A FAN $X$ ADMITS A WHITNEY MAP FOR $C(X)$ IFF IT IS METRIZABLE

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Abstract. Let $X$ be a non-metric continuum, and $C(X)$ be the hyperspace of subcontinua of $X$. It is known that there is no Whitney map on the hyperspace $2^X$ for non-metrizable Hausdorff compact spaces $X$. On the other hand, there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$. In this paper we will show that a generalized fan $X$ admits a Whitney map for $C(X)$ if and only if it is metrizable.

1. Introduction

Introduction contains some basic definitions, results and notations. An external characterization of non-metric continua which admit a Whitney map is given in Section 2 (Theorem 2.3). In Section 3 we study hereditarily irreducible mappings onto a fan. The main theorem of this paper is Theorem 4.20.

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space $X$ is denoted by $w(X)$. The cardinality of a set $A$ is denoted by $\text{card}(A)$. We shall use the notion of inverse system as in [3, pp. 135-142]. An inverse system is denoted by $X = (X_a, p_{ab}, A)$.

A generalized arc is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

For a compact space $X$ we denote by $2^X$ the hyperspace of all nonempty closed subsets of $X$ equipped with the Vietoris topology. $C(X)$ and $X(n)$, where $n$ is a positive integer, stand for the sets of all connected members of $2^X$ and of all nonempty subsets consisting of at most $n$ points, respectively, both considered as subspaces of $2^X$, see [6].

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For a mapping \( f : X \to Y \) define \( 2f : 2^X \to 2^Y \) by \( 2f(F) = f(F) \) for \( F \in 2^X \). By \([14, 5, 10]\) \( 2f \) is continuous, \( 2f(C(X)) \subset C(Y) \) and \( 2f(X(n)) \subset Y(n) \). The restriction \( 2f|C(X) \) is denoted by \( C(f) \).

An element \( \{x_a\} \) of the Cartesian product \( \prod \{X_a : a \in A\} \) is called a thread of \( X \) if \( p_{ab}(x_b) = x_a \) for any \( a, b \in A \) satisfying \( a \leq b \). The subspace of \( \prod \{X_a : a \in A\} \) consisting of all threads of \( X \) is called the limit of the inverse system \( X = \{X_a, p_{ab}, A\} \) and is denoted by \( \lim X \) or by \( \lim\{X_a, p_{ab}, A\} \) \([3, p. 135]\).

Let \( X = \{X_a, p_{ab}, A\} \) be an inverse system of compact spaces with the natural projections \( p_a : \lim X \to X_a \), for \( a \in A \). Then \( 2^X = \{2^{X_a}, 2^{p_{ab}}, A\}, C(X) = \{C(X_a), C(p_{ab}), A\} \) and \( X(n) = \{X_a(n), 2^{p_{ab}}|X_b(n), A\} \) form inverse systems. For each \( F \in 2^{\lim X} \), i.e., for each closed \( F \subset \lim X \) the set \( p_a(F) \subset X_a \) is closed and compact. Thus, we have a mapping \( 2p_a : 2^{\lim X} \to 2^{X_a} \) induced by \( p_a \) for each \( a \in A \). Define a mapping \( M : 2^{\lim X} \to 2^X \) by \( M(F) = \{p_a(F) : a \in A\} \). Since \( \{p_a(F) : a \in A\} \) is a thread of the system \( 2^X \), the mapping \( M \) is continuous and one-to-one. It is also onto since for each thread \( \{F_a : a \in A\} \) of the system \( 2^X \) the set \( F' = \bigcap\{p_a^{-1}(F_a) : a \in A\} \) is non-empty and \( p_a(F') = F_a \). Thus, \( M \) is a homeomorphism. If \( P_a : \lim 2^X \to 2^{X_a}, a \in A \), are the projections, then \( P_aM = 2^{p_a} \). Identifying \( F \) with \( M(F) \) we have \( P_a = 2^{p_a} \).

**Lemma 1.1 ([6, Lemma 2.])**. Let \( X = \lim X \). Then \( 2^X = \lim 2^X, C(X) = \lim C(X) \) and \( X(n) = \lim X(n) \).

An arboroid is an hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a dendroid. If \( X \) is an arboroid and \( x, y \in X \), then there exists a unique arc \( [x, y] \) in \( X \) with endpoints \( x \) and \( y \). If \( [x, y] \) is an arc, then \( [x, y] \setminus \{x, y\} \) is denoted by \( (x, y) \).

A point \( t \) of an arboroid \( X \) is said to be a ramification point of \( X \) if \( t \) is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point \( e \) of an arboroid \( X \) is said to be end point of \( X \) if there exists no arc \( [a, b] \) in \( X \) such that \( x \in [a, b] \setminus \{a, b\} \).

If an arboroid \( X \) has only one ramification point \( t \), it is called a generalized fan with the top \( t \). A metrizable generalized fan is called a fan.

We say that an inverse system \( X = \{X_a, p_{ab}, A\} \) is \( \sigma \)-directed if for each sequence \( a_1, a_2, ..., a_k, ... \) of the members of \( A \) there is an \( a \in A \) such that \( a \geq a_k \) for each \( k \in \mathbb{N} \).

In the sequel we shall use the following theorem.

**Theorem 1.2 ([7, Lemma 2.]).** Let \( X = \{X_a, p_{ab}, A\} \) be a \( \sigma \)-directed inverse system of compact spaces with surjective bonding mappings and the limit \( X \). Let \( Y \) be a metric compact space. Then for each surjective mapping \( f : X \to Y \) there exists an \( a \in A \) such that for each \( b \geq a \) there exists a mapping \( g_b : X_b \to Y \) such that \( f = g_b p_a \).
If the bonding mappings are not surjective, then we consider the inverse system \( \{ p_a(X), p_{ab}(X), A \} \) which has surjective bonding mappings. Moreover, \( p_a(X) = \cap \{ p_{ab}(X_b) : b \geq a \} \). Applying Theorem 1.2 we obtain the following theorem.

**Theorem 1.3.** Let \( X = \{ X_a, p_{ab}, A \} \) be a \( \sigma \)-directed inverse system of compact spaces with the limit \( X \). Let \( Y \) be a metric compact space. Then for each surjective mapping \( f : X \to Y \) there exists an \( a \in A \) such that for each \( b \geq a \) there exists a mapping \( g_b : p_b(X) \to Y \) such that \( f = g_b p_b \).

In the sequel we shall use the following results.

**Lemma 1.4** ([3, Corollary 2.5.7]). Any closed subspace \( Y \) of the limit \( X \) of an inverse system \( X = \{ X_a, p_{ab}, A \} \) is the limit of the inverse system \( X_Y = \{ Cl(p_a(Y)), p_{ab}[Cl(p_b(Y))], A \} \).

**Lemma 1.5** ([3, Corollary 2.5.11]). Let \( X = \{ X_a, p_{ab}, A \} \) be an inverse system and \( B \) a subset cofinal in \( A \). The mapping consisting in restricting all threads from \( X = \lim X \) to \( B \) is a homeomorphism of \( X \) onto the space \( \lim \{ X_b, p_{bc}, B \} \).

Now we will prove some expanding theorems of non-metric compact spaces into \( \sigma \)-directed inverse systems of compact metric spaces.

