GLASNIK MATEMATIČKI
Vol. 38(58)(2003), 299 – 309

A DISCRETE BOUNDARY VALUE PROBLEM WITH PARAMETERS IN BANACH SPACE

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Abstract. In the present note we are concerned with the question of the existence and uniqueness of solutions of two- point boundary value problem for nonlinear difference equations with parameters in Banach space.The proof of the above problem is based on the method of successive approximation and fixed-point method.

1. Introduction

In this paper we present some existence and uniqueness results for discrete boundary value problems where the values of the solution lie in a Banach space E.

The motivation for the present work comes from many recent investigations. In fact, the continuous versions have been studied by many authors, e.g. see [4, 5, 6, 11, 12] and the references cited therein.Discrete boundary value problems have been discussed widely in the literature, see [1, 3, 7, 10] and their references.

In the present note we are concerned with the question of the existence and uniqueness of solutions of two-point boundary value problem for nonlinear difference equations with parameters in Banach space.

The proof of the existence is based on the method of successive approximation and fixed-point methods. We use conditions expressed in terms of the measure of strong or weak noncompactness and weakly–weakly sequentially continuity.

2000 *Mathematics Subject Classification.* 39A10, 47H10.

Key words and phrases. Difference equations, boundary value problems, Banach space.

Consider the Banach spaces E and F with the norms $\|\cdot\|$. Let Z denotes the set of integers. Given $a < b$ in Z, let $[a, b] = \{a, a+1, \dots, b\}$. Also the symbol Δ denotes the forward difference operator with stepsize 1.

2. Main results

2.1. In the first part of this section we discusses the nonlinear discrete equation of the form

$$\Delta x(n) = f(n, x(n), \lambda), \tag{1}$$

with the boundary conditions

$$x(0) = x_0, \quad x(T+1) = X, \tag{2}$$

where T is a fixed positive integer, $f : [0, T] \times B_r \times C \to E$, $B_r = \{x \in E : \|x - x_0\| \leq r\}$, $C \subset F$ some closed, bounded set. For this equation we consider the following problem (A):

(A) There exists parameter $\lambda \in C$ such that equation (1) has a solution $x : [0, T] \to E$ satisfying conditions (2).

Equation (1) is equivalent to the equation

$$x(n+1) = x_0 + \sum_{s=0}^{n} f(s, x(s), \lambda) \tag{i}$$

for every $n \in [0, T]$.

Suppose that the function f satisfies the following conditions:

(H_1) there exists a function $\alpha : [0, T] \to R_+$ such that

$$\sum_{s=0}^{T} \alpha(s) \leq r$$

and

$$|f(n, x, \lambda)| \leq \alpha(n)$$

for every $n \in [0, T], x \in B_r, \lambda \in C$,

(H_2) there exists a function $\beta : [0, T] \to R_+$ such that

$$\|f(n, u_1, \lambda_1) - f(n, u_2, \lambda_2)\| \leq \beta(n)(\|u_1 - u_2\| + \|\lambda_1 - \lambda_2\|),$$

for every $n \in [0, T],\ u_1, u_2 \in B_r,\ \lambda_1, \lambda_2 \in C$,

(H_3) there exists a constant $L > 0$ such that

$$\left\| \sum_{s=0}^{T} (f(s, x(s), \lambda_1) - f(s, x(s), \lambda_2) \right\| \geq L \|\lambda_1 - \lambda_2\|$$

for $\lambda_1, \lambda_2 \in C, x : [0, T] \to B_r$ and $q\left(1 + \frac{q}{L}\right) < 1$ where $q = \sum_{s=0}^{T} \beta(s)$,

(H_4) for every $x : [0,T] \to B_r$ there exists $\lambda \in C$ such that

$$x_0 + \sum_{s=0}^{T} f(s, x(s), \lambda) = X.$$

We are now ready to present the main results.

