BANACH-STEINHAUS THEOREMS FOR BOUNDED LINEAR OPERATORS WITH VALUES IN A GENERALIZED 2-NORMED SPACE

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Abstract. In this paper we will prove Banach-Steinhaus Theorems for some families of bounded linear operators from a normed space into a generalized 2-normed space.

1. Introduction

In 1964 S. Gahler introduced the concept of linear 2-normed spaces and he has investigated many important properties and examples for the above spaces ([1, 2]).

Definition 1.1 ([1]). Let $X$ be a real linear space of dimension greater than 1 and let $\| \cdot , \cdot \|$ be a real valued function on $X \times X$ satisfying the following four properties:

(G1) $\| x, y \| = 0$ if and only if the vectors $x$ and $y$ are linearly dependent;
(G2) $\| x, y \| = \| y, x \| ;$
(G3) $\| x, \alpha y \| = |\alpha| \| x, y \|$ for every real number $\alpha ;$
(G4) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$ for every $x, y, z \in X .

The function $\| \cdot , \cdot \|$ will be called a 2-norm on $X$ and the pair $(X, \| \cdot , \cdot \|)$ a linear 2-normed space.

In [3] and [4] we gave a generalization of the Gähler’s 2-normed space. Namely a generalized 2-norm need not be symmetric and satisfy the first condition of the above definition.

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Definition 1.2 ([3]). Let $X$ and $Y$ be real linear spaces. Denote by $\mathcal{D}$ a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets $D_x = \{ y \in Y; (x, y) \in \mathcal{D} \}$ and $D^y = \{ x \in X; (x, y) \in \mathcal{D} \}$ are linear subspaces of the space $Y$ and $X$, respectively.

A function $\| \cdot, \cdot \| : \mathcal{D} \to [0, \infty)$ will be called a generalized 2-norm on $\mathcal{D}$ if it satisfies the following conditions:

(N1) $\| \alpha x, y \| = |\alpha| \| x, y \|$ for any real number $\alpha$ and all $(x, y) \in \mathcal{D}$;
(N2) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$ for $x \in X$, $y, z \in Y$ such that $(x, y), (x, z) \in \mathcal{D}$;
(N3) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$ for $x, y \in X$, $z \in Y$ such that $(x, z), (y, z) \in \mathcal{D}$.

The set $\mathcal{D}$ is called a 2-normed set.

In particular, if $\mathcal{D} = X \times Y$, the function $\| \cdot, \cdot \|$ will be called a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, \| \cdot, \cdot \|)$ a generalized 2-normed space. Moreover, if $X = Y$, then the generalized 2-normed space will be denoted by $(X, \| \cdot, \cdot \|)$.

In [3] and [4] we considered properties of generalized 2-normed spaces on $X \times Y$. In what follows we shall use the following results:

Theorem 1.3 ([3]). Let $(X \times Y, \| \cdot, \cdot \|)$ be a generalized 2-normed space. Then the family $\mathcal{B}$ of all sets defined by

$$\bigcap_{i=1}^{n} \{ x \in X; \| x, y_i \| < \varepsilon \},$$

where $y_1, y_2, \ldots, y_n \in Y, n \in \mathbb{N}$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in $X$.

We will denote it by the symbol $T(X, Y)$. Similarly, we have the preceding theorem for a topology $T(Y, X)$ in the space $Y$. In the case when $X = Y$ we will denote the above topologies as follows: $T_1(X) = T(X, Y)$ and $T_2(X) = T(Y, X)$.

Theorem 1.4 ([4]). Let $(X \times Y, \| \cdot, \cdot \|)$ be a generalized 2-normed space. Let $\Sigma$ be a directed set.

(a) A net $\{ x_\sigma; \sigma \in \Sigma \}$ is convergent to $x_\omega \in X$ in $(X, T(X, Y))$ if and only if for all $y \in Y$ and $\varepsilon > 0$ there exists $\sigma_\omega \in \Sigma$ such that $\| x_\sigma - x_\omega, y \| < \varepsilon$ for all $\sigma \geq \sigma_\omega$.

(b) A net $\{ y_\sigma; \sigma \in \Sigma \}$ is convergent to $y_\omega \in Y$ in $(Y, T(Y, X))$ if and only if for all $x \in X$ and $\varepsilon > 0$ there exists $\sigma_\omega \in \Sigma$ such that $\| x, y_\sigma - y_\omega \| < \varepsilon$ for all $\sigma \geq \sigma_\omega$.