**Theorem 1.6.** If \( X \) is the Cartesian product \( X = \prod \{ X_s : s \in S \} \), where \( \text{card}(S) > \aleph_0 \) and each \( X_s \) is compact, then there exists a \( \sigma \)-directed inverse system \( X = \{ Y_a, P_{ab}, A \} \) of the countable products \( Y_a = \prod \{ X_{\mu} : \mu \in a \} \), \( \text{card}(a) = \aleph_0 \), such that \( X \) is homeomorphic to \( \lim X \).

**Proof.** Let \( A \) be the set of all subsets of \( S \) of the cardinality \( \aleph_0 \) ordered by inclusion. If \( a \subseteq b \), then we write \( a \leq b \). It is clear that \( A \) is \( \sigma \)-directed.

For each \( a \in A \) there exists the product \( Y_a = \prod \{ X_{\mu} : \mu \in a \} \). If \( a, b \in A \) and \( a \leq b \), then there exists the projection \( P_{ab} : Y_b \to Y_a \). Finally, we have the system \( X = \{ Y_a, P_{ab}, A \} \). Let us prove that \( X \) is homeomorphic to \( \lim X \).

Let \( x \in X \). It is clear that \( P_a(x) = x_a \) is a point of \( Y_a \) and that \( P_{ab}(x_b) = x_a \) if \( a \leq b \). This means that \( (x_a) \) is a thread in \( X = \{ Y_a, P_{ab}, A \} \). Set \( H(x) = (x_a) \). We have the mapping \( H : X \to \lim X \). It is clear that \( H \) is continuous, 1-1 and onto. Hence, \( H \) is a homeomorphism.

**Corollary 1.7.** For each Tychonoff cube \( F^n \), \( m \geq \aleph_1 \), there exists a \( \sigma \)-directed inverse system \( I = \{ I^a, P_{ab}, A \} \) of the Hilbert cubes \( F^a \) such that \( F^n \) is homeomorphic to \( \lim I \).

**Theorem 1.8.** Let \( X \) be a compact Hausdorff space such that \( w(X) \geq \aleph_1 \). There exists a \( \sigma \)-directed inverse system \( X = \{ X_a, p_{ab}, A \} \) of metric compacta \( X_a \) such that \( X \) is homeomorphic to \( \lim X \).
Proof. By [3, Theorem 2.3.23.] the space $X$ is embeddable in $I^w(X)$.

From Corollary 1.7 it follows that $I^w(X)$ is a limit of $I = \{I^a, P_{ab}, A\}$, where every $I^a$ is the Hilbert cube. Now, $X$ is a closed subspace of $\lim I$. Let $X_a = P_{ma}(X)$, where $P_{ma}: I^m \to I^a$ is a projection of the Tychonoff cube $I^m$ onto the Hilbert cube $I^a$. Let $p_{ab}$ be the restriction of $P_{ab}$ on $X_b$. We have the inverse system $X = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$. By virtue of Lemma 1.4 $X$ is homeomorphic to $\lim X$. Moreover, $X$ is a $\sigma$-directed inverse system since $I = \{I^a, P_{ab}, A\}$ is a $\sigma$-directed inverse system. \qed

2. Whitney map and hereditarily irreducible mappings

The notion of an irreducible mapping was introduced by Whyburn [21, p. 162]. If $X$ is a continuum, a surjection $f : X \to Y$ is irreducible provided no proper subcontinuum of $X$ maps onto all of $Y$ under $f$. Some theorems for the case when $X$ is semi-locally-connected are given in [21, p. 163].

A mapping $f : X \to Y$ is said to be hereditarily irreducible [15, p. 204, (1.212.3)] provided that for any given subcontinuum $Z$ of $X$, no proper subcontinuum of $Z$ maps onto $f(Z)$.

A mapping $f : X \to Y$ is light (zero-dimensional) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [3, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

**Lemma 2.1.** Every hereditarily irreducible mapping is light.

**Lemma 2.2.** If $f : X \to Y$ is monotone and hereditarily irreducible, then $f$ is 1-1.

Let $\Lambda$ be a subspace of $2^X$. By a Whitney map for $\Lambda$ [15, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If $X$ is a metric continuum, then there exists a Whitney map for $2^X$ and $C(X)$ ([15, pp. 24-26], [5, p. 106]). On the other hand, if $X$ is non-metrizable, then it admits no Whitney map for $2^X$ [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2]. Moreover, if $X$ is a non-metrizable locally connected or a rim-metrizable continuum, then $X$ admits no Whitney map for $C(X)$ [9, Theorem 8, Theorem 11]. In what follows we shall show that a generalized fan $X$ does not admit any Whitney map for $C(X)$.

The first step in proving this statement is an external characterization of non-metric continua which admit a Whitney map.

**Theorem 2.3.** Let $X$ be a non-metric continuum. Then $X$ admits a Whitney map for $C(X)$ if and only if for each $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$. By virtue of Lemma 1.4 $X$ is homeomorphic to $\lim X$. Moreover, $X$ is a $\sigma$-directed inverse system since $I = \{I^a, P_{ab}, A\}$ is a $\sigma$-directed inverse system. \qed
\[ \{X_a, p_{ab}, A\} \text{ of continua which admit Whitney maps for } C(X_a) \text{ and } X = \lim X \text{ there exists a cofinal subset } B \subset A \text{ such that for every } b \in B \text{ the projection } p_b : \lim X \to X_b \text{ is hereditarily irreducible.} \]

**Proof.** Necessity. Consider inverse system \( C(X) = \{C(X_a), C(p_{ab}), A\} \) whose limit is \( C(X) \) (Lemma 1.1). If \( \mu : C(X) \to [0, \infty) \) is a Whitney map for \( C(X) \), then, by Theorem 1.3, there exists a cofinal subset \( B \subset A \) such that for every \( b \in B \) there is a mapping \( \mu_b : C(p_b)(X) \to [0, \infty) \) with \( \mu = \mu_bC(p_b) \). Suppose that \( p_b \) is not hereditarily irreducible. Then there exists a pair \( F, G \) of subcontinua of \( X \) with \( F \subseteq G, F \neq G \), (i.e., \( F \) is a proper subcontinuum of \( G \)) such that \( p_b(F) = p_b(G) \). It is clear that \( C(p_b)((F)) = C(p_b)((G)) \). This means that \( \mu(C(p_b)((F))) = \mu(C(p_b)((G))) \). From \( \mu = \mu_bC(p_b) \) it follows that \( \mu((F)) = \mu((G)) \). This is impossible since \( \mu \) is a Whitney map for \( C(X) \) and from \( F \subseteq G, F \neq G \) it follows \( \mu((F)) < \mu((G)) \).

**Sufficiency.** Suppose that there exists a cofinal subset \( B \subset A \) such that for every \( b \in B \) the projection \( p_b : \lim X \to X_b \) is hereditarily irreducible. Consider inverse system \( C(X) = \{C(X_a), C(p_{ab}), A\} \) whose limit is \( C(X) \) (Lemma 1.1). Let \( \mu_b : C(X_b) \to [0, \infty) \) be a Whitney map for \( C(X_b) \), where \( b \in B \) is fixed. We shall prove that \( \mu = \mu_bC(p_b) : C(X) \to [0, \infty) \) is a Whitney map for \( C(X) \). Let \( F, G \) be a pair of subcontinua of \( X \) with \( F \subseteq G, F \neq G \). We must prove that \( \mu((F)) < \mu((G)) \). Now, \( p_b(F) \subset p_b(G) \) and \( p_b(F) \neq p_b(G) \) since \( p_b \) is hereditarily irreducible. We infer that \( \mu_b(p_b(F)) < \mu_b(p_b(G)) \) since \( \mu_b \) is a Whitney map for \( C(X_b) \). Moreover, \( \{p_b(F)\} = C(p_b)((F)) \) and \( \{p_b(G)\} = C(p_b)((G)) \). From \( \mu_b(p_b(F)) < \mu_b(p_b(G)) \) we have \( \mu_b(C(p_b)((F))) < \mu_b(C(p_b)((G))) \), i.e., \( \mu(C(p_b)((F))) < \mu(C(p_b)((G))) \). Finally, \( \mu((F)) < \mu((G)) \) since \( \mu = \mu_bC(p_b) \).