THEOREM 2.1. *Suppose $(H_1) - (H_4)$ hold. Then the problem (A) has exactly one solution.*

PROOF. In order to apply method of successive approximations we define sequences $\{x_m(\cdot)\}_{m=0}^{\infty}$ and $\{\lambda_m\}_{m=0}^{\infty}$ in the following way:

Let $x_0(\cdot) : [0,T] \to B_r, x_0(n) = x_0$ for every $n \in [0,T]$. Then by (H_4) there exists $\lambda_0 \in C$ such that

$$x_0 + \sum_{s=0}^{T} f(s, x_0(s), \lambda_0) = X.$$

Next we define the function $x_1(\cdot)$ as follows:

$$x_1(0) = x_0$$

and

$$x_1(n+1) = x_0 + \sum_{s=0}^{n} f(s, x_0(s), \lambda_0) \quad for \ \ n \in [0,T].$$

By (H_1) $x_1(n) \in B_r$ and $x_1(T+1) = X \in B_r$. Repeating the procedure, we define the function $x_2(\cdot)$ by the formula

$$x_2(n+1) = x_0 + \sum_{s=0}^{n} f(s, x_1(s), \lambda_1).$$

In this manner we obtain two sequences $\{x_m(\cdot)\}$ and $\{\lambda_m\}$, whose m-th term are defined by relations

$$x_m(n+1) = x_0 + \sum_{s=0}^{n} f(s, x_{m-1}(s), \lambda_{m-1}), \ n \in [0,T] \tag{3}$$

where λ_{m-1} satisfy equality

$$x_0 + \sum_{s=0}^{T} f(s, x_{m-1}(s), \lambda_{m-1}) = X. \tag{4}$$

By assumptions we have

$$x_m(n) \in B_r \quad \text{for} \ \ n \in [0,T], x_m(T+1) = X \in B_r.$$

It will be now verified that sequences $\{x_m(\cdot)\}$ and $\{\lambda_m\}$ are convergent.

From (3) and (4) we have

$$
\begin{aligned}
&x_m(T+1) - x_{m-1}(T+1) = 0 = \\
&= \sum_{s=0}^{T} \{ f(s, x_{m-1}(s), \lambda_{m-1}) - f(s, x_{m-1}(s), \lambda_{m-2}) \} + \\
&+ \sum_{s=0}^{T} \{ f(s, x_{m-1}(s), \lambda_{m-2}) - f(s, x_{m-2}(s), \lambda_{m-2}) \}.
\end{aligned} \tag{5}
$$

So

$$
\begin{aligned}
&\left\| - \sum_{s=0}^{T} \{ f(s, x_{m-1}(s), \lambda_{m-1}) - f(s, x_{m-1}(s), \lambda_{m-2}) \} \right\| = \\
&= \left\| \sum_{s=0}^{T} \{ f(s, x_{m-1}(s), \lambda_{m-2}(s)) - f(s, x_{m-2}(s), \lambda_{m-2} \} \right\|.
\end{aligned}
$$

By (H_3) left hand side is greater then $L \left\| \lambda_{m-1} - \lambda_{m-2} \right\|$ and by (H_2) right hand side is less then

$$
\sum_{s=0}^{T} \beta(s) \left| x_{m-1}(s) - x_{m-2}(s) \right|,
$$

hence

$$
\begin{aligned}
\left\| \lambda_{m-1} - \lambda_{m-2} \right\| &\le \frac{1}{L} \left(\sum_{s=0}^{T} \beta(s) \right) \max_{0 \le s \le T} \left\| x_{m-1}(s) - x_{m-2}(s) \right\| \le \\
&\le \frac{q}{L} \max_{0 \le s \le T} \left\| x_{m-1}(s) - x_{m-2}(s) \right\|.
\end{aligned} \tag{6}
$$

From (3),(6) and assumption (H_1), we have

$$
\begin{aligned}
&\| x_m(n+1) - x_{m-1}(n+1) \| \le \\
&\quad \le \sum_{s=0}^{n} \beta(s) \| \lambda_{m-1} - \lambda_{m-2} \| + \sum_{s=0}^{n} \beta(s) \max_{0 \le n \le T} \left\| x_{m-1}(n) - x_{m-2}(n) \right\| \le \\
&\quad \le q \left(1 + \frac{q}{L} \right) \max_{0 \le n \le T} \left\| x_{m-1}(n) - x_{m-2}(n) \right\| \le \cdots \le \\
&\quad \le q^{m-1} \left(1 + \frac{q}{L} \right)^{m-1} \max_{0 \le n \le T} \left\| x_1(n) - x_0(n) \right\|.
\end{aligned}
$$