Theorem 1.5 ([4]). Let $(X \times Y, \| \cdot, \cdot \|)$ be a generalized 2-normed space. If the generalized 2-norm $\| \cdot, \cdot \| : X \times Y \to [0, \infty)$ is jointly continuous and
a sequence \( \{(x_n, y_n); n \in N\} \subset X \times Y \) is convergent, then the sequence of 2-norms \( \{\|x_n, y_n\|; n \in N\} \) is bounded.

**Definition 1.6 ([4]).** Let \( (X \times Y, \| \cdot \cdot \|) \) be a generalized 2-normed space. A sequence \( \{x_n; n \in N\} \subset X \) is called a Cauchy sequence if for every \( y \in Y \) and \( \varepsilon > 0 \) there exists a number \( n_0 \in N \) such that inequality \( n, m > n_0 \) implies \( \|x_n - x_m, y\| < \varepsilon \).

**Definition 1.7 ([4]).** Let \( (X \times Y, \| \cdot \cdot \|) \) be a generalized 2-normed space. A space \( (X, T(X, Y)) \) is called sequentially complete if every Cauchy sequence in \( X \) is convergent in this space.

By analogy we obtain definitions of a Cauchy sequence in the space \( Y \) and the sequential completeness of the space \( (Y, T(Y, X)) \).

In what follows \( L(X, Y) \) stands for the linear space of all linear operators from \( X \) with values in \( Y \), where \( X, Y \) are real linear spaces.

**Definition 1.8 ([5]).** Let \( X \) be a real normed space and \( Y \subset Y \times Y \) be a 2-normed set, where \( Y \) denotes a real linear space. A set \( \mathcal{M} \) is defined as follows:

\[
\mathcal{M} = \{(f, g) \in L(X, Y)^2; \forall x \in X (f(x), g(x)) \in Y \\
\land \exists M > 0 \forall x \in X \|f(x), g(x)\| \leq M \cdot \|x\|^2 \}.
\]

The set \( \mathcal{M} \) defined in Definition 1.8 has the following property:

For every \( f, g \in L(X, Y) \) the sets

\[
\mathcal{M}^g = \{f' \in L(X, Y); (f', g) \in \mathcal{M}\} \quad \text{and} \quad \mathcal{M}_f = \{g' \in L(X, Y); (f, g') \in \mathcal{M}\}
\]

are linear subspaces of the space \( L(X, Y) \).

For \( (f, g) \in \mathcal{M} \) we introduce the number

\[
(1.1) \quad \|f, g\| = \inf\{M > 0; \forall x \in X \|f(x), g(x)\| \leq M \cdot \|x\|^2 \}.
\]

Then

\[
(1.2) \quad \|f(x), g(x)\| \leq \|f, g\| \cdot \|x\|^2 \quad \text{for all} \quad x \in X;
\]

\[
(1.3) \quad \|f, g\| = \sup\{\|f(x), g(x)\|; x \in X \land \|x\| = 1 \}
\]

\[
= \sup\{\|f(x), g(x)\|; x \in X \land \|x\| \leq 1 \}
\]

\[
= \sup\left\{\frac{\|f(x), g(x)\|}{\|x\|^2}; x \in X \land \|x\| \neq 0 \right\}.
\]

Moreover, the set \( \mathcal{M} \) is a 2-normed set with the 2-norm defined by the formula (1.1) (cf. [5]).
**Definition 1.9 ([5]).** Let $X$ be a real normed space and $\mathcal{Y} \subset Y \times Y$ be a 2-normed set, where $Y$ denotes a real linear space. A set $\mathcal{N}$ is defined as follows:

$$\mathcal{N} = \left\{ (f, g) \in L(X, Y)^2; \forall_{x, y \in X} (f(x), g(y)) \in \mathcal{Y}\right\}$$

$$\wedge \exists_{M > 0} \forall_{x, y \in X} \|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\|.$$

The set $\mathcal{N}$ defined in Definition 1.9 has similar properties:

For every $f, g \in L(X, Y)$ the sets

$$\mathcal{N}^g = \{ f' \in L(X, Y); (f', g) \in \mathcal{N} \} \quad \text{and} \quad \mathcal{N}_f = \{ g' \in L(X, Y); (f, g') \in \mathcal{N} \}$$

are linear subspaces of the space $L(X, Y)$.