**Remark 2.4.** It follows from Theorem 2.3 and Lemma 2.1 that the projections \( p_b \) are light for every \( b \in B \). It is a question are the bonding mappings \( p_{ab} \) light mappings. The following theorem shows that it is possible to find such inverse system which has the light bonding mappings.

**Theorem 2.5.** If \( X \) is a non-metric continuum which admits a Whitney map for \( C(X) \), then there exists a \( \sigma \)-directed inverse system \( X = \{X_a, p_{ab}, A\} \) of metric continua \( X_a \) such that the bonding mappings \( p_{ab} \) are light and \( X = \lim X \).

**Proof.** By virtue of Theorem 1.8 there exists a \( \sigma \)-directed inverse system \( Y = \{Y_a, q_{ab}, B\} \) of metric compact spaces \( Y_a \) such that \( X = \lim Y \). From Remark 2.4 it follows that there exists a metric space \( Y_b \) such that the projection \( q_b : X \to Y_b \) is light. Using [18, p. 204, Theorem 7.10] we obtain an inverse system \( X = \{X_a, p_{ab}, A\} \) of metric compact spaces and zero-dimensional bonding mappings such that \( X = \lim X \). Since every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide [3, p. 450], we infer that \( p_{ab} \) are
light. Applying Theorem 2.3 we conclude that there exists a $B \subset A$ which is cofinal in $A$ and such that the projections $p_b$ are light for every $b \in B$. □

We close this section with the following theorem.

**Theorem 2.6.** If $X$ is the Cartesian product $X = \prod \{X_s : s \in S\}$, where $\text{card}(S) > \aleph_0$ and each $X_s$ is a continuum, then there is no Whitney map for $C(X)$.

**Proof.** By virtue of Theorem 1.6 it follows that for the Cartesian product $X = \prod \{X_s : s \in S\}$, $\text{card}(S) > \aleph_0$, there exists a $\sigma$-directed inverse system $X = \{Y_a, P_{ab}, A\}$ of the products $Y_a = \prod \{X_\mu : \mu \in a\}$, $\text{card}(a) = \aleph_0$, such that $X$ is homeomorphic to $\lim X$. If every $X_s : s \in S$, is a continuum, then every bonding mapping $P_{ab}$ in $X = \{Y_a, P_{ab}, A\}$ is monotone since $P_{ab}^{-1}(x)$ is the product of all $X_s$ which are factors in $Y_b$, but not factors in $Y_a$.

The statement of Theorem follows from Theorem 2.3. □

3. HEREDITARILY IRREDUCIBLE MAPPINGS ONTO ARBOROIDS

Theorem 2.3 suggests the study of hereditarily irreducible mappings. In this section we will consider hereditarily irreducible mappings onto arboroids.

A continuum $X$ is said to be *arcwise connected* provided for every two points $x, y \in X, x \neq y$, there is a generalized or a metrizable arc $[x, y] \subset X$.

**Lemma 3.1.** If $X$ is an arboroid and if $Y$ is an arboroid which contains finitely many ramification points, then every hereditarily irreducible mapping $f : X \to Y$ is a homeomorphism.

**Proof.** Suppose that $f$ is not a homeomorphism. Then there exists a point $y \in Y$ such that $f^{-1}(y)$ is not a single point. This means that there exist points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$. Since $X$ is an arboroid there exists the unique generalized arc $Z$ in $X$ such that $x_1, x_2$ are end points of $Z$.

**Claim 1.** There exists a segment $[a, b]$ of $Z$ such that $f^{-1}(y) \cap (a, b) = \emptyset$ and $f^{-1}(y) \cap [a, b] = \{a, b\}$.

It is clear that $f^{-1}(y)$ is not dense in $Z$. In the opposite case we have that $Z$ is a proper subcontinuum of $f^{-1}(y)$. This is impossible since $f^{-1}(y)$ contains no continuum. It follows that there exists a segment $[c, d] \subset Z$ such that $f^{-1}(y) \cap Z \subset [c, d]$ and $[c, d] \subset f^{-1}(y) \cap Z$. It is again clear that there exists a subinterval $(a_1, b_1)$ of $[c, d]$ such that $f^{-1}(y) \cap (a_1, b_1) = \emptyset$. Let $A$ be a family of all segments $(a_\alpha, b_\alpha)$ which contains $(a_1, b_1)$ and $f^{-1}(y) \cap (a_\alpha, b_\alpha) = \emptyset$. It is clear that the union of all elements of $A$ is a subsegment $(a, b)$ of $[c, d]$. Let us prove that $a, b \in f^{-1}(y)$. Suppose that $a \notin f^{-1}(y)$. Then $f(a) \neq y$. There exists an open set $U$ containing $a$ such that $f(U)$ does not contain the point $y$. It is clear that there exists a segment $(e, h)$ contained in
U. Then \((a, b) \cup (e, h)\) is a segment which contains \((a_1, b_1)\). It is clear that
\((a, b) \cup (e, h)\) is not in \(A\), a contradiction. Hence, \(a \in f^{-1}(y)\). Similarly, one
can prove that \(b \in f^{-1}(y)\).

In the remaining part of the proof we shall consider the restriction \(g = f|_{[a, b]}\). Let us recall that \(g\) is hereditarily irreducible and that \(W = f([a, b])\), as
a subcontinuum of \(Y\), is an arboroid. Thus we have a hereditarily irreducible
surjection \(g\) of the arc \([a, b]\) onto a dendroid \(W\) such that \(g^{-1}(y) = \{a, b\}\).

**Claim 2.** There exist subarcs \([a, x]\) and \([z, b]\) such that \(g([a, x]) \subset g([z, b])\)
or \(g([a, x]) \supseteq g([z, b])\).

Let \(U_y\) be a neighborhood of \(y\) such that \(U_y \setminus \{y\}\) does not contain rami-
fication points. There exist segments \([a, x]\) and \([z, b]\) such that \(g([a, x]) \subset U_y\)
and \(g([z, b]) \subset U_y\). It follows that \(g([a, x])\) and \(g([z, b])\) are arcs since \(g((a, x])\)
and \(g([z, b])\) do not contain ramification points. Suppose that
\(g([a, x]) \cap g([z, b]) = \{y\}\). Then \(C = g([a, x]) \cup g([z, b])\) is a continuum. Be-
cause of Claim 1, \(g([x, z])\) is a continuum not containing the point \(y\). It follows that
\(C \cap g([x, z])\) is not a continuum since \(C \cap g([x, z])\) contains \(\{y\}\) and two
disjoint subsets \(g([a, x]) \cap g([x, z]) \supseteq \{g(x)\}\) and \(g([x, z]) \cap g([z, b]) \supseteq \{g(z)\}\)
ot containing \(y\). This is impossible since \(W\) is hereditarily unicoherent.
Hence, \(D = g([a, x]) \cap g([z, b])\) is a non-degenerate continuum containing the
point \(y\). It is clear that \(D\) does not contain ramification points. It follows that
\(g([a, x]) \subset g([z, b])\) or \(g([a, x]) \supseteq g([z, b])\) since in the opposite case we
obtain a triod in \(U_y\).

