THE DEMEYER-KANZAKI GALOIS EXTENSION AND ITS
SKEW GROUP RING

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Abstract. Several characterizations are given for a ring $B$ being a DeMeyer-Kanzaki Galois extension with Galois group $G$ in terms of the skew group ring $B \rtimes G$. Consequently, the results of S. Ikehata on commutative Galois algebras are generalized.

1. Introduction

In [5], the class of commutative Galois algebras $B$ with Galois group $G$ was characterized in terms of the Azumaya skew group ring $B \rtimes G$ over $B^G$ and the $H$-separable skew group ring $B \rtimes G$ of $B$ respectively, where $B^G = \{a \in B \mid g(a) = a \text{ for all } g \in G\}$. In [3], a broader class of DeMeyer-Kanzaki Galois extensions $B$ with Galois group $G$ was investigated where $B$ is called a DeMeyer-Kanzaki Galois extension with Galois group $G$ if $B$ is an Azumaya algebra over its center $C$ and $C$ is a Galois algebra with Galois group induced by and isomorphic with $G$. Further generalizations to Azumaya Galois extensions and to Hopf Azumaya Galois extensions were also given (see [2, 7]). The purpose of the present paper is to generalize the characterizations of a commutative Galois algebra $B$ in terms of the skew group ring $B \rtimes G$ as given by S. Ikehata (see [5]). We shall show the following equivalent statements:

$(1)$ $B$ is a DeMeyer-Kanzaki Galois extension of $B^G$ with Galois group $G$. 

2000 Mathematics Subject Classification. 16S35, 16W20.

Key words and phrases. Galois extensions, DeMeyer-Kanzaki Galois extensions, commutative Galois algebras, Azumaya algebras, $H$-separable extensions, skew group rings.

This work was done under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank Caterpillar Inc. for the support.

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(2) The skew group ring $B \ast G$ is an Azumaya $C^G$-algebra and $C$ is a maximal commutative separable subalgebra of $V_{B \ast G}(B^G)$, the commutator subring of $B^G$ in $B \ast G$, over $C^G$.

(3) The skew group ring $B \ast G$ is an $H$-separable extension of $B$ (= the Harata separable), $B$ is a separable algebra over $C^G$, and $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \neq 1$ in $G$ where $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$.

(4) $B$ is a separable $C^G$-algebra, $C^G$ is a direct summand of $C$ as a $C^G$-submodule, and $C \otimes_{C^G} (B \ast G) \cong M_n(B)$ where $M_n(B)$ is the matrix ring of order $n$ over $B$ and $n$ is the order of $G$.

(5) $B$ is a separable $C^G$-algebra, $C^G$ is a direct summand of $C$ as a $C^G$-submodule, and $C \otimes_{C^G} V_{B \ast G}(B^G) \cong M_n(C)$ where $M_n(C)$ is the matrix ring of order $n$ over $C$ and $n$ is the order of $G$.

2. Basic definitions and notations

Throughout, $B$ will represent a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$ of order $n$ for some integer $n$, $B^G$ the set of elements fixed under each element in $G$, and $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$. For a subring $A$ of $B$ with the same identity 1, we denote the commutator subring of $A$ in $B$ by $V_B(A)$. Following the definitions given in [10], we call $B$ a separable extension of $A$ if there exist \{a_i, b_i \in B, i = 1, 2, \ldots, m\} such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b \in B$ where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. A ring $B$ is called an $H$-separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule. $B$ is called a Galois extension of $B^G$ with Galois group $G$ if there exist elements \{a_i, b_i \in B, i = 1, 2, \ldots, m\} such that $\sum_{i=1}^m a_i g(b_i) = \delta_{i,g}$ for each $g \in G$. A Galois extension $B$ with Galois group $G$ is called an Azumaya Galois extension if $B^G$ is an Azumaya algebra over $C^G$ (see [2, 7]), and a DeMeyer-Kanzaki Galois extension if $B$ is an Azumaya algebra over $C$ which is a Galois algebra over $C^G$ with Galois group induced by and isomorphic with $G$ (see [3, 6]).

Let $P$ be a finitely generated and projective module over a commutative ring $R$. Then for a prime ideal $p$ of $R$, $P_p := P \otimes_R R_p$ is a free module over $R_p = (\text{the local ring of } R \text{ at } p)$, and the rank of $P$ over $R_p$ is the number of copies of $R_p$ in $P_p$, that is, $\text{rank}_{R_p}(P_p) = m$ for some integer $m$. It is known that the rank$_R(P)$ is a continuous function (rank$_R(P)(p) = m$) from Spec$(R)$ to the set of nonnegative integers with the discrete topology (see [4, Corollary 4.11, page 31]). We shall use the rank$_R(P)$-function for a finitely generated and projective module $P$ over a commutative ring $R$. 
3. Characterizations

In this section, keeping all notations as given in section 2, we shall generalize the characterizations of a commutative Galois algebra as given by S. Ikehata (see [5]) to a DeMeyer-Kanzaki Galois extension $B$ with Galois group $G$ in terms of the skew group ring $B \ast G$. We begin with an equivalent condition for a commutative Galois algebra $C$ with Galois group $G$.

