod_{i=1}^{r+1} S_{n_i}) \ltimes \mathbb{Z}_2^{n_1 + \cdots + n_{r+1}}$$

(b) *If $\alpha_n \in \theta$, we have*

$$W \cong \prod_{i \neq r} (S_{n_i} \ltimes \mathbb{Z}_2^{n_i})$$

Proof. The claim is obvious if we keep in mind the structure of Levi subgroups. In other words, relative Weyl group W_θ acts as permutation of blocks of the same size and sign change on the blocks inside the torus A_θ. □

Because we are interested in the representations of the groups of F-rational points of the reductive algebraic groups, from now on, to simplify notation, we will denote by M_θ the group previously denoted by $M_\theta(F)$.

Definition 3.2. *Let $P_\theta = M_\theta N_\theta$ be the standard parabolic subgroup of $G_n(D)$. If $M_\theta \cong GL(m_1, D)^{n_1}$ or $M_\theta \cong GL(m_1, D)^{n_1} \times G_k(D)$, the parabolic subgroup P_θ is called basic.*

3.1. Keeping in mind 3.1 and 3.2 we want to express the relative Weyl group W_θ in terms of the relative Weyl groups of the basic parabolic subgroups. We are keeping our usual assumption, i.e. there are no gaps between the components of θ and they are arranged in such a way that the components of the same length are grouped together. If $\alpha_n \notin \theta$ suppose M_θ is of the form (3.1). For $i = 1, \dots, r+1$ (where $n_{r+1} \geq 0$) we define a group G_i as $G_i = G_{m_i n_i}(D)$. Then the group G_i has a basic parabolic subgroup P_i with Levi subgroup $M_i \cong GL(m_i, D)^{n_i}$. If we denote by A_i the split component of M_i from (3.1) follows

Lemma 3.3. *If $\alpha_n \notin \theta$ then*

$$W_\theta \cong \prod W(G_i/A_i).$$

It is useful to realize the tori A_i as subgroups of the torus A_θ and the groups M_i as subgroups of the group M_θ. Now, we follow [6]. If $X_i = \{\theta_{i1}, \theta_{i2}, \dots, \theta_{in_i}\}$ then X_i (as set of roots) has cardinality $(m_i - 1)n_i$. Denote by Θ_i, $i \in \{1, 2, \dots, r\}$ the (unique) subroot system of $\Phi(G, A_0)$ of the type $C_{m_i n_i}$ containing X_i. Basis Δ_i of Θ_i is obtained by joining to X_i intermediate roots and one long root (i.e. $\varepsilon_{m_1} - \varepsilon_{m_1+1}, \dots \varepsilon_{m_1(n_1-1)} - \varepsilon_{m_1(n_1-1)+1}$ and $2\varepsilon_{m_1 n_1}$ for X_1). Also, let $\Theta_{r+1} = \{\alpha \in \Phi(G, A_0) : (\alpha, \beta) = 0, \forall \beta \in \theta\}$. Now, the torus A_i in G_i is the split torus which corresponds to set of roots X_i, i.e. $A_i = A_{X_i}$ and then let $P_i = P_{X_i}$ in G_i.

Let us turn our attention to the representations again. If $\sigma \in \mathcal{E}_2(M_\theta)$ then $\sigma \cong \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{r+1}$ where $\sigma_i \in \mathcal{E}_2(M_i)$. As before, let $W_i(\sigma_i)$ denote the stabilizer of σ_i in $W(G_i/A_i)$. We consider representations $\mathrm{Ind}_{P_i}^{G_i}(\sigma_i)$. Then, from the fact that $M_\theta \cong \prod_{i=1}^{r+1} M_i$ and from Lemma 3.3 the content of the following lema is straightforward

Lemma 3.4. *If $\alpha_n \notin \theta$ then*

$$W(\sigma) \cong \prod_{i=1}^{r+1} W_i(\sigma_i).$$

In the following lemmas, we relate the subset Δ' from the definition of the R group to the analogous subsets Δ'_i which correspond to Levi subgroups.