**Claim 3.** We may assume that \(g([a, x]) \supseteq g([z, b])\).

Now, \(g([a, z]) = g([a, b])\) since \(g([a, x]) \supseteq g([z, b])\). This is impossible since
\(g\) is hereditarily irreducible.

**Corollary 3.2.** If \(X\) is an arboroid and if \(Y\) is a generalized fan, then
every hereditarily irreducible mapping \(f : X \rightarrow Y\) is a homeomorphism.

Now we are ready to prove the following theorem.

**Theorem 3.3.** Let \(X = \{X_\alpha, p_\beta, A\}\) be a \(\sigma\)-directed inverse system of
fans. If \(X = \lim X\) is arcwise connected, then \(X\) admits a Whitney map for \(C(X)\) if and only if \(X\) is metrizable.

**Proof.** If \(X\) is metrizable, then it admits a Whitney map for \(C(X)\) [15, pp. 24-26]. Suppose now that \(X\) admits a Whitney map for \(C(X)\). From
Theorem 2.3 it follows that there exists a cofinal subset \(B\) of \(A\) such that for
every \(b \in B\) the projection \(p_b\) is hereditarily irreducible. By Corollary 3.2 we
infer that \(p_b\) is a homeomorphism. Hence, \(X\) is metrizable since each \(X_b\) is
metrizable.
4. AM-fans

We say that an arboroid $X$ is an $AM$-arboroid if each arc in $X$ is metrizable. Now we shall prove that every arboroid is a limit of a $\sigma$-directed inverse systems of $AM$-arboroids.

**Theorem 4.1.** Let $X$ be an arboroid. There exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is an $AM$-arboroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Proof.** If $X$ is an $AM$-dendroid, then it has metrizable arcs and Theorem is obvious. If $X$ is not an $AM$-dendroid, then there exists an inverse $\sigma$-system $Y = \{Y_a, q_{ab}, A\}$ of metric continua $X_a$ such that $X$ is homeomorphic to $\lim Y$ (Theorem 1.8). It is clear that the projections $q_a$ are not light since then the restrictions $q_a|L$ are light for every arc $L$ in $X$. Then from [10, Theorem 1] it follows that $L$ is metrizable. Hence, $q_a$ is not light. Let $q_a$ be the natural projection of $X$ onto $Y_a$. Applying the monotone-light factorization [3, pp. 450-451] to $q_a$, we get compact spaces $X_a$, monotone surjections $m_a : X \to X_a$ and light surjections $l_a : X_a \to Y_a$ such that $q_a = l_a m_a$. By [10, Lemma 8] there exist monotone surjections $p_{ab} : X_b \to X_a$ such that $p_{ab} m_a = m_a, a \leq b$. It follows that $X = \{X_a, p_{ab}, A\}$ is an inverse system such that $X$ is homeomorphic to $\lim X$. Let us prove that $X_a$ is an $AM$-arboroid. The space $X_a$ is hereditarily unicoherent since $m_a$ is monotone. Moreover, $X_a$ is arcwise connected. Namely, if $x_a, y_a$ are distinct points of $X_a$, then there exists a pair $x, y$ of points of $X$ such that $x_a = m_a(x)$ and $y_a = m_a(y)$. Let $L$ be the arc with end points $x$ and $y$. Now, $m_a(L)$ is a continuous image of an arc and, consequently, arcwise connected [19]. Hence, $X_a$ is an arboroid. Since every map $l_a$ is light, we infer that each arc in $X_a$ is metrizable (by [20, Theorem 1.2, p. 464] saying that if $X$ is rim-metrizable and a surjective mapping $l : X \to Y$ is light, then $w(X) = w(Y)$; compare also [10, Theorem 1]). Hence, every $X_a$ is an $AM$-arboroid.

**Corollary 4.2.** Let $X$ be a generalized fan. There exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is an $AM$-fan, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Proof.** By Theorem 4.1 there exists an inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is an $AM$-arboroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$. Let us observe that the projections $p_a : X \to X_a$ are monotone [3, 6.3.16.(a), pp. 462-463]. It remains to prove that each $X_a$ is an $AM$-fan. Suppose that some $X_a$ is not $AM$-fan. This means that $X_a$ has two different ramification points. It follows that $X_a$ contains two different triods $T_1$ and $T_2$. Hence, there is a triod, say $T_2$, such that $p_a^{-1}(T_2)$ is a subset of some arc $L$ in $X$ since $X$ is a generalized fan. It is clear that this impossible since $p_a^{-1}(T_2)$ is a continuum.
THEOREM 4.3. If a generalized fan \( X \) admits a Whitney map for \( C(X) \), then \( X \) is an AM-fan.

Proof. It follows from Corollary 4.2 that there exists an inverse system \( \mathbf{X} = \{X_a, p_{ab}, A\} \) such that each \( X_a \) is an AM-fan, every \( p_{ab} \) is monotone and \( X \) is homeomorphic to \( \lim \mathbf{X} \). If \( X \) admits a Whitney map for \( C(X) \), then there exists a cofinal subset \( B \) of \( A \) such that \( p_b \) is hereditarily irreducible for every \( b \in B \) (Theorem 2.3). From Lemma 2.2 we infer that \( p_b \) is 1-1. Hence, \( p_b \) is a homeomorphism. This means that \( X \) is an AM-fan. \( \square \)

Now we shall expand every non-metric AM-fan into inverse system of a metric finite fan. This is done in Theorem 4.19. The proof of this Theorem requires some preliminary definitions and results which are straightforward modifications of [4].

A chain, in a topological space, is a collection \( \mathcal{E} = \{E_1, \ldots, E_m\} \) of open sets \( E_i \) such that \( E_i \cap E_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). The elements of \( \mathcal{E} \) are links. Let \( \mathcal{U} \) be an open cover of a space \( X \). We say that a chain \( \mathcal{E} = \{E_1, \ldots, E_m\} \) is a \( \mathcal{U} \)-chain if each link \( E_i \) of \( \mathcal{E} \) is contained in some member \( U \) of \( \mathcal{U} \).

Let \( \mathcal{E} = \{E_1, \ldots, E_m\} \) be a chain; frequently we denote \( \mathcal{E} \) by \( E(1, m) \) and denote \( \bigcup\{E_i : 1 \leq i \leq m\} \) by \( E^*(1, m) \) or by \( \mathcal{E}^* \).

Definition 4.4. If \( [a, b] \) is an arc and \( \mathcal{E} = E(1, m) \) is a chain covering \( [a, b] \) then \( [a, b] \) is straight in \( \mathcal{E} \) provided:

1. \( \mathcal{E} \) is a chain from \( a \) to \( b \) i.e. \( a \in E_1 \setminus \text{Cl} E_2, b \in E_m \setminus \text{Cl} E_{m-1} \),
2. \( (\partial E_i \cap [a, b]) \) is a one point set if \( i = 1 \) or \( i = m \) and a two point set otherwise.