Thus the sequence $\{x_m(i)\}$ forms a Cauchy sequence for each $i \in [0, T]$ and there is $x^*(\cdot)$ such that

$$
\lim_{m \to \infty} x_m(n) = x^*(n) \quad \text{for each} \quad n \in [0, T].
$$

Since B_r is closed, $x^*(n)$ is in B_r. From this and (6) it is clear that

$$\|\lambda_m - \lambda_{m-1}\| \leq \frac{q}{L}\left[q\left(1+\frac{q}{L}\right)\right]^{m-1} \max_{0\leq n\leq T} \|x_1(n) - x_0(n)\|.$$

So the sequence $\{\lambda_m\}$ Cauchy sequence and converges to some λ^* as $m \to \infty$. Take the limit as $m \to \infty$ in (3) and (4) we obtain that $x^*(n)$ satisfies problem (A).

Now, we prove the uniqueness of the solution of (A).

In fact if there exist two functions $x(\cdot), y(\cdot)$ and two parameters $\lambda, \overline{\lambda}$ satisfying the problem (A), then

$$\begin{aligned} x(n+1) &= x_0 + \sum_{s=0}^{n} f(s, x(s), \lambda) \\ y(n+1) &= x_0 + \sum_{s=0}^{n} f(s, y(s), \bar{\lambda}) \end{aligned} \tag{7}$$

and

$$x(0) = y(0) = x_0,\ x(T+1) = X = y(T+1). \tag{8}$$

From (7) and assumption (H_2) we have

$$\|x(n) - y(n)\| \leq \sum_{s=0}^{n} \beta(s)[\|x(s) - y(s)\| + \|\lambda - \overline{\lambda}\|]. \tag{9}$$

From (8)

$$\left\|\sum_{s=0}^{T} f(s, x(s), \lambda) - \sum_{s=0}^{T} f(s, y(s), \overline{\lambda})\right\| = 0.$$

Hence analogously to (6) we obtain that

$$\|\lambda - \bar{\lambda}\| \leq \frac{q}{L} \max_{0\leq s\leq T} \|x(s) - y(s)\|. \tag{9'}$$

Applying this inequality to (9) we can show that

$$\max_{0\leq n\leq T} \|x(n) - y(n)\| \leq q\left(1+\frac{q}{L}\right) \max_{0\leq n\leq T} \|x(n) - y(n)\|,$$

a contradiction. So $x(n) = y(n)$ for $n \in [0, T]$ and by (9') $\lambda = \bar{\lambda}$. Thus the proof is complete. ∎

2.2. In this part we will consider problem (A) for the difference equation with delay.

Let $\tau \in N,\ \varphi : [-\tau, 0] \to B_r,\ \varphi(0) = x_0,\ f : [0, T] \times B_r \times C \to E$. Instead of the (1) and (2) we take the equation

$$\Delta x(n) = f(n, x(n-\tau), \lambda), \quad n \in [0, T], \tag{10}$$

with the boundary conditions

$$x(n) = \varphi(n) \quad \text{for} \quad n \in [-\tau, 0] \tag{11}$$
$$x(T+1) = X.$$

Analogously to Theorem 1 we can prove the following theorem:

THEOREM 2.2. *Under the assumptions* $(H_1) - (H_4)$ *the problem* (A) *for(10) and (11) has exactly one solution.*

2.3. Now we consider the boundary value problem which is essentially related to the equation with delayed argument: there exist a solution $x(n, \lambda)$ of the problem

$$\Delta x(n) = f(n, x(n), x(n-\tau)), \quad n \in [0, T], \tag{12}$$

$$x(n) = \varphi(n, \lambda) \quad \text{for} \quad n \in [-\tau, 0] \tag{13}$$

(τ–is nonnegative integer) which satisfies the condition

$$x(T+1, \lambda^*) = X, \tag{14}$$

where (T, X) are given, λ^* will be definite.