For $(f, g) \in \mathcal{N}$ we introduce the number

$$\|f, g\| = \inf \{ M > 0; \forall_{x, y \in X} \|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\| \}.$$  

Then

$$\|f(x), g(y)\| \leq \|f, g\| \cdot \|x\| \cdot \|y\|$$

for all $x, y \in X$;

$$\|f, g\| = \sup \{ \|f(x), g(y)\|; x, y \in X \wedge \|x\| = \|y\| = 1 \}$$

$$= \sup \{ \|f(x), g(y)\|; x, y \in X \wedge \|x\| \leq 1, \|y\| \leq 1 \}$$

$$= \sup \left\{ \frac{\|f(x), g(y)\|}{\|x\| \cdot \|y\|}; x, y \in X \wedge \|x\| \neq 0, \|y\| \neq 0 \right\}.$$  

Moreover, the set $\mathcal{N}$ is a 2-normed set with the 2-norm defined by the formula (1.4) (cf. [5]).

**2. Banach-Steinhaus Theorems for bounded linear operators**

In this section we will consider properties of sequences of operators, which are contained in $\mathcal{M}^g, \mathcal{M}_f$ or $\mathcal{N}^g, \mathcal{N}_f$ for some $f, g \in L(X, Y)$. Moreover we will investigate sequences $\{(f_n, g_n); n \in \mathbb{N}\}$ from $\mathcal{M}$ or $\mathcal{N}$. In every case we will formulate Banach-Steinhaus Theorems. Because any theorem for sequences of operators from $\mathcal{M}^g$ or $\mathcal{N}^g$ is also true (after making necessary changes) for sequences of operators from $\mathcal{M}_f$ or $\mathcal{N}_f$, we will give only one version of theorems.

**Theorem 2.1.** Let $(X, \| \cdot \|)$ be a normed space, $(Y, \| \cdot \cdot \|)$ a generalized 2-normed space and $g \in L(X, Y)$. Then:

(a) If a sequence $\{f_n; n \in \mathbb{N}\} \subset \mathcal{M}^g$ and the sequence of 2-norms $\{\|f_n\|; n \in \mathbb{N}\}$ is bounded, then for every $x \in X$ the sequence $\{\|f_n(x), g(x)\|; n \in \mathbb{N}\}$ is bounded.

(b) If a sequence $\{f_n; n \in \mathbb{N}\} \subset \mathcal{N}^g$ and the sequence of 2-norms $\{\|f_n\|; n \in \mathbb{N}\}$ is bounded, then for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\|; n \in \mathbb{N}\}$ is bounded.
Proof. (a) Let \( \|f_n, g\| \leq M \) for every \( n \in N \). Then for \( x \in X \) we have
\[
\|f_n(x), g(x)\| \leq \|f_n, g\| \cdot \|x\|^2 \leq M \cdot \|x\|^2.
\]
Hence for every \( x \in X \) the sequence \( \{\|f_n(x), g(x)\|; n \in N\} \) is bounded by the number \( M \cdot \|x\|^2 \).

(b) If \( \|f_n, g\| \leq M \) for every \( n \in N \), then for \( x, y \in X \) we have
\[
\|f_n(x), g(y)\| \leq \|f_n, g\| \cdot \|x\| \cdot \|y\| \leq M \cdot \|x\| \cdot \|y\|.
\]
Thus for every \( x, y \in X \) the sequence \( \{\|f_n(x), g(y)\|; n \in N\} \) is bounded by the number \( M \cdot \|x\| \cdot \|y\| \).

**Theorem 2.2.** Let \((X, \| \cdot \|)\) be a Banach space, \((Y, \| \cdot \|)\) a generalized 2-normed space and \(\{f_n; n \in N\}\) a sequence of elements from \(\mathcal{N}^g\) for some \(g \in L(X, Y)\). Then the following conditions are equivalent:

(a) The sequence of 2-norms \(\{\|f_n, g\|; n \in N\}\) is bounded.
(b) \(\exists M > 0 \forall x, y \in X, \|x\| \leq 1, \|y\| \leq 1 \forall n \in N \|f_n(x), g(y)\| \leq M\).
(c) The following conditions are true:

(i) \(\forall x \in X \exists M_x > 0 \forall y \in X, \|y\| \leq 1 \forall n \in N \|f_n(x), g(y)\| \leq M_x\).
(ii) \(\forall y \in Y \exists M_y > 0 \forall x \in X, \|x\| \leq 1 \forall n \in N \|f_n(x), g(y)\| \leq M_y\).

Proof. At first let us suppose that the sequence of 2-norms \(\{\|f_n, g\|; n \in N\}\) is bounded. From this it follows that there exists a positive number \(M\) such that \(\|f_n, g\| \leq M\) for each \(n \in N\). Thus for \(x, y \in X, \|x\| \leq 1, \|y\| \leq 1\) and \(n \in N\) we have \(\|f_n(x), g(y)\| \leq \|f_n, g\| \cdot \|x\| \cdot \|y\| \leq M\).