Lemma 4.5. Suppose that \( X \) is an AM-arboroid, \( Y \) is a finite tree, \( Y \subset X \) and \( p \in Y \). Let \( \mathcal{K} = \{K : K \text{ is a component of } Y \setminus \{p\}\} \). Then for each open set \( U \) such that \( p \in U \) there exists an open set \( V \) such that \( p \in V \subset U \) and \( \text{card}(Y \cap \partial V) = \text{card}(\mathcal{K}) \).

Proof. The proof is the same as the proof of Lemma 1 of [4] since \( X \) has metrizable arcs and \( Y \) is metrizable. Namely, \( \mathcal{K} \) is a finite set, since each component of \( Y \setminus \{p\} \) contains an end point of \( Y \). This follows from the fact that if \( K \in \mathcal{K} \), then \( K \) is arcwise connected, because \( Y \) is locally connected. The end points of \( Y \) are precisely the end points of maximal arcs in \( Y \). Since \( K \cup \{p\} \) is a tree and \( K \) is arcwise connected, then if \( A \) is a maximal arc in \( K \cup \{p\} \), at least one end point of \( A \) is an end point of \( Y \). Suppose \( \mathcal{K} = \{K_1, \ldots, K_n\} \). According to [21, p. 88] there is a set \( V' \), open in \( Y \) such that \( p \in V' \subset U \), and \( \partial_Y V' \), the boundary of \( V' \) relative to \( Y \), contains exactly \( n \) points. Now \( V' \) must be connected, since if \( V'' \) is the component of \( V' \) containing \( p \), then \( V'' \) is open in \( Y \) and \( \partial_Y V'' \subset \partial_Y V' \). Since we may assume that for each \( i, K_i \not\subset \text{Cl} U \), \( \partial_Y V'' \) contains a point from each \( K_i \).
Since $\partial Y V'$ contains only $n$ points, $V' = V''$. Thus $Y \setminus \partial Y V'$ is the union of two separated sets, one of which is $V'$ and the other contains $Y \setminus U$. There are disjoint sets $S$ and $T$, open in $X$, such that $V' \subset S$ and $Y \setminus U \subset T$. Now let $V = U \setminus \text{Cl} T$. Then $(\partial V) \cap Y = (\partial T) \cap Y = \partial Y V'$, an $n$-point set.

**Lemma 4.6.** Suppose $[a, b]$ is straight in $E = E(1, m)$ and $W$ is an open set containing $[a, b]$. Then $[a, b]$ is straight in $\{E_1 \cap W, E_2 \cap W, ..., E_m \cap W\}$.

**Proof.** This is Lemma 2 of [4]. It is clear from the definition of straightness that for each $i$, $\partial(E_i \cap W)$ contains at least as many points of $[a, b]$ as $\partial E_i$ does. Conversely, since $\partial(E_i \cap W) \subset (\partial E_i) \cap (\partial W)$ and $[a, b] \subset W$, $(\partial(E_i \cap W)) \cap [a, b] \subset (\partial E_i) \cap [a, b]$. Thus $\partial(E_i \cap W)$ contains exactly as many points of $[a, b]$ as $\partial E_i$ does. That is, $[a, b]$ is straight in $\{E_1 \cap W, E_2 \cap W, ..., E_m \cap W\}$.

We now show that each arc in $\text{AM}$-arboroid can be covered by chains in which that arc is straight.

**Lemma 4.7.** If $[a, b]$ is an arc in an $\text{AM}$-dendroid $X$ and $U$ an open covering of $X$, then there an chain $F = E(1, m)$ of sets open in $X$ such that $E = E(1, m)$ refines $U$ and $[a, b]$ is straight in $E$.

**Proof.** The proof is a straightforward modification of the proof of [4, Proposition 1]. Suppose, to the contrary, that there is an arc $[a, b]$ in $X$ such that $[a, b]$ is not straight in any chain which refines $U$. For fixed $U$ and fixed arc $[a, b]$, we say that a subarc $[a', b']$ of $[a, b]$ has property $P$ iff $[a', b']$ is not straight in any chain which refines $U$. Clearly $[a, b]$ has property $P$. We now show that property $P$ is inductive. Let $L = \{L_\alpha : \alpha < \omega_\tau\}$ be a transfinite sequence such that, for each ordinal $\alpha < \omega_\tau$, $L_\alpha$ has property $P$ and $L_\beta \subset L_\alpha$ if $\alpha < \beta < \omega_\tau$. We must show that $L = \cap\{L_\alpha : \alpha < \omega_\tau\}$ has property $P$. If it does not then, then $L$ is not degenerate, hence it is a subarc $[c, d]$ of $[a, b]$. Since $[a, b]$ has property $P$, $[c, d] \neq [a, b]$. Without loss of generality, we may assume that $a < c < d < b$, $< \!\!\!\!\!\!\!\!$ denoting the usual order from $a$ to $b$ on $[a, b]$. Since $[c, d]$ does not have property $P$, there is a chain $F = F(1, n)$ of open sets in $X$ such that $[c, d]$ is straight in $F$ and $F$ refines $U$. Let $U$ be an open set such that $c \in U$ and $\text{Cl} U \subset F_1 \setminus \text{Cl} F_2$. According to Lemma 4.5, there is an open set $V$ such that $c \in V \subset U$ and $(\partial V) \cap [c, d]$ is degenerate. Similarly, there is an open set $R$ such that $d \in R \subset \text{Cl} R \subset F_n \setminus \text{Cl} F_{n-1}$ and $(\partial R) \cap [c, d]$ is degenerate. Now $(V \cup [c, d] \cup R) \cap [a, b]$ is open in $[a, b]$ and contains $L$. Hence there is an $\alpha < \omega_\tau$ such that $L_\alpha \subset (V \cup [c, d] \cup R) \cap [a, b]$. If $L_\alpha = [a_\alpha, b_\alpha]$, then we may assume that $a_\alpha \in V$ and $b_\alpha \in R$, since $L_\alpha \setminus [c, d] \subset V \cup R$. Since $V \subset F_1 \setminus \text{Cl} F_2$ and $R \subset F_n \setminus \text{Cl} F_{n-1}$, $F$ is a chain from $a_\alpha$ to $a_\beta$ covering $[a_\alpha, b_\alpha]$. Since, for each $\alpha$, $(\partial F_\alpha) \cap (V \cup R) = \emptyset$, $\partial F_\alpha \cap [a_\alpha, b_\alpha] = \partial F_\alpha \cap [c, d]$, which is degenerate if $F_\alpha$ is an end link of $F$ and a two point set otherwise. Thus $L_\alpha = [a_\alpha, b_\alpha]$ is straight in $F$. This is impossible, for $L_\alpha$ was assumed
to have property $\mathcal{P}$. It follows that $[c, d]$ must have property $\mathcal{P}$, hence that property $\mathcal{P}$ is inductive.

Since $[a, b]$ has property $\mathcal{P}$, there is a subcontinuum of $[a, b]$ which is irreducible with respect to having property $\mathcal{P}$. This subcontinuum must be non-degenerate; we shall simply assume that $[a, b]$ is irreducible with respect to having property $\mathcal{P}$. Let $x$ be a non-end point of $[a, b]$. Since $[a, x]$ and $[x, b]$ are proper subarcs of $[a, b]$, neither has property $\mathcal{P}$. Hence there are $U$-chains $\mathcal{G} = G(1, j)$ and $\mathcal{H} = H(1, k)$ of open sets in $X$ such $[a, x]$ is straight in $\mathcal{G}$ and $[x, b]$ is straight in $\mathcal{H}$.