Lemma 3.5. *If $\alpha_n \notin \theta$ then, with the above notation, we have:*

$$\text{if } \alpha \notin \bigcup \Theta_i \text{ then } \alpha|_{A_\theta} \notin \Delta'.$$

Proof. The roots of the form $\alpha = \pm 2\varepsilon_l$ for some $l \in \{1, 2 \dots, n\}$ are necessarily in some $\Theta_i, i \in \{1, 2 \dots, r+1\}$ by the construction: they are the roots added X_i to form $\Theta_i, i \in \{1, 2 \ldots, r\}$, or they are in Θ_{r+1}. So, for $\alpha \notin \bigcup \Theta_i$ we can suppose that $\alpha = \varepsilon_j \pm \varepsilon_k$ for some j, k, that is, α is a shorter root. Because $\alpha \notin \bigcup \Theta_i$ there exist l and s, $l \neq s$ such that $\varepsilon_j - \varepsilon_{j+1} \in \Delta_l$ and $\varepsilon_k - \varepsilon_{k+1} \in \Delta_s$. Then, from the Figure 1 that presents a case $\alpha = \varepsilon_j - \varepsilon_k$ and Figure 2 that presents a case $\alpha = \varepsilon_j + \varepsilon_k$ it is easy to see that

$$M_\alpha \cong \prod_{f \neq l,s} M_f \ \times GL(m_l, D)^{n_l - 1} \ \times GL(m_s, D)^{n_s - 1} \ \times GL(m_l + m_s, D).$$

Consider now the representation $\mathrm{Ind}_{P_\alpha^*}^{M_\alpha}(\sigma)$, where P_α^* is a maximal parabolic subgroup in M_α obtained as $M_\alpha \cap P_\theta$. Then

$$\text{(3.3)} \quad \mathrm{Ind}_{P_\alpha^*}^{M_\alpha}(\sigma) \cong \bigotimes_{f \neq l,s} \sigma_f \otimes \bigotimes_{p \neq p_0} \sigma_{l,p} \otimes \bigotimes_{t \neq t_0} \sigma_{s,t} \otimes \\ \mathrm{Ind}_{GL(m_l,D) \times GL(m_s,D)}^{GL(m_l+m_s,D)}(\sigma_{l,p_0} \otimes \sigma_{s,t_0}),$$

where

$$\sigma_l \cong \bigotimes_{p=1}^{n_l} \sigma_{l,p} \text{ and } \sigma_s \cong \bigotimes_{t=1}^{n_s} \sigma_{s,t},$$

and $\varepsilon_j - \varepsilon_{j+1}$ is between θ_{l,p_0} and θ_{l,p_0+1} in X_l and analogously $\varepsilon_k - \varepsilon_{k+1}$ in X_s. Because M_θ is maximal in M_α, for $\mu_\alpha(\sigma) = 0$ to be fulfilled i.e. $\alpha \in \Delta'$ it is necessary and sufficient that the representation σ ramifies in M_α and that the induced representation is irreducible [7]. Now, for the ramification of σ in M_α it is necessary that $m_l = m_s$ but this is not the case. ∎

On the figures we denote $B_{ij} = J_{m_i}(A_{ij}^{-1})^* J_{m_i}$.

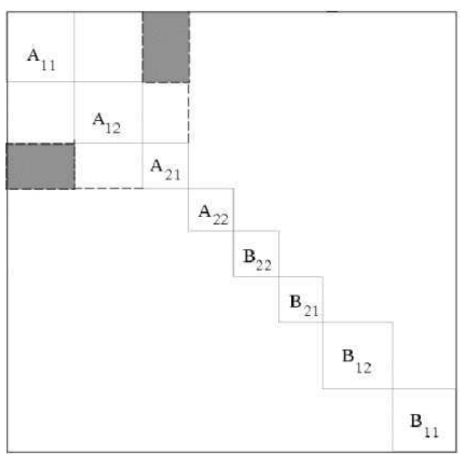

Figure 1:

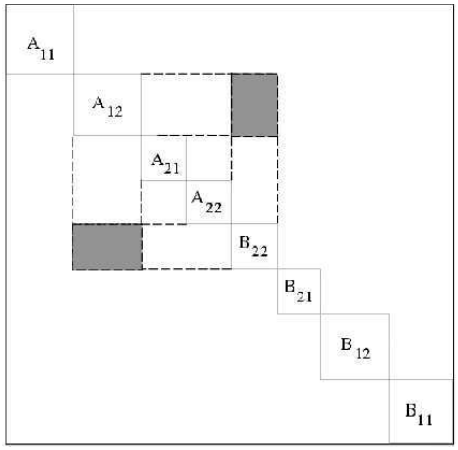

Figure 2:

From Lemma 3.5 follows that we can restrict our attention to each Θ_i in order to determine Δ'. So, let $\alpha \in \Theta_i$. We can consider α as an element of $\Phi(P_i, A_i)$ and as an element of $\Phi(P_\theta, A_\theta)$. Let us denote $A_{i,\alpha} = (A_i \cap \mathrm{Ker}\alpha)^\circ$, $M_{i,\alpha} = Z_{G_i}(A_{i,\alpha})$ and $P^*_{i,\alpha} = P_i \cap M_{i,\alpha}$. Let $\mu_\alpha(\sigma_i)$ be the Plancherel measure associated with the representation $\mathrm{Ind}_{P^*_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)$. Then the following lemma holds.

Lemma 3.6. *If $\alpha_n \notin \theta$ then for $i \in \{1, 2, \dots r+1\}$ and $\alpha \in \Theta_i$ we have*

(*i*)

$$M_\alpha \cong \prod_{k \neq i}^{r+1} M_k \ \times M_{i,\alpha},$$

(*ii*)

$$P^*_\alpha \cong \prod_{k \neq i}^{r+1} M_k \ \times P^*_{i,\alpha},$$

(*iii*)

$$W(M_\alpha/A_\theta) = W(M_{i,\alpha}/A_i),$$

(*iv*)

$$\mathrm{Ind}_{P^*_\alpha}^{M_\alpha}(\sigma) \cong \sigma_1 \otimes \cdots \otimes \sigma_{i-1} \otimes (\mathrm{Ind}_{P^*_{i,\alpha}}^{M_{i,\alpha}}(\sigma_i)) \otimes \cdots \otimes \sigma_{r+1}.$$

Proof. When α is shorter root (*i*) and (*ii*) follow immediately from Figure 1 and Figure 2 but with the constraint that everything is happening in the square corresponding to X_i. If $\alpha = \pm 2\varepsilon_l$ then the torus $A_{i,\alpha}$ has a m_i

block consisting of all 1's, so

$$M_{i,\alpha} \cong GL(m_i, D)^{n_i-1} \times \ G_{m_i}(D),$$

and

$$M_\alpha \cong \prod_{j\neq i}^{r+1} M_j \ \times GL(m_i, D)^{n_i-1} \times \ G_{m_i}(D),$$

so (i) and (ii) follow in this case too. Claims (iii) and (iv) are now straightforward. ∎

We can now finally attach R-groups to Levi subgroups. As before, let

$$\sigma \cong \bigotimes_i \sigma_i \in \mathcal{E}_2(M_\theta).$$

Let

$$\Delta'_i = \{\alpha \in \Phi(P_i, A_i) : \mu_\alpha(\sigma_i) = 0\} \text{ and } R_i = \{w \in W_i(\sigma_i) : w(\Delta'_i) = \Delta'_i\}.$$

We have the following proposition

Proposition 3.7. *If $\alpha_n \notin \theta$, let us denote by R the R-group of the representation $\mathrm{Ind}_{P_\theta}^{G_n(D)}(\sigma)$ and by R_i the R-group associated to the representation $\mathrm{Ind}_{P_i}^{G_i}(\sigma_i)$. Then we have*

$$R = R_1 \times \ \cdots \ \times R_{r+1}.$$

Proof. The proof is straightforward from the preceding lemmas. ∎

Discussion in the case when $\alpha_n \in \theta$ is very similar, but with the following difference: in the decomposition into the connected components of Dynkin diagram now X_r has only one component θ_k which contains α_n. Now, for each $i \in \{1, 2, \ldots, r-1\}$ let Θ_i be the unique subroot system of type $C_{n_i m_i + m}$ containing $X_i \cup X_r$. Hence, $M_i \cong GL(m_i, D)^{n_i} \times \ G_m(D)$. Then, we have

$$M_\theta \cong \tilde{M}_i \ \cdots \ \times \tilde{M}_{r-1} \times \ G_m(D)$$

where $M_i \cong \tilde{M}_i \times \ G_m(D)$.