We shall assume that the function

$$f : [0, T] \times E \times E \to E$$

satisfies the following conditions:

(G_1) there exists a function $\alpha : [0, T] \to R_+$ such that

$$\|f(n, x, y)\| \leq \alpha(n), \quad \sum_{s=0}^{T} \alpha(s) \leq r$$

for every $n \in [0, T],\ x, y \in B_r$,

(G_2) there exists a function $\beta : [0, T] \to R_+$ such that

$$\|f(n, u_1, v_1) - f(n, u_2, v_2)\| \leq \beta(n)\{\|u_1 - u_2\| + \|v_1 - v_2\|\},$$

for every $n \in [0, T],\ u_i v_i \in B_r,\ i = 1, 2,$

$$\sum_{s=0}^{T} \beta(s) = q < \frac{1}{4},$$

(G_3) the function

$$\varphi : [-\tau, 0] \times C \to B_r, \quad \varphi(0, \lambda) = x_0,$$

is continuous,

(G_4) there exists a constant $L > 0$ such that

$$\left\| \sum_{s=0}^{\tau} \{f(s, x(s), \varphi(s-\tau, \lambda_1)) - f(s, x(s), \varphi(s-\tau, \lambda_2))\} \right\| \geq L\|\lambda_1 - \lambda_2\|$$

for $\lambda_1, \lambda_2 \in C$,

(G_5) for every $y \in B_r$ and $x : [0,T] \to B_r$ there exists $\lambda \in C$ such that

$$y + \sum_{s=0}^{\tau} f(s, x(s), \varphi(s-\tau, \lambda)) = X.$$

THEOREM 2.3. *Suppose that the conditions $(G_1) - (G_5)$ hold. Then the problem (12)-(14) has solution.*

PROOF. Let $x_0(\cdot) : [0,T] \to B_r,\ x_0(n) = x_0$ for every $n \in [0,T]$. Then, by condition (G_5) there exists $\lambda_0 \in C$ such that

$$x_0 + \sum_{s=0}^{\tau} f(s, x_0(s), \varphi(s-\tau, \lambda_0)) + \sum_{s=\tau+1}^{T} f(s, x_0(s), x_0(s-\tau)) = X,$$

because

$$y = x_0 + \sum_{s=\tau+1}^{T} f(s, x_0(s), x_0(s-\tau)) \in B_r.$$

We build a sequence of $x_1, x_2, \ldots$ according to the following law:

$$\begin{aligned} x_1(n+1) =& x_0 + \sum_{s=0}^{\tau} f(s, x_0(s), \varphi(s-\tau, \lambda_0)) \\ &+ \sum_{s=\tau+1}^{n} f(s, x_0(s), x_0(s-\tau)) = X, \end{aligned}$$

for $n \in [0,T]$,

$$x_1(n) = \varphi(n, \lambda_0) \quad \text{for} \quad n \in [-\tau, 0],$$

$$\begin{aligned} x_m(n+1) &= x_0 + \sum_{s=0}^{\tau} f(s, x_{m-1}(s), \varphi(s-\tau, \lambda_{m-1})) \\ &\quad + \sum_{s=\tau+1}^{n} f(s, x_{m-1}(s), x_{m-1}(s-\tau)), \quad n \in [0,T], \end{aligned} \tag{15}$$

$$x_m(n) = \varphi(n, \lambda_{m-1}) \quad \text{for} \quad n \in [-\tau, 0]. \tag{16}$$

Evidently, $x_1(n) \in B_r$ and $x_1(T+1) = X \in B_r$,

$$x_m(n) \in B_r \quad \text{and} \quad x_m(T+1) = X \in B_r. \tag{17}$$

Now, we shall prove that the sequences $\{x_m(\cdot)\}$ and $\{\lambda_m\}$ are convergent.