Using regularity and Lemma 4.5, we obtain an open set $Q$ such that $x \in Q \subset \text{Cl } Q \subset (G_j \setminus \text{Cl } G_{j-1}) \cap (H_i \setminus \text{Cl } H_{i+1})$ and $(\partial Q) \cap [a, b]$ contains exactly two points, one in $[a, x]$, the other in $[x, b]$. Clearly, $[a, x] \setminus Q$ and $[x, b] \setminus Q$ are disjoint closed sets. It follows that $X \setminus Q$ is the union of two disjoint closed sets $A$ and $B$, with $[a, x] \setminus Q \subset A$ and $[x, b] \setminus Q \subset B$. From the normality of $X$ we infer that there exist open sets $S$ and $T$ such that $A \subset S$, $B \subset T$ and $\text{Cl } S \cap T = \emptyset$. We now define chains $\mathcal{G}' = G'(1, j)$ and $\mathcal{H}' = H'(1, k)$, one-to-one refinements of $\mathcal{G}$ and $\mathcal{H}$, respectively, by $G'_i = G_i \cap (S \cup Q)$, $H'_i = H_i \cap (T \cup Q)$. Lemma 4.6 shows that $[a, x]$ is straight in $\mathcal{G}'$ and $[x, b]$ is straight in $\mathcal{H}'$. Since the only points in a link of $\mathcal{G}'$ and a link of $\mathcal{H}'$ are those in $Q$, we may define a chain $\mathcal{E} = E(1, m)$ by $E_i = G'_i$, if $1 \leq i \leq j$; $G_i = H'_{i-j}$, if $j + 1 \leq i \leq j + k$. One can prove (see [4, p. 116]) that $[a, b]$ is straight in $\mathcal{E}$.

Lemma 4.7 shows that one can cover each arc from the top of an $AM$-fan to an end point by a chain in which the arc is straight and a finite collection of these chains cover the $AM$-fan. However, different chains may intersect very badly. In order to cut them apart, we will need some control over boundaries of the links. Hence we establish

**Lemma 4.8.** Suppose $X$ is an $AM$-fan, $t$ is the top of $X$ and $W$ is the set of end points of $X$. For each cover $\mathcal{U}$ of $X$ and each $w \in W$ there is a $U$-chain $\mathcal{E} = E(1, m)$ of sets open in $X$ such that $[t, w]$ is straight in $\mathcal{E} = E(1, m)$ and $\partial E^n(2, m) \subset E_1$.

Given an $AM$-fan $X$ and an open cover $\mathcal{U}$ of $X$ we want to cover $X$ with a $U$-tree chain whose nerve is a triangulation of a finite fan as does Figure 3 in [4]. The following Lemma shows that we can do this for a finite subfan $Y$ of $X$.

**Lemma 4.9 ([4, Proposition 3]).** Suppose $X$ is an $AM$-fan, $Y$ is a finite subfan of $X$, the top of $X$, $t$, is the top of $Y$ and each end point $w$ of $Y$, $w \neq t$, is an end point of $X$. If $Y = \bigcup\{[t, w_i] : i \in \{1, 2, \ldots, n\}\}$ and $\mathcal{U}$ is a cover of $X$, then there exists a finite collection $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ such that:

(i) each $\mathcal{F}_j = F_j(1, r_j) = \{F_{j1}, F_{j2}, \ldots, F_{jr_j}\}$ is a $U$-chain consisting of at least 3 links,
Since \( \partial F^\ast (2, r_j) \subset F_{j1} \),

(iv) for each \( j \), \( F_{j1} = F_{1j} \),

(v) if \( i \neq j \) then \((\{t, w_j\} \cup F^\ast (2, r_j)) \cap C_F^\ast (2, r_i) = \emptyset \).

Let \( \mathcal{U} \) be a cover of a space \( X \). We shall write \((x, y) < \mathcal{U}\) if there is an element \( U \in \mathcal{U} \) such that \( x, y \in U \).

Once we have covered the AM-fan \( X \) as in Figure 3 of [4], we use the cover to construct the retraction. To do this, we will piece together the retractions of chains onto straight arcs. We therefore prove

**Lemma 4.10.** Let a compact space \( X \) contain an arc \([a, b]\) that is straight in a \( \mathcal{U}\)-chain \( \mathcal{E} = E(1, m) \), \( \mathcal{E}^\ast \subset X \), \( \partial \mathcal{E}^\ast (2, m) \subset E_1 \) and \( p = (\partial E_1) \cap [a, b] \).

Then there is a continuous function \( f : (\mathcal{E}^\ast \setminus E_1) \to [p, b] \) such that \( f \) is a retraction onto \([p, b]\), \( f([\partial E_1] \cap E_2) = p \) and for each \( x \in \mathcal{E}^\ast \setminus E_1 \), \((x, f(x)) < \mathcal{U}\).

**Proof.** This is actually Proposition 4 of [4] whose proof is valid in the case of AM-fans.

Since \( \partial \mathcal{E}^\ast (2, m) \subset E_1 \), \( \mathcal{E}^\ast \setminus E_1 \) is compact and for each \( i \in \{2, ..., m - 1\} \), \( \partial E_i \) is the union of two disjoint closed sets, \((\partial E_i) \cap E_{i-1} \) and \((\partial E_i) \cap E_{i+1} \). Since \([a, b]\) is straight in \( \mathcal{E} \), for each \( i \in \{1, ..., m - 1\} \), \((\partial E_i) \cap E_{i+1} \cap [a, b] \) is a single point, \( r_i \). Then \( p = r_1 \). Let \( b = r_m \). Again, straightness guarantees that \( p = r_1 < r_2 < ... < r_m = b \), where \(<\) denotes the usual order from \( a \) to \( b \) on \([a, b] \).

For each \( i \in \{1, ..., m - 2\} \), we define a function

\[
g_i : ((\partial E_i) \cap E_{i+1}) \cup [r_i, r_{i+1}] \cup ((\partial E_{i+1}) \cap E_{i+2}) \to [r_i, r_{i+1}]
\]

by

\[
g_i(x) = \begin{cases} 
   r_i & \text{if } x \in (\partial E_i) \cap E_{i+1}, \\
   x & \text{if } x \in [r_i, r_{i+1}], \\
   r_{i+1} & \text{if } x \in (\partial E_{i+1}) \cap E_{i+2}.
\end{cases}
\]

Clearly, each \( g_i \) is a continuous retraction onto \([r_i, r_{i+1}] \). Since metric arcs are absolute retracts, for each \( i \) there is a continuous extension \( h_i \) of \( g_i \), \( h_i : C_l E_{i+1} \setminus E_i \to [r_i, r_{i+1}] \). The function \( f = h_1 \cup h_2 \cup ... \cup h_{m-1} \) is a continuous retraction of \( \mathcal{E}^\ast \setminus E_1 \) onto \([p, b]\) such that, for each \( x \in \mathcal{E}^\ast \setminus E_1 \), \((x, f(x)) < \mathcal{U}\).

The final step is the following theorem.