Now, let the condition (b) be satisfied. We fix \(x \in X \setminus \{0\}\). Then for each \(y \in X, \|y\| \leq 1\) and \(n \in N\) we obtain the inequalities:
\[
\|f_n(x), g(y)\| = \|f_n\left(\frac{x}{\|x\|} \cdot \|y\|\right), g(y)\| = \|x\| \cdot \|f_n\left(\frac{x}{\|x\|}\right), g(y)\| \leq M \cdot \|x\|.
\]
If we choose \(M_x = M \cdot \|x\|\), then we have the condition (i). Moreover, for \(x = 0\) the condition (i) is satisfied for every positive number \(M_x\). Analogously, taking \(M_y = M \cdot \|y\|\) for each \(y \in X \setminus \{0\}\) and any positive number for \(y = 0\) we obtain (ii).

Conversely, let (i) and (ii) be satisfied. In \(X \times X\) let us define a norm by the formula:
\[
\|(x, y)\|_* = \|x\| + \|y\| \text{ for each } (x, y) \in X \times X.
\]
It is easy to verify that \((X \times X, \|(\cdot, \cdot)_*\|)\) is a Banach space. Put
\[
A_{nm} = \{(x, y) \in X \times X; \|f_n(x), g(y)\| \leq m\}
\]
and
\[
B_m = \bigcap_{n=1}^{\infty} A_{nm}
\]
for \(m, n \in N\). We shall show that sets \(B_m\) are closed in \((X \times X, \|(\cdot, \cdot)_*\|)\) for each \(m \in N\).
At first we shall show that sets $A_{nm}$ are closed in this space. Let $m, n \in N$ and let $\{(x_k, y_k); k \in N\} \subset A_{nm}$ be a sequence converging to $(x', y') \in X \times X$. Then
\[
\|f_n(x_k), g(y_k)\| \leq m \text{ and } \|(x_k, y_k) - (x', y')\|_* \rightarrow 0, k \rightarrow \infty.
\]
The last condition is equivalent to the following: $\|x_k - x'\| \rightarrow 0$ and $\|y_k - y'\| \rightarrow 0$, which implies the convergence of the sequences $\{x_k; k \in N\}, \{y_k; k \in N\}$. As a consequence these sequences are bounded. There exists $K > 0$ such that the inequalities $\|x_k\| \leq K$, $\|y_k\| \leq K$ are true for each $k \in N$. Using these results we get
\[
\|f_n(x'), g(y')\| \leq m + K \cdot \|f_n, g\| \cdot \|x_k - x'\| + K \cdot \|f_n, g\| \cdot \|y_k - y'\|
\]
\[
+ \|f_n, g\| \cdot \|x_k - x'\| \cdot \|y_k - y'\|.
\]
Letting $k \rightarrow \infty$ we obtain $\|f_n(x'), g(y')\| \leq m$, which means that $(x', y') \in A_{nm}$. Therefore the sets $A_{nm}$ are closed for each $n, m \in N$, and hence the sets $B_m$ are also closed in $(X \times X, \|\cdot\|_*)$.

Now, we shall show that the equality
\[
X \times X = \bigcup_{m=1}^{\infty} B_m
\]
is true. Let $x, y \in X, x \neq 0$. Then $\|\frac{x}{|x|}\| = 1$. By virtue (ii) there exists $M_y > 0$ such that
\[
\|f_n\left(\frac{x}{|x|}\right), g(y)\| \leq M_y \text{ for each } n \in N.
\]
Thus $\|f_n(x), g(y)\| \leq M_y \cdot \|x\|$ for each $n \in N$.

If $x = 0$, then $\|x\| \leq 1$ and $\|f_n(x), g(y)\| = \|0, g(y)\| = 0 = M_y \cdot \|0\|$. As a consequence, for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\|; n \in N\}$ is bounded. From this it follows that for any point $(x, y) \in X \times X$ there exists $n \in N$ such that $\|f_n(x), g(y)\| \leq m$ for every $m \in N$, i.e.
\[
(x, y) \in \bigcup_{m=1}^{\infty} B_m.
\]
Thus
\[
X \times X = \bigcup_{m=1}^{\infty} B_m.
\]
By the well known Baire theorem there exists a set $B_{m_0}$ with non-empty interior. Therefore $B_{m_0}$ contains some closed ball with the center $(x_o, y_o)$ and radius $r$. Denote it by $K((x_o, y_o), r)$. Thus for each $n \in N$ and $(x, y) \in K((x_o, y_o), r)$ we have $\|f_n(x), g(y)\| \leq m_o$.