**Theorem 3.1.** Let $C$ be a commutative ring with a finite automorphism group $G$. Then, $C$ is a commutative Galois algebra with Galois group $G$ if and only if $C^G$ is a direct summand of $C$ as a $C^G$-submodule, and $C \otimes_{C^G} (C \ast G) \cong M_n(C)$.

**Proof.** $(\Longrightarrow)$ By Corollary 1.3 on page 85 in [4], $C^G$ is a direct summand of $C$ as a $C^G$-submodule, and that $C \otimes_{C^G} (C \ast G) \cong M_n(C)$ is a consequence of Theorem 2 in [5].

$(\Longleftarrow)$ Since $C \otimes_{C^G} (C \ast G) \cong M_n(C)$, $C \otimes_{C^G} (C \ast G)$ is an Azumaya algebra over $C$. But $C^G$ is a direct summand of $C$ as a $C^G$-submodule by hypothesis, so $C \ast G$ is an Azumaya $C^G$-algebra (see [4, Corollary 1.10, page 45]). Hence $C$ is a commutative Galois algebra with Galois group $G$ (see [5, Theorem 2]).

Next we characterize a DeMeyer-Kanzaki Galois extension $B$ in terms of the skew group ring $B \ast G$.

**Theorem 3.2.** The following statements are equivalent:

1. $B$ is a DeMeyer-Kanzaki Galois extension of $B^G$ with Galois group $G$.
2. The skew group ring $B \ast G$ is an Azumaya $C^G$-algebra and $C$ is a maximal commutative separable subalgebra of $V_{B \ast G}(B^G)$ over $C^G$.
3. The skew group ring $B \ast G$ is an $H$-separable extension of $B$, $B$ is a separable algebra over $C^G$, and $J_g = \{0\}$ for each $g \neq 1$ in $G$.

**Proof.** $(1) \implies (2)$ Since $B$ is a DeMeyer-Kanzaki Galois extension of $B^G$ with Galois group $G$, $B \cong B^G \otimes_{C^G} C$ such that $B^G$ is an Azumaya $C^G$-algebra (see [3, Lemma 2]). Hence $B$ is an Azumaya Galois extension with Galois group $G$, and so $B \ast G$ is an Azumaya $C^G$-algebra (see [2, Theorem 1]). Moreover, $C$ is a commutative Galois algebra with Galois group $G$ by hypothesis, so $C$ is a maximal commutative separable subalgebra of $C \ast G$ over $C^G$ (see [5, Theorem 2]). But $V_{B \ast G}(B^G) = V_B(B^G) \ast G = C \ast G$, so $C$ is a maximal commutative separable subalgebra of $V_{B \ast G}(B^G)$ over $C^G$.

$(2) \implies (1)$ Since $B \ast G$ is an Azumaya $C^G$-algebra, $B$ is an Azumaya Galois extension with Galois group $G$ (see [2, Theorem 1]). Hence $V_B(B^G)$ is a Galois algebra over $C^G$ with Galois group $G$ (see [1, Theorem 2]). Thus $V_B(B^G) \ast G \cong \text{Hom}_{C^G}(V_B(B^G), V_B(B^G))$. But $C$ is a maximal commutative separable subalgebra of $V_B(B^G) \ast G (= V_{B \ast G}(B^G))$ over $C^G$ by hypothesis,
so by the proof of Theorem 5.5 on page 64 in [4],

\[ C \otimes_{C^G} (V_B(B^G) \star G) \cong \text{Hom}_C(V_B(B^G) \star G, V_B(B^G) \star G). \]

Then we have

\[
\begin{align*}
\text{Hom}_C(V_B(B^G) \star G, V_B(B^G) \star G) & \cong \\
& \cong C \otimes_{C^G} (V_B(B^G) \star G) \\
& \cong C \otimes_{C^G} \text{Hom}_{C^G}(V_B(B^G), V_B(B^G)) \\
& \cong \text{Hom}_C(C \otimes_{C^G} V_B(B^G), C \otimes_{C^G} V_B(B^G)).
\end{align*}
\]

Thus \( V_B(B^G) \star G \cong (C \otimes_{C^G} V_B(B^G)) \otimes_C P \) as a \( C \)-module for some finitely generated and projective \( C \)-module \( P \) such that \( \text{rank}_C(P) = 1 \). Since the rank of a Galois algebra is the order of the Galois group, applying the rank function on both sides of the above isomorphism, we have that

\[
\text{rank}_C(V_B(B^G)) \cdot n = \text{rank}_C(V_B(B^G) \star G) = \text{rank}_C(C \otimes_{C^G} V_B(B^G)) = \text{rank}_{C^G}(V_B(B^G)) = n.
\]

This implies that \( \text{rank}_C(V_B(B^G)) = 1 \). Noting that \( V_B(B^G) \) is an Azumaya \( C \)-algebra and a finitely generated projective \( C^G \)-module, we conclude that \( V_B(B^G) = C \); and so \( C \) is a Galois algebra over \( C^G \) with Galois group \( G \). Consequently, \( B \) is a DeMeyer-Kanzaki Galois extension of \( B^G \) with Galois group \( G \) because \( B \) is also an Azumaya \( C \)-algebra.