**Theorem 4.11.** Suppose \( X \) is an AM-fan and \( \mathcal{U} \) is a cover of \( X \). Then there is a finite fan \( Y \subset X \) and a retraction \( r : X \to Y \) such that if \( x \in X \), then \((x, f(x)) < \mathcal{U}\).

**Proof.** Let \( t \) denote the top of \( X \) and let \( W \) denote the set of end points of \( X \). Then \( X = \cup \{[t, w] : w \in W\} \).

For each \( w \in W \), we apply Lemma 4.8 to obtain a chain \( \mathcal{E}_w \) such that \([t, w]\) is straight in \( \mathcal{E}_w \) and \( \partial (\mathcal{E}_w \setminus E_{w1})^\ast \subset E_{w1} \).
There is a finite subset $W' \subset W$ such that $\{E_w' : w \in W'\}$ covers $X$. If $W' = \{w_1, \ldots, w_n\}$, let us relabel the corresponding chains $E_1, E_2, \ldots, E_n$.

For each $j \in \{1, 2, \ldots, n\}$ let $E_j = \{E_{j1}, E_{j2}, \ldots, E_{jm_j}\} = E_j(1, m_j)$. As in Step III of the proof of Theorem 1 of [4] one can construct the new $U$-chains $K_1 = \{K_{11}, K_{12}, \ldots, K_{1p_1}\}$, $K_2 = \{K_{21}, K_{22}, \ldots, K_{2p_2}\}, \ldots, K_n = \{K_{n1}, K_{n2}, \ldots, K_{np_n}\}$ such that:

1. $\bigcup\{K_j : j \in \{1, 2, \ldots, n\}\}$ covers $X$,
2. For each $j \in \{1, 2, \ldots, n\}$ the arc $[t, w_j]$ is straight in $K_j$,
3. If $j \in \{1, 2, \ldots, n\}$, then $K_{j1} = K_{11}$, and
4. If $j \neq i$, then $K_i^* (2, p_i) \cap K_j^* (2, p_j) = \emptyset$.

See Figure 3 in [4, p. 124]. We now construct a retraction $r$ of $X$ onto $Y = \cup\{[t, w_j] : j \in \{1, \ldots, n\}\}$. We will assume that each $K_j$ has more than one link; if this is not true, the needed modifications in the definition of $r$ are obvious.

For each $j \in \{1, 2, \ldots, n\}$, there exists a point $s_j \in [t, w_j]$ such that $(\partial K_{j1}) \cap [t, w_j] = (\partial K_{j1}) \cap Y \cap K_{j2} = \{s_j\}$. Since each $[t, w_j]$ is straight in the $U$-chain $K_j$, we apply Lemma 4.10 to obtain a retraction $f_j : (K_{j1}^* \setminus K_{j1}) \to [s_j, w_j]$ such that $f_j((\partial K_{j1}) \cap K_{j2}) = \{s_j\}$ and $f_j$ moves each point less than $U$. If $i \neq j$, then $(\text{domain } f_i) \cap (\text{domain } f_j) \subset K_i^* (2, p_i) \cap K_j^* (2, p_j) = \emptyset$.

Hence we may define $f = \cup\{f_i : i \in \{1, 2, \ldots, n\}\}$. Clearly, $f$ is a retraction of $X \setminus K_{11}$ onto $\cup\{[s_i, w_i] : i \in \{1, 2, \ldots, n\}\}$ moving each point less than $U$.

Now $Y \cap K_{11} = \cup\{[t, s_i] : i \in \{1, 2, \ldots, n\}\}$ and $(\text{Cl } K_{11}) \cap Y = \cup\{[t, s_i] : i \in \{1, 2, \ldots, n\}\}$ since each $[t, w_i]$ is straight in $K_i$. We define $g : (\partial K_{11}) \cup ((\text{Cl } K_{11}) \cap Y) \to (\text{Cl } K_{11}) \cap Y$ by $g(x) = x$ if $x \in Y$ and $g(x) = s_i$ if $x \in (\partial K_{11}) \cap K_{j2}$. Since $(\text{Cl } K_{11}) \cap Y$ is a metric tree, it is an absolute retract. Hence $g$ can be extended to a map $h : Cl K_{11} \to (\text{Cl } K_{11}) \cap Y$. Since $Cl K_{11}$ is contained in some member of $U$, $f$ moves each point less than $U$. Finally, let $r = h \cup f$. Since $f$ and $h$ agree on the intersection of their domains, $\partial K_{11}$, $r$ is well-defined and continuous. Obviously, $r$ is a retraction of $X$ onto $Y$.

Since neither $f$ nor $h$ moves any point as much as $U$, neither does $r$.

**Remark 4.12.** Theorem 4.11 is a modification of Theorem 1 of [4]. The proof is valid for AM-fans. Let us observe that from the proof of this Theorem it follows that if $t$ is the top of $X$, then $r(t) = t$.

In the case that $X$ is a fan, we obtain [4, Theorem 2].

**Theorem 4.13.** Each fan is an inverse limit of a sequence of finite fans.

Given an open covering $U$ of a compact space $X$, we say that a mapping $f : X \to Y$ is a $U$-mapping provided there is an open covering $V$ of $Y$ such that $f^{-1}(V)$ refines $U$, written as $f^{-1}(V) \supseteq U$.

Let $\mathcal{P}$ be a class of compact polyhedra. We say that a compact space $X$ is $\mathcal{P}$-like provided for every open covering $U$ of $X$ there is a polyhedron $P \in \mathcal{P}$ and a $U$-mapping $f : X \to P$ which is surjective.
Now let $\mathcal{F}$ be a class of finite metrizable fans. From Lemmas 4.5-4.10 and Theorems 4.11-4.13 it follows the following theorem.

**Theorem 4.14.** Every AM-fan is $\mathcal{F}$-like.

In what follows we shall use the notion of approximate inverse systems in the sense of S. Mardesić [11]. Cov($X$) is the set of all normal coverings of a topological space $X$. If $\mathcal{U}$, $\mathcal{V} \in \text{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} < \mathcal{U}$.

An approximate inverse system is a collection $X = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, $X_a, a \in A$, is a topological space and $p_{ab} : X_b \to X_a, a \leq b$, are mappings such that $p_{aa} = id$ and the following condition (A2) is satisfied:

(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \text{Cov}(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cb}, p_{ad}) < \mathcal{U}$, whenever $a \leq b \leq c \leq d$.

An approximate map [13, Definition (1.9), p. 592] $p = \{p_a : a \in A\} : X \to X$ into an approximate system $X = \{X_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a, a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $\mathcal{U} \in \text{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_{bc}, p_{ad}) < \mathcal{U}$, for each $c \geq b$.

Let $X = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let $p = \{p_a : a \in A\} : X \to X$ be an approximate map. We say that $p$ is a limit of $X$, written as $\lim X$, provided it has the following universal property:

(UL) For any approximate map $q = \{q_a : a \in A\} : Y \to X$ of a space $Y$ there exists a unique map $g : Y \to X$ such that $p_a g = q_a$.

Let $X = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called an approximate thread of $X$ provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall \mathcal{U} \in \text{Cov}(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in \text{st}(x_a, \mathcal{U})$.

If $X_a$ is a $T_{3.5}$-space, then the sets $\text{st}(x_a, \mathcal{U}), \mathcal{U} \in \text{Cov}(X_a)$, form a basis of the topology at the point $x_a$. Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L)* $(\forall a \in A) \lim \{p_{ac}(x_c) : c \geq a\} = x_a$.