Let us take $x, y \in X$ such that $\|x\| \leq \frac{r}{2}$ and $\|y\| \leq \frac{r}{2}$. Then $\|(x, y)\|_* = \|x\| + \|y\| \leq r$ and $\|(x, y)\|_* = \|(x + x_o, y + y_o) - (x_o, y_o)\|_* \leq r$. 
Therefore \( \|f_n(x + x_o), g(y + y_o)\| \leq m_o. \) In particular \( \|f_n(x_o), g(y_o)\| \leq m_o. \) Thus

\[
\|f_n(x), g(y)\| \leq \|f_n(x + x_o), g(y + y_o)\| + \|f_n(x + x_o), g(y_o)\|
\]
\[
+ \|f_n(x_o), g(y + y_o)\| + \|f_n(x_o), g(y_o)\|
\]
\[
\leq 2m_o + \|f_n(x) + f_n(x_o), g(y_o)\| + \|f_n(x_o), g(y) + g(y_o)\|
\]
\[
\leq 4m_o + \|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|
\]

So we have shown that the inequalities \( \|x\| \leq \frac{r}{2} \) and \( \|y\| \leq \frac{r}{2} \) imply the condition

\[
\|f_n(x), g(y)\| \leq 4m_o + \|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|
\]

Now, let \( x, y \in X, \|x\| \leq 1 \) and \( \|y\| \leq 1. \) Because \( \|\frac{r}{2}x\| \leq \frac{r}{2} \) and \( \|\frac{r}{2}y\| \leq \frac{r}{2}, \)
then

\[
\|f_n(\frac{r}{2}x), g(\frac{r}{2}y)\| \leq 4m_o + \|f_n(\frac{r}{2}x), g(y_o)\| + \|f_n(x_o), g(\frac{r}{2}y)\|
\]

As a consequence we obtain

\[
\|f_n(x), g(y)\| \leq \frac{16m_o}{r^2} + \frac{2}{r}(\|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|)
\]

for each \( n \in N. \) Applying (i) we have that there exists \( M_{x_o} > 0 \) such that for every \( y \in X, \|y\| \leq 1 \) and \( n \in N \) the inequality \( \|f_n(x), g(y)\| \leq M_{x_o} \) is true. However the assumption (ii) implies there exists \( M_{y_o} > 0 \) such that for every \( x \in X, \|x\| \leq 1 \) and \( n \in N \) the inequality \( \|f_n(x), g(y_o)\| \leq M_{y_o} \) is satisfied. So

\[
\|f_n(x), g(y)\| \leq \frac{16m_o}{r^2} + \frac{2}{r} \cdot (M_{y_o} + M_{x_o})
\]

for each \( n \in N \) and \( x, y \in X \) such that \( \|x\| \leq 1, \|y\| \leq 1. \) Therefore

\[
\|f_n, g\| = \sup\{\|f_n(x), g(y)\|, x, y \in X \wedge \|x\| \leq 1, \|y\| \leq 1\}
\]
\[
\leq \frac{16m_o + 2r(M_{x_o} + M_{y_o})}{r^2}
\]

for each \( n \in N. \) So the sequence \( \{\|f_n, g\|, n \in N\} \) is bounded and the proof is completed.

Let \( g \in L(X, Y). \) A sequence \( \{f_n; n \in N\} \subset N^g \) is pointwise convergent to \( f \in L(X, Y), \) if

\[
\forall x \in X \forall z \in Y \lim_{n \to \infty} \|f_n(x) - f(x), z\| = 0
\]

(cf. [4]). However, if \( g \) is the operator from \( X \) on \( Y, \) then the sequence \( \{f_n; n \in N\} \subset N^g \) is pointwise convergent to \( f \in L(X, Y), \) if

\[
\forall x \in X \forall y \in Y \lim_{n \to \infty} ||f_n(x) - f(x), g(y)|| = 0
\]

We will use the above note in the following theorem.
Theorem 2.3. Let \((X, \| \cdot \|)\) be a Banach space, \((Y, \| \cdot \cdot \|)\) a generalized 2-normed space and \(g\) a linear operator from \(X\) on \(Y\). If \(\{f_n; n \in N\} \subset N^9\) is pointwise convergent to \(f \in L(X,Y)\) and satisfies one of the conditions (a), (b), (c) from Theorem 2.2, then \(f \in N^9\).

Proof. From Theorem 2.2 the sequence of 2-norms \(\{\|f_n, g\|; n \in N\}\) is bounded. Thus there exists \(M > 0\) such that \(\|f_n, g\| \leq M\) for each \(n \in N\). For points \(x, y \in X\) we have

\[
\|f_n(x), g(y)\| \leq \|f_n, g\| \cdot \|x\| \cdot \|y\| \leq M \cdot \|x\| \cdot \|y\|
\]

So \(\|f(x), g(y)\| \leq \|f(x) - f_n(x), g(y)\| + M \cdot \|x\| \cdot \|y\|.\) Letting \(n \to \infty\) in the above inequality we obtain

\[
\|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\|
\]

which implies \(f \in N^9\). 