Now let $\sigma \cong \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_{r-1} \otimes \sigma_{r+1} \otimes \rho$ where $\rho \in \mathcal{E}_2(G_m(D))$ and $\sigma_i \in \mathcal{E}_2(\tilde{M}_i)$. Analogously as before, we let $W_i(\sigma_i)$ be the stabilizer of the representation $\sigma_i \otimes \rho$ in G_i.

Proposition 3.8. *If $\alpha \in \theta$ then*

(i)

$$W(\sigma) \cong W_1(\sigma_1) \ \times \cdots \times \ W_{r-1}(\sigma_{r-1}) \ \times W_{r+1}(\sigma_{r+1}),$$

(ii)

$$R = R_1 \ \times R_2 \ \cdots \times R_{r-1} \times \ R_{r+1}.$$

So, we have reduced the calculation of R-groups to the case of basic parabolic subgroups.

4. R-groups for basic parabolic subgroups

4.1. First we present an observation of Muić and Savin ([11]) on representations that will occur.

DEFINITION 4.1. *Let π be an admissible representation of $GL(n,D)$. Let us define representation π^* in the following way: $\pi^*(g) = \pi((g^*)^{-1})$.*

LEMMA 4.2. $\pi^* \cong \tilde{\pi}$.

For computation of R-groups we must have some information about induction from maximal parabolic subgroup. So, Levi subgroup of maximal F-parabolic subgroup of $G_n(D)$ is of the following form:

$$\begin{pmatrix} g & & \\ & g' & \\ & & J_k(g^{-*})J_k \end{pmatrix}.$$

Here $g \in GL(k,D)$ and $g' \in G_m(D)$ are such that $m \geq 0$ and $m+k=n$. As we have seen, the only non-trivial element of the relative Weyl group acts as a sign change, i.e

$$w \begin{pmatrix} g & & \\ & g' & \\ & & J_k(g^{-*})J_k \end{pmatrix} w^{-1} = \begin{pmatrix} g^{-*} & & \\ & g' & \\ & & J_k g J_k \end{pmatrix}.$$

We conclude that $\sigma \otimes \rho$ ramifies in $G_n(D)$ if and only if $\sigma^* \cong \sigma$.

4.2. To calculate R-groups for basic parabolic subgroups we follow the arguments of D.Keys [9] which prove that only possible elements in R-group are sign changes. We have the following lemma

LEMMA 4.3. *Let $w = sc$ be an element of the R-group of the representation $\mathrm{Ind}_{P_\theta}^{G_n(D)}(\sigma)$, where P_θ is a basic parabolic subgroup, $\sigma \in \mathcal{E}_2(M_\theta)$ with $s \in S_r$ and $c \in \mathbb{Z}_2^r$. Then $s = 1$.*

PROOF. See [6], [9]. ∎

LEMMA 4.4. *Let $M \cong GL(m,D) \times GL(k,D)$ be Levi subgroup of the maximal standard parabolic $P = MN$ in $GL(m+k,D)$. Let $\sigma_1 \in \mathcal{E}_2(GL(m,D))$ and $\sigma_2 \in \mathcal{E}_2(GL(k,D))$. Then the representation $\mathrm{Ind}_P^{GL(m+k,D)}(\sigma_1 \otimes \sigma_2)$ is irreducible.*

PROOF. The proof of this lemma can be found in [5]. ∎

Now we can state our main proposition about R groups:

PROPOSITION 4.5. *Let P_θ be the basic parabolic subgroup of $G_n(D)$ such that $M_\theta \cong GL(k,D)^r \times G_m(D)$, where $m \geq 0$, and let $\sigma = \sigma_1 \otimes \cdots \sigma_r \otimes \rho \in \mathcal{E}_2(M_\theta)$. Then R-group of the representation $\mathrm{Ind}_{P_\theta}^{G_n(D)}(\sigma)$ is isomorphic to $\mathbb{Z}_2^d$, where d is number of inequivalent σ_i's such that $\mathrm{Ind}(\sigma_i \otimes \rho)$ is reducible.*

PROOF. The proof of this proposition follows from the proof of the analogous result of Goldberg [6] in the split case combined with Lemma 4.4. $\square$

5. Multiplicity one

Finally, we prove that the cocycle relation mentioned in the preliminaries is, in our case, in fact, trivial. As a result, there exists an algebra isomorphism between the commuting algebra $C(\sigma)$ of the representation $\mathrm{Ind}_P^G(\sigma)$ and group algebra $\mathbb{C}[R]$. Because R is abelian this means that $\mathrm{Ind}_P^G(\sigma)$ is multiplicity one representation. Recall the operators T_w which realize the isomorphisms between the representations $w(\sigma)$ and σ. If V is the representation space of the representation σ, then $T_w : V \to V$ and $T_w w(\sigma)(m) = \sigma(m)T_w$ holds.