The existence of the limit of any approximate system was proved in [13, (1.14) Theorem].

**Theorem 4.15.** Let $X = \{X_a, p_{ab}, A\}$ be an approximate inverse system. Let $X \subseteq \prod\{X_a : a \in A\}$ be the set of all threads of $X$ and let $p_a : X \to X_a$ be the restriction $p_a = \pi_a|X$ of the projection $\pi_a : \prod\{X_a : a \in A\} \to X_a, a \in A$. Then $p = \{p_a : a \in A\} : X \to X$ is a limit of $X$.

We call this limit the canonical limit of $X = \{X_a, p_{ab}, A\}$. In the sequel limit means the canonical limit.
A preordered set \((A, \leq)\) is cofinite provided each \(a \in A\) the set of all predecessors of \(a\) is a finite set.

We shall use the following theorem from [12, Theorem 3].

**Theorem 4.16.** Let \(P\) be a class of polyhedra with no isolated points. Let \(X\) be a compact Hausdorff space which is \(P\) - like. Then there exists an approximate inverse system of compact polyhedra \(\mathbf{P} = \{P_a, \varepsilon_a, p_{ab}, A\}\) such that \(P_a \in P\), all the bonding mappings \(p_{ab}\) are surjective and the limit \(\lim \mathbf{P}\) is homeomorphic to \(X\). Moreover, \(A\) is cofinite and \(\text{card}(A) \leq w(X)\).

**Theorem 4.17.** For every AM-fan \(X\) there exists an approximate inverse system \(\mathbf{F} = \{F_a, \varepsilon_a, p_{ab}, B\}\) of finite metric fans such that \(F_a \in \mathcal{F}\), all the bonding mappings \(p_{ab}\) are surjective and the limit \(\lim \mathbf{F}\) is homeomorphic to \(X\).

**Proof.** Theorem follows from Theorems 4.14 and 4.16.

**Remark 4.18.** Let us observe that from the proof of [12, Theorem 3], in particular, from the proof [12, Lemma 2] it follows that \(p_{ab} : P_b \to P_a\) is a simplicial map such that \(p_{ab}(r_b(t)) = r_a(t)\), where \(t\) is the top of the fan \(X\) and \(r_a : X \to P_a\) is a retraction from Theorem 4.11.

Now we shall expand each non-metrizable AM-fan into usual inverse systems of metric fans.

**Theorem 4.19.** For every AM-fan \(X\) there exists a \(\sigma\)-directed inverse system \(\mathbf{X} = \{X_a, p_{ab}, A\}\) of metric fans such that all the bonding mappings \(p_{ab}\) are surjective and the limit \(\lim \mathbf{X}\) is homeomorphic to \(X\).

**Proof.** By Theorem 4.17 there exists an approximate inverse system \(\mathbf{F} = \{F_a, \varepsilon_a, q_{ab}, B\}\) of finite metric fans such that \(F_a \in \mathcal{F}\), all the bonding mappings \(q_{ab}\) are surjective and the limit \(\lim \mathbf{F}\) is homeomorphic to \(X\). By forgetting the meshes \(\varepsilon_a\) [13, (1.7) Definition] and using Corollary 1 of [8] we obtain a usual \(\sigma\)-directed inverse system \(\mathbf{X} = \{X_a, p_{ab}, A\}\), where each \(X_a\) is the limit of an approximate inverse subsystem \(\{F_a, q_{ab}, \Phi\}\), \(\text{card}(\Phi) = \aleph_0\), of the system \(\mathbf{F}^* = \{F_a, q_{ab}, B\}\). Let us prove that every \(X_a\) is a metric fan. Firstly, each \(X_a\) is arcwise connected since there exists the projection \(p_a : X \to X_a\) and \(X\) is arcwise connected. Now we shall prove that \(X\) is hereditarily unicoherent. From Lemma 3 of [8] it follows that we may assume that \(\Phi\) is order isomorphic to the set of natural numbers \(\mathbb{N}\). Then from Proposition 8 of [1] it follows that there exists an inverse sequence \(\{F_n, q_{nm}^*, N\}\) such that \(\lim \{F_a, q_{ab}, \Phi\} = \lim \{F_n, q_{nm}^*, N\}\). It is known that \(\lim \{F_n, q_{nm}^*, N\}\) is hereditarily unicoherent [16, Corollary 1, p. 228] since each \(F_n\) is hereditarily unicoherent. It remains to prove that \(\lim \{F_n, q_{nm}^*, N\}\) is a fan. For each \(n \in \mathbb{N}\) let \(t_n\) be the top of \(F_n\). From Remark 4.12 it follows that \(t = (t_n)\) is a point of \(\lim \{F_n, q_{nm}^*, N\}\). It is clear that \(t\) is a ramification point of \(\lim \{F_n, q_{nm}^*, N\}\). Suppose that there exists a ramification
point \( u \) of \( \lim \{ F_n, q_n^*, \mathbb{N} \} \) such that \( u \neq t \). Then there exists a triod \( T \) in \( \lim \{ F_n, q_n^*, \mathbb{N} \} \) which contains \( u \) and \( t \notin T \). There exists an \( n \in \mathbb{N} \) such that \( T_n = q_n^*(T) \) contains no \( t_n = q_n^*(t) \). This means that \( T_n \) is an arc since \( F_n \) is a fan. Now, \( \lim \{ T_n, q_n^*[T_m, m > n] \} \) is chainable. Hence, \( \lim \{ T_n, q_n^*[T_m, m > n] \} \) is atriodic [17, Theorem 12.4]. This is impossible since \( T = \lim \{ T_n, q_n^*[T_m, m > n] \} \). Thus, \( \lim \{ F_n, q_n^*[\mathbb{N}] \} \) contains only one ramification point \( t \). Hence, \( \lim \{ F_n, q_n^*[\mathbb{N}] \} \) is a fan.

Now we are ready to prove the main result of this paper.

**Theorem 4.20.** A generalized fan \( X \) admits a Whitney map for \( C(X) \) if and only if it is metrizable.

**Proof.** If \( X \) is metrizable, then \( X \) admits a Whitney map for \( C(X) \). Conversely, if \( X \) admits a Whitney map for \( C(X) \), then, by Theorem 4.3 \( X \) is an \( AM \)-fan. From Theorem 4.19 it follows that there exists a \( \sigma \)-directed inverse system \( X = \{ X_a, p_{ab}, A \} \) of metric fans such that all the bonding mappings \( p_{ab} \) are surjective and the limit \( \lim X \) is homeomorphic to \( X \). Theorem 3.3 completes the proof.

Let \( AM \) be a class of \( AM \)-arboroids. From Theorem 4.1 it follows that each arboroid is \( AM \)-like. Using Theorem 4.14 we obtain the following result.

**Corollary 4.21.** Each generalized fan is \( \mathcal{F} \)-like.

By a similar method of proof as in the proof of Theorem 4.19 we obtain the following theorem.

**Theorem 4.22.** For every generalized fan \( X \) there exists a \( \sigma \)-directed inverse system \( X = \{ X_a, p_{ab}, A \} \) of metric fans such that all the bonding mappings \( p_{ab} \) are surjective and the limit \( \lim X \) is homeomorphic to \( X \).

**References**

A FAN $X$ ADMITS A WHITNEY MAP FOR $C(X)$


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