A GENERALIZATION OF A RESULT ON MAXIMUM MODULUS OF POLYNOMIALS

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Abstract. For an arbitrary entire function $f(z)$, let

$$M(f, d) = \max_{|z|=d} |f(z)|.$$

It is known that if the geometric mean of the moduli of the zeros of a polynomial $p(z)$ of degree $n$ is at least 1, and $M(p, 1) = 1$, then for $R > 1$

$$M(p, R) \leq \left\{ \begin{array}{ll}
\frac{R}{2} + \frac{1}{2}, & \text{if } n = 1, \\
\frac{R^n}{n} + \frac{(n+2)nR^n - 2}{2} & \text{if } n \geq 2.
\end{array} \right.$$  

We have obtained a generalization of this result, by assuming the geometric mean of the moduli of the zeros of the polynomial to be at least $k$, ($k > 0$).

1. Introduction and statement of result

For a polynomial $p(z)$ of degree $n$, we have, as a simple consequence [4, Part III, Chapter 6, Problem no. 269] of maximum modulus principle

Theorem 1.1. If $p(z)$ is a polynomial of degree $n$ such that $M(p, 1) = 1$, then for $R > 1$

$$M(p, R) \leq R^n.$$  

Equality holds in (1.1) for $p(z) = az^n$, with $|a| = 1$.

Ankeny and Rivlin [1] considered a restricted class of polynomials and obtained the following refinement

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THEOREM 1.2. If the moduli of the zeros of a polynomial \( p(z) \) of degree \( n \) are all \( \geq 1 \) and \( M(p, 1) = 1 \), then for \( R > 1 \)

\[
M(p, R) \leq \frac{R^n + 1}{2}.
\]

Equality holds in (1.2) for \( p(z) = (bz^n + d)/2 \), with \( |b| = |d| = 1 \).

Frappier and Rahman [3] in a somewhat different context, obtained a similar type of result for a broader class of polynomials and proved

THEOREM 1.3. If the geometric mean of the moduli of the zeros of a polynomial \( p(z) \) of degree \( n \) is at least 1 and \( M(p; 1) = 1 \), then for \( R > 1 \)

\[
M(p, R) \leq \left\{
\begin{array}{ll}
\frac{R^n}{2} + \frac{1}{2}, & n = 1, \\
\frac{R^n}{2} + \frac{(3+2\sqrt{2})R^{n-2}}{2}, & n \geq 2.
\end{array}
\right.
\]

In this note, we have obtained a generalization of Theorem 1.3, by assuming the geometric mean of the moduli of the zeros of the polynomial \( p(z) \) to be at least \( k \), \((k > 0)\). More precisely, we prove

THEOREM 1.4. If the geometric mean of the moduli of the zeros of a polynomial \( p(z) \) of degree \( n \) is at least \( k \), \((k > 0)\), and \( M(p, 1) = 1 \), then for \( R > 1 \)

\[
M(p, R) \leq \left\{
\begin{array}{ll}
\frac{R^n}{1+k^n} + k^{1+k^n}, & n = 1, \\
\frac{R^n}{1+k^n} + \frac{R^{n-2}}{4} \left[(5+k^n) + \frac{1}{1+k^n} \sqrt{D}\right], & n \geq 2.
\end{array}
\right.
\]

where

\[
D = k^{4n} + 4k^{3n} + 30k^{2n} + 52k^n + 41.
\]

Equality holds in (1.3) for \( p(z) = (z+k)/(1+k) \).

2. LEMMAS

For the proof of the theorem, we require following lemmas.

**LEMMA 2.1.** If \( p(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial of degree \( n \) such that \( M(p, 1) = 1 \), then

\[
|a_0| + |a_n| \leq 1.
\]

This lemma is due to Visser [5].

**LEMMA 2.2.** If \( p(z) = \sum_{k=0}^{n} a_k z^k \) is a polynomial of degree \( n \) such that \( M(p, 1) = 1 \), then

\[
2|a_0| \cdot |a_n| + \sum_{k=0}^{n} |a_k|^2 \leq 1.
\]

This lemma is due to van der Corput and Visser [2].
3. Proof of Theorem 1.4

If

\[ p(z) = a_0 + a_1 z, \]

then

\[ \frac{M(p, R)}{M(p, 1)} = \frac{|a_0| + |a_1|R}{|a_0| + |a_1|} \leq \frac{R + k}{1 + k}, \]

thereby proving the theorem for this particular case. Therefore we now assume that

\[ n \geq 2, \]

and

\[ p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0, \]

As the geometric mean of the moduli of the zeros of the polynomial is at least \( k \), we have

\[ |a_0| \geq k^n |a_n|, \]

and therefore, by Lemma 2.1

\[ \alpha := |a_n| \leq \frac{1}{1 + k^n}. \]

Further, by Lemma 2.2, we have

\[ (|a_0| + |a_n|)^2 + |a_{n-1}|^2 \leq 1, \]

which, by (3.2) and (3.3), implies

\[ (k^n \alpha + \alpha)^2 + |a_{n-1}|^2 \leq 1, \]

i.e.

\[ |a_{n-1}| \leq \sqrt{1 - \alpha^2 (1 + k^n)^2}. \]

Using (3.3) and (3.4), we can now say that

\[ |a_n z^n + a_{n-1} z^{n-1}| \leq \alpha |z|^n + |z|^{n-1} \sqrt{1 - \alpha^2 (1 + k^n)^2} \]

\[ \leq \frac{1}{1 + k^n} |z|^n + \frac{(1 + k^n) + \alpha (1 + k^n)^2}{4} |z|^{n-2}, \]

by (3.3). And, by (3.1)

\[ r(z) = p(z) - a_n z^n - a_{n-1} z^{n-1} \]

is a polynomial, of degree at most \( n - 2 \), with

\[ M(r, 1) \leq 1 + \alpha + \sqrt{1 - \alpha^2 (1 + k^n)^2}, \]

(by (3.3) and (3.4)), thereby implying, by Theorem 1.1, for \( R > 1 \)

\[ M(r, R) \leq \left[ 1 + \alpha + \sqrt{1 - \alpha^2 (1 + k^n)^2} \right] R^{n-2}. \]
Hence, by (3.1) and (3.5), we have, for \( R > 1 \)
\[
M(p, R) \leq \frac{R^n}{1 + k^n} + \left[ \frac{5 + k^n}{4} + \alpha \left\{ 1 + \frac{(1 + k^n)^2}{4} \right\} \right] + \sqrt{\{1 - \alpha^2(1 + k^n)^2\}} R^{n-2},
\]
from which, the inequality (1.32) follows, on finding the maximum value of the function
\[
\phi(\alpha) = \alpha \left\{ 1 + \frac{(1 + k^n)^2}{4} \right\} + \sqrt{\{1 - \alpha^2(1 + k^n)^2\}},
\]
on the interval \([0, 1/(1 + k^n)]\). This completes the proof of Theorem 1.4.

REFERENCES