Definition 2.4 ([6]). A set \(A\) of elements of a normed space \(X\) is said to be linearly dense in \(X\), if the set \(X_0\) of all linear combinations of elements from \(A\) is dense in \(X\). 

Theorem 2.5. Let \(A\) be a linearly dense set in a Banach space \((X, \| \cdot \|), (Y, \| \cdot \cdot \|)\) a generalized 2-normed space such that \((Y, T_1(Y))\) is a Hausdorff sequentially complete space. Let \(g\) be a linear operator from \(X\) on \(Y\) and \(\{f_n; n \in N\} \subset N^9\). The following conditions are equivalent:

(a) The sequence \(\{f_n; n \in N\}\) is pointwise convergent to \(f \in L(X,Y)\) and the conditions (i), (ii) from Theorem 2.2 are satisfied.

(b) The sequence \(\{f_n; n \in N\}\) is pointwise convergent to \(f \in N^9\) on the set \(A\) and the sequence of 2-norms \(\{\|f_n, g\|; n \in N\}\) is bounded.

Proof. If the sequence \(\{f_n(x); n \in N\}\) is convergent to \(f(x) \in Y\) for each \(x \in X\), then it is convergent also for \(x \in A \subset X\). Moreover - this follows from Theorem 2.2 and Theorem 2.3 - the sequence \(\{\|f_n, g\|; n \in N\}\) is bounded and \(f \in N^9\).

Now, we will suppose that the sequence \(\{f_n; n \in N\}\) is pointwise convergent to \(f \in N^9\) on the set \(A\) and the sequence of 2-norms \(\{\|f_n, g\|; n \in N\}\) is bounded. By Theorem 2.2 the conditions (i), (ii) hold. Let \(X_0\) be the vector subspace of the Banach space \(X\) generated by \(A\). So \(X_0\) is a normed space.

Let \(x, y \in X_0\). Then \(x = a_1x_1 + \cdots + a_kx_k, y = b_1y_1 + \cdots + b_ly_l\), where \(a_i, b_j \in \mathbb{R}, x_i, y_j \in A, i = 1, 2, \ldots, k, j = 1, 2, \ldots, t, k, t \in N\). Thus, it follows from assumptions on \(f_n, f, g\) that

\[
\|f_n(x) - f(x), g(y)\| =
\]

\[
= \|a_1(f_n(x_1) - f(x_1)) + \cdots + a_k(f_n(x_k) - f(x_k)), b_1g(y_1) + \cdots + b_lg(y_l)\|.
\]
Using properties of 2-norms we get:
\[
\|f_n(x) - f(x), g(y)\| \leq \sum_{i=1}^{k} \sum_{j=1}^{t} |a_ib_j| \cdot \|f_n(x_i) - f(x_i), g(y_j)\|.
\]
Because
\[\lim_{n \to \infty} \|f_n(x_i) - f(x_i), g(y_j)\| = 0\]
for each \(x_i, y_j \in A\), then
\[\lim_{n \to \infty} \|f_n(x) - f(x), g(y)\| = 0,
\]
i.e. the sequence \(\{f_n; n \in N\}\) is convergent to \(f\) on \(X_o\).

Let \(\|f_n, g\| \leq M\) for every \(n \in N\). Let us take a number \(\varepsilon > 0\), \(x \in X\) and \(y \in X\) such that \(y \neq 0\). Since \(X_o\) is a dense set in \(X\), we can choose \(x_o \in X_o, x_o \neq 0\) such that
\[\|x - x_o\| < \frac{\varepsilon}{6M \cdot \|y\|}.
\]
Moreover there exists \(y_o \in X_o\) with the property
\[\|y - y_o\| < \frac{\varepsilon}{6M \cdot \|x_o\|}.
\]
The sequence \(\{f_n(x_o); n \in N\}\) is convergent in \((Y, T_1(Y))\), so it is a Cauchy sequence in this space. Therefore there exists a number \(n_o\) such that
\[\|f_n(x_o) - f_m(x_o), g(y_o)\| < \frac{\varepsilon}{3}\]
for each \(n, m \geq n_o\).