(1) \( \implies \) (3) Since \( B \) is a DeMeyer-Kanzaki Galois extension of \( B^G \) with Galois group \( G \), \( B \cong B^G \otimes_{C^G} C \) such that \( B^G \) is an Azumaya \( C^G \)-algebra and \( C \) is a Galois algebra with Galois group induced by and isomorphic with \( G \) (see [3, Lemma 2]). Hence \( B \star G \) is an \( H \)-separable extension of \( B \) (see [9, Lemma 3.1 and Theorem 3.2]) and \( B \) is a separable algebra over \( C^G \). Noting that \( V_B(B^G) = C = J_1 \) and that \( V_B(B^G) = \oplus_{g \in G} J_g \) (see [6, Proposition 1]), we conclude that \( J_g = \{0\} \) for each \( g \neq 1 \) in \( G \).

(3) \( \implies \) (1) Since \( B \) is a separable algebra over \( C^G \), \( B \) is an Azumaya algebra over \( C \). Next we claim that \( C \) is a Galois algebra with Galois group induced by and isomorphic with \( G \). In fact, since \( B \ast G \) is an \( H \)-separable extension of \( B \) by hypothesis and \( B \) is a direct summand of \( B \ast G \) as a left (or right) \( B \)-module, \( V_{B \ast G}(V_{B \ast G}(B)) = B \) (see [8, Proposition 1.2]). This implies that the center of \( B \ast G \) is \( C^G \). Moreover, \( B \) is a separable algebra over \( C^G \), so \( B \ast G \) is a separable algebra over \( C^G \) by the transitivity of separable extensions. Thus \( B \ast G \) is an Azumaya \( C^G \)-algebra; and so \( B \) is an Azumaya Galois extension with Galois group \( G \) (see [2, Theorem 1]). Therefore \( V_B(B^G) \) is a Galois algebra over \( C^G \) with Galois group induced by and isomorphic with \( G \) (see [1, Theorem 2]). But then, by Proposition 1 in [6], \( V_B(B^G) = \oplus_{g \in G} J_g \). Since \( J_g = \{0\} \) for each \( g \neq 1 \) in \( G \) by hypothesis, so \( V_B(B^G) = J_1 = C \). This proves that \( C \) is a Galois algebra with Galois group induced by and isomorphic with \( G \). Thus statement (1) holds.
By generalizing Theorem 3.1, we obtain another two characterizations of a DeMeyer-Kanzaki Galois extension.

**Theorem 3.3.** The following statements are equivalent:

1. \( B \) is a DeMeyer-Kanzaki Galois extension of \( B^G \) with Galois group \( G \).
2. \( B \) is a separable \( C^G \)-algebra, \( C^G \) is a direct summand of \( C \) as a \( C^G \)-submodule, and \( C \otimes_{C^G} (B \ast G) \cong M_n(B) \).
3. \( B \) is a separable \( C^G \)-algebra, \( C^G \) is a direct summand of \( C \) as a \( C^G \)-submodule, and \( C \otimes_{C^G} V_{B \ast G}(B^G) \cong M_n(C) \).

**Proof.** (1) \( \implies \) (2) Since \( B \) is a DeMeyer-Kanzaki Galois extension of \( B^G \) with Galois group \( G \), \( B \cong B^G \otimes_{C^G} C \) where \( B^G \) is an Azumaya \( C^G \)-algebra and \( C \) is a Galois algebra with Galois group induced by and isomorphic with \( G \) (see [3, Lemma 2]). Hence \( C^G \) is a direct summand of \( C \) as a \( C^G \)-submodule (see [4, Corollary 1.3, page 85]), and \( V_{B \ast G}(B^G) = C \ast G \) such that \( C \otimes_{C^G} (C \ast G) \cong M_n(C) \) (see [5, Theorem 2]); and so

\[
C \otimes_{C^G} (B \ast G) \cong C \otimes_{C^G} (B^G \otimes_{C^G} C \ast G) \cong C \otimes_{C^G} (C \ast G) \otimes_{C^G} B^G \\
\cong M_n(C) \otimes_{C^G} B^G \cong M_n(B).
\]

(2) \( \implies \) (1) Since \( B \) is a separable \( C^G \)-algebra, \( B \) is an Azumaya algebra over \( C \). Moreover, \( M_n(B) \cong B \otimes_C M_n(C) \), so \( M_n(B) \) is a Galois algebra over \( C \). By hypothesis, \( C \otimes_{C^G} (B \ast G) \cong M_n(B) \), so \( C \otimes_{C^G} (B \ast G) \) is an Azumaya algebra over \( C \). But \( C \) contains \( C^G \) as a direct summand as a \( C^G \)-submodule by hypothesis, so \( B \ast G \) is an Azumaya \( C^G \)-algebra (see [4, Corollary 1.10, page 45]). Hence \( B \) is an Azumaya Galois extension with Galois group \( G \) (see [2, Theorem 1]). Thus \( V_B(B^G) \) is a