From (15), (16), (17) and assumptions $(G_1), (G_4)$ we have

$$\begin{aligned} 0 =& \sum_{s=0}^{\tau} \{f(s, x_{m-1}(s), \varphi(s-\tau, \lambda_{m-1})) - f(s, x_{m-2}(s), \varphi(s-\tau, \lambda_{m-2}))\} \\ &+ \sum_{s=\tau+1}^{T} \{f(s, x_{m-1}(s), x_{m-1}(s-\tau)) - f(s, x_{m-2}(s), x_{m-2}(s-\tau))\}, \end{aligned}$$

$$\begin{aligned}L\|\lambda_{m-1}-\lambda_{m-2}\| &\le \sum_{s=0}^{\tau}\beta(s)\|x_{m-1}(s)-x_{m-2}(s)\| \\ &\quad+\sum_{s=\tau+1}^{T}\beta(s)\{\|x_{m-1}(s)-x_{m-2}(s)\| \\ &\quad+\|x_{m-1}(s-\tau)-x_{m-2}(s-\tau)\|\} \\ &\le 3\sum_{s=0}^{T}\beta(s)\max_{0\le s\le T}\|x_{m-1}(s)-x_{m-2}(s)\| \\ &= 3q\max_{0\le s\le T}\|x_{m-1}(s)-x_{m-2}(s)\|.\end{aligned}$$

Hence

$$\|\lambda_{m-1}-\lambda_{m-2}\| \le \frac{3q}{L}\max_{[0,T]}\|x_{m-1}(s)-x_{m-2}(s)\|. \tag{18}$$

By (15) and (18) we get

$$\begin{aligned}&\|x_m(n+1)-x_{m-1}(n+1)\| = \\ &= \Bigg\|\sum_{s=0}^{\tau}[f(s,x_{m-1}(s),\varphi(s-\tau,\lambda_{m-1}))-f(s,x_{m-2}(s),\varphi(s-\tau,\lambda_{m-2}))] \\ &\quad+\sum_{s=\tau+1}^{n}[f(s,x_{m-1}(s),x_{m-1}(s-\tau))-f(s,x_{m-2}(s),x_{m-2}(s-\tau))]\Bigg\| \\ &\le \sum_{s=\tau+1}^{T}\|f(s,x_{m-1}(s),x_{m-1}(s-\tau))-f(s,x_{m-2}(s),x_{m-2}(s-\tau))\| \\ &\quad+\sum_{s=\tau+1}^{n}\|f(s,x_{m-1}(s),x_{m-1}(s-\tau))-f(s,x_{m-2}(s),x_{m-2}(s-\tau))\| \\ &\le 4\sum_{s=\tau+1}^{T}\beta(s)\max_{0\le s\le T}\|x_{m-1}(s)-x_{m-2}(s)\| \\ &= q_1\max_{0\le s\le T}\|x_{m-1}(s)-x_{m-2}(s)\|,\end{aligned}$$

where $q_1=4q$. Let $V_m=\max_{0\le s\le T}\|x_m(s)-x_{m-1}(s)\|$, then

$$V_m\le q_1V_{m-1} \quad \text{for} \ \ m=2,3,\ldots$$

Thus, by induction, the inequality

$$V_m\le q_1^{m-1}V_1$$

is true for $m=2,3,\ldots$ Since by (G_2) $q_1<1$, we have that $\{x_m(n)\}$ is uniformly convergent to $x^*(n)$ for each $n\in[0,T]$ as $m\to\infty$.

This implies from (18) that

$$\|\lambda_m - \lambda_{m-1}\| \leq \frac{3q}{L} q_1^{m-1} V_1,$$

so $\{\lambda_m\}$ converges to λ^*. The relations (15), (16), (17) implies that $x^*(n)$ is a solution of the problem (12)-(14). $\square$

2.4. In this part we prove existence theorems for the problem (A). We will use the measure of noncompactness and weak noncompactness. We now gather together some definitions and results which we will be needed in this section.