As a consequence we obtain
\[
\|f_n(x) - f_m(x), g(y)\| \leq
\]
\[
\leq \|f_n(x) - f_n(x_o), g(y)\| + \|f_n(x_o) - f_m(x_o), g(y)\|
\]
\[
+ \|f_m(x_o) - f_m(x), g(y)\|
\]
\[
\leq \|f_n, g\| \cdot \|x - x_o\| \cdot \|y\| + \|f_n(x_o) - f_m(x_o), g(y - y_o) + g(y_o)\|
\]
\[
+ \|f_m, g\| \cdot \|x - x_o\| \cdot \|y\|
\]
\[
\leq 2M \|x - x_o\| \cdot \|y\| + \|f_n(x_o) - f_m(x_o), g(y - y_o)\|
\]
\[
+ \|f_m(x_o) - f_m(x), g(y_o)\|
\]
\[
< 2M \|x - x_o\| \cdot \|y\| + \|f_n(x_o), g(y - y_o)\| + \|f_m(x_o), g(y - y_o)\| + \frac{\varepsilon}{3}
\]
\[
< 2M \frac{\|y\|}{6M \|y\|} \|y\| + \|f_n, g\| \cdot \|x_o\| \cdot \|y - y_o\|
\]
\[
+ \|f_m, g\| \cdot \|x_o\| \cdot \|y - y_o\| + \frac{\varepsilon}{3}
\]
\[
< \frac{2}{3} \varepsilon + 2M \|x_o\| \cdot \|y - y_o\| < \frac{2}{3} \varepsilon + 2M \|x_o\| \frac{\varepsilon}{6M \|x_o\|} = \varepsilon
\]
for \(n, m \geq n_o\). If \(y = 0\), then the inequality \(\|f_n(x) - f_m(x), g(y)\| = 0 < \varepsilon\) is also true.
Hence we have shown that \( \{f_n(x); n \in N\} \) is a Cauchy sequence in 
\((Y, T_1(Y))\) for every \( x \in X \). Because \((Y, T_1(Y))\) is a sequentially complete 
space, then the sequence \( \{f_n; n \in N\} \) is pointwise convergent.

Let us denote

\[ h(x) = \lim_{n \to \infty} f_n(x) \] for every \( x \in X \).

The fact that \((Y, T_1(Y))\) is a Hausdorff space implies \( h(x) = f(x) \) for \( x \in A \), i.e. \((h - f)(x) = 0\) for \( x \in A \). The operator \( h - f \) is linear, thus \((h - f)(x) = 0\) for every \( x \in X_0 \). Using Theorem 2.3 we see that \( h \in \mathcal{N}^9 \). Because \( \mathcal{N}^9 \) is a linear subspace, then \( h - f \in \mathcal{N}^9 \). Thus there exists a positive number \( K \) such that

\[ \|(h - f)(x), g(y)\| \leq K \cdot \|x\| \cdot \|y\| \] for every \( x, y \in X \).

Let \( \varepsilon > 0, x, y \in X, y \neq 0 \). Since the set \( X_0 \) is dense in \( X \) we can choose \( x_o \in X_0 \) such that

\[ \|x - x_o\| < \frac{\varepsilon}{K \cdot \|y\|} \]

Then

\[ 0 \leq \|(h - f)(x), g(y)\| = \|(h - f)(x - x_o) + (h - f)(x_o), g(y)\|
= \|(h - f)(x - x_o), g(y)\| \leq K \cdot \|x - x_o\| \cdot \|y\| < \varepsilon \]

This gives \( \|(h - f)(x), g(y)\| = 0 \) for each \( x \in X, y \in X \setminus \{0\} \). Thus \( h(x) = f(x) \) for every \( x \in X \). As a consequence we have shown that the sequence \( \{f_n; n \in N\} \) is pointwise convergent to \( f \), which finishes the proof.

**Theorem 2.6.** Let \((X, \| \cdot \|)\) be a Banach space, \((Y, \| \cdot \|, \cdot \|)\) a 
generalized 2-normed space such that \((Y, T_1(Y))\) is a Hausdorff sequentially 
complete space. Let \( g \) be a linear operator from \( X \) on \( Y \). If a sequence 
\( \{f_n; n \in N\} \subset \mathcal{N}^9 \) is pointwise convergent to \( f \in \mathcal{N}^9 \) on a linearly dense 
set \( A \) in \( X \) and the sequence of 2-norms \( \{\|f_n, g\|; n \in N\} \) is bounded, then 
\( \{f_n; n \in N\} \) is pointwise convergent to \( f \) and \( \|f, g\| \leq \sup\{\|f_n, g\|; n \in N\} \).