PROPOSITION 5.1. *We can choose operators T_w in such a way that $T_{w_1w_2} = T_{w_1}T_{w_2}$ for all $w_1, w_2 \in R$.*

PROOF. Recall that the representation $\sigma = \sigma_1 \otimes \cdots \otimes \rho$ is acting on the space $V = V_1 \otimes \cdots \otimes V_{r+1} \otimes W$. The representation σ_i is representation of group $M_i \cong GL(m_i, D)^{n_i}$ on the representation space $V_{i_1} \otimes \cdots \otimes V_{in_i}$. We know from (3.7) and (3.8) that $R = R_1 \ \times \cdots \ \times R_{r+1}$. R is generated by the certain number of sign changes, say $c_{j_1}, c_{j_2}, \dots, c_{j_d}$, and each c_{j_k} acts on the corresponding GL- block of the Levi subgroup by sending $g \mapsto g^{-*}$. So, let operator $T_{c_{j_k}}$ act on the space V_{j_k}. The operator $T_{c_{j_k}}$ establishes isomorphism between σ_{i_j} and $\sigma_{i_j}^*$ which acts on the same space. Because of the fact that $(\sigma_{ij}^*)^* = \sigma_{ij}$, follows that $T_{i_j}^2$ is a scalar complex operator which we can normalize in such a way that $T_{i_j}^2 = 1$. Now we extend it trivially (as identity) on the whole space V. It is easy to see that now these operators commute, because they act nontrivially on the different components of V. We conclude that the mapping $c_{i_j} \mapsto T_{c_{i_j}}$ defines desired homomorphism. $\square$

We can state our main theorem:

THEOREM 5.2. *Let $M \cong GL(n_1, D) \times GL(n_2, D) \times \cdots \times GL(n_k, D) \times G_m(D)$ be Levi subgroup of the standard parabolic subgroup P of $G_n(D)$ with $m \geq 0$. Let $\sigma \cong \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k \otimes \rho$ be discrete series representation of M. Then*

$$\mathrm{Ind}_P^{G_n(D)}(\sigma) = \bigoplus_{i=1}^{2^d} \pi_i.$$

Here $\{\pi_i,\ i = 1, \dots 2^d\}$ form the set of mutually inequivalent irreducible tempered representations of $G_n(D)$, and d is number of mutually inequivalent σ_i such that $\mathrm{Ind}_{GL(n_i,D)\times G_m(D)}^{G_{n_i+m}(D)}(\sigma_i \otimes \rho)$ reduces.

REMARK 5.3. The question of the group $G(F, \varepsilon)$ in the case $\varepsilon = 1$ resolves in the same way. We can take the same maximal F-split torus as in the case $\varepsilon = -1$, and we obtain the same type of the relative root system–that is, type C_n. Then, Weyl group of the system is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$ and for the representatives of the Weyl group we can take elements of $O(2n, F) \subset G(F, 1)$ embedded in $G(\overline{F})$. The Levi subgroups have virtually the same form, so the rest of the calculation and the results are the same as in the case $\varepsilon = -1$.

REMARK 5.4. The question of the reducibility of the induced representation now reduces to examining reducibility when inducing from the maximal parabolic. In the Siegel case, we can calculate Plancherel measure $\mu(\sigma, s)$ where σ is discrete series representation of maximal Levi subgroup M isomorphic to $GL(n, D)$. Here s corresponds to the character $|det|_F^s$. We can do that combining equality of Plancherel measures when σ is cuspidal representation with corresponding measure on the split form, see ([11]), with the fact that σ discrete series is subrepresentation of $\nu_\rho^{\frac{k-1}{2}} \rho \times \cdots \times \nu_\rho^{\frac{-(k-1)}{2}} \rho$ for some $k \in \mathbb{N}$ and ρ cuspidal representation. For notation and reference see [12].

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