Definition 2.4 (K. Kuratowski). *For any bounded subset A of a metric space E we denote by $\alpha(A)$ the infimum of all $\varepsilon > 0$, such that there exists a finite covering of A by sets of diameter $\leq \varepsilon$. The number $\alpha(A)$ is called the measure of noncompactness of set A. (For properties of α see* [9] *and their references.)*

Definition 2.5. *Let $B = \{x \in E : \|x\| \leq 1\}$ and let A be a bounded subset of E. The $\beta(A)$ measure of weak noncompactness of A is defined by*

$$\beta(A) = \inf\{t \geq 0 : A \subset K + tB \quad \text{for some weakly compact} \quad K \subset E\}.$$

(For properties of β see [7]*.)*

In particular if $\mu = \alpha$ or $\mu = \beta$, A and B are bounded subset of E, then

(1) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(2) $\mu(A \cup B) = \max(\mu(A), \mu(B))$,
(3) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(4) $\mu(e + A) = \mu(A)$ for $e \in E$,
(5) $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact,
(6) $\beta(A) = 0 \Leftrightarrow \overline{A}^w$ weakly compact.

Since $[0, T] = \{0, 1, \ldots, T\}$ is a discrete space, then any mapping of $[0, T]$ to a topological space E is continuous. We shall denote the set of all such mappings by $C([0, T], E)$. Let $\widetilde{B}_r = \{x : [0, T] \to B_r\}$.

Lemma 2.6 (Ambrosetti). *Let μ be a measure of strong (or weak) noncompactness, $V \subset C([0, T], E)$ be bounded. Then*

$$\mu(V) = \mu(V([0, T])) = \sup\{\mu(V(i)) : i \in [0, T]\},$$

where $V(i) = \{v(i) : v \in V\}$,

$$V([0, T]) = \bigcup_{i=0}^{T} V(i) = V(0) \cup V(1) \cup \cdots \cup V(T),$$

$\mu(V)$ denotes the measure of strong (or weak) noncompactness in $C([0, T], E)$.

Our results will be proved by the following fixed point theorem of Darbo [2]:

Theorem 2.7. *Let D be a nonempty, closed, convex and bounded subset of a Banach space E. Let $G : D \to D$ be a continuous mapping, which is condensing with respect to the measure of weak noncompactness α i.e.*

$$\alpha(F(V)) < \alpha(V)$$

for $\alpha(V) > 0, V \subset D$. Then G has a fixed point.

Theorem 2.8. *Let C be subset of F and $f : [0,T] \times B_r \times C \to E$ be continuous and satisfies H_1, and*

(H_4') *for every $x \in \widetilde{B}_r$ their exists $\lambda(x) \in C$ such that*

$$x_0 + \sum_{s=0}^{T} f(s, x(s), \lambda(x)) = X$$

and $\lambda(x)$ is continuous with respect x,

(H_5)

$$\alpha(f(n, V, C) \leq k\alpha(V)$$

for each $n \in [0,T]$ and for every subset $V \subset B_r$, with $Tk < 1$.

Then the problem (A) has at last one solution.

Proof. By H_4' for each function $x \in \widetilde{B}_r$ there exists $\lambda(x)$ such that

$$x_0 + \sum_{s=0}^{T} f(s, x(s), \lambda(x)) = X.$$

We define the operator G as follows:

$$(Gx)(i+1) = x_0 + \sum_{s=0}^{i} f(s, x(s), \lambda(x)).$$

By our assumptions $G : \widetilde{B}_r \to \widetilde{B}_r$ is continuous and the fixed point of G is a solution of the problem (A).

By the properties of α and by Ambrosetti lemma we have

$$\alpha((GV)(i+1)) \leq \alpha\left(x_0 + \sum_{s=0}^{i} f(s, V(s), C)\right) \leq$$
$$\leq \sum_{s=0}^{i} \alpha(f(s, V(s), C)) \leq \sum_{s=0}^{i} k\alpha(V(s)) \leq kT\alpha(V)$$

so $\alpha(G(V)) \leq kT\alpha(V)$.

By the Darbo fixed point theorem G has a fixed point. ∎

Remark 2.9. Using the above method and Darbo theorem of β- measure of weak noncompactness (see [3, Thm. 2]) we can prove analogical theorem under conditions expressed in terms of measure of weak noncompactness and weakly-weakly sequentically continuity.

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Received: 06.09.2002.
Revised: 21.01.2003.