**Proof.** It follows from Theorem 2.5 that the sequence \( \{f_n(x); n \in N\} \) is
convergent in \( Y \) to \( f(x) \) for every \( x \in X \). Let us denote \( M = \sup\{\|f_n, g\|; n \in N\} \).
Then for every \( n \in N \) and \( x, y \in X \) such that \( \|x\| \leq 1, \|y\| \leq 1 \) we have
\( \|f_n(x), g(y)\| \leq M \). Thus

\[ \|f(x), g(y)\| \leq \|f_n(x) - f(x), g(y)\| + \|f_n(x), g(y)\| \leq \|f_n(x) - f(x), g(y)\| + M. \]

By letting \( n \to \infty \) we obtain

\[ \|f(x), g(y)\| \leq M \] for \( x, y \in X, \|x\| \leq 1, \|y\| \leq 1 \).

This implies \( \|f, g\| \leq M \), which finishes the proof.

Now, let us consider sequences \( \{(f_n, g_n); n \in N\} \) from \( \mathcal{M} \) or \( \mathcal{N} \). Using analogous arguments as in proofs of the foregoing theorems we can show that the following theorems are true.
Theorem 2.7. Let \((X, \| \cdot \|)\) be a normed space and \((Y, \| \cdot , \cdot \|)\) a generalized 2-normed space.

(a) If \(\{(f_n, g_n); n \in N\} \subset \mathcal{M}\) and the sequence of 2-norms \(\{\|f_n, g_n\|; n \in N\}\) is bounded, then for every \(x \in X\) the sequence \(\{\|f_n(x), g_n(x)\|; n \in N\}\) is bounded.

(b) If \(\{(f_n, g_n); n \in N\} \subset \mathcal{N}\) and the sequence of 2-norms \(\{\|f_n, g_n\|; n \in N\}\) is bounded, then for every \(x, y \in X\) the sequence \(\{\|f_n(x), g_n(y)\|; n \in N\}\) is bounded.

Theorem 2.8. Let \((X, \| \cdot \|)\) be a Banach space, \((Y, \| \cdot , \cdot \|)\) a generalized 2-normed space and \(\{(f_n, g_n); n \in N\}\) a sequence of elements from \(\mathcal{N}\). Then the following conditions are equivalent:

(a) The sequence of 2-norms \(\{\|f_n, g_n\|; n \in N\}\) is bounded;

(b) \(\exists M > 0 \forall x, y \in X, \|x\| \leq 1, \|y\| \leq 1 \forall n \in N \|f_n(x), g_n(y)\| \leq M\);

(c) The following conditions are satisfied:

(i) \(\forall x \in X \exists M_x > 0 \forall y \in X, \|y\| \leq 1 \forall n \in N \|f_n(x), g_n(y)\| \leq M_x\);

(ii) \(\forall y \in X \exists M_y > 0 \forall x \in X, \|x\| \leq 1 \forall n \in N \|f_n(x), g_n(y)\| \leq M_y\).

Theorem 2.9. Let \((X, \| \cdot \|)\) be a Banach space, \((Y, \| \cdot , \cdot \|)\) a generalized 2-normed space with the continuous 2-norm. If a sequence \(\{(f_n, g_n); n \in N\} \subset \mathcal{N}\) is pointwise convergent to \((f, g) \in L(X, Y)^2\) and one of three conditions (a), (b), (c) of Theorem 2.8 is true, then \((f, g) \in \mathcal{N}\).

Proof. Using Theorem 2.8 we have that the sequence of 2-norms \(\{\|f_n, g_n\|; n \in N\}\) is bounded, i.e. there exists \(M > 0\) such that \(\|f_n, g_n\| \leq M\) for each \(n \in N\). Let \(x, y \in X\) be arbitrary. Then

\[
\|f_n(x), g_n(y)\| \leq \|f_n, g_n\| \cdot \|x\| \cdot \|y\| \leq M \|x\| \cdot \|y\|
\]

Since the 2-norm is continuous, then

\[
\|f(x), g(y)\| = \lim_{n \to \infty} \|f_n(x), g_n(y)\| \leq M \|x\| \cdot \|y\|
\]

i.e. \((f, g) \in \mathcal{N}\). \(\square\)

From Theorem 1.5 the following follows

Theorem 2.10. Let \((X, \| \cdot \|)\) be a normed space, \((Y, \| \cdot , \cdot \|)\) a generalized 2-normed space. If a sequence \(\{(f_n, g_n); n \in N\} \subset \mathcal{N}\) is pointwise convergent to \((f, g) \in L(X, Y) \times L(X, Y)\) and the 2-norm is continuous, then the sequence \(\{\|f_n(x), g_n(y)\|; n \in N\}\) is bounded for each \(x, y \in X\).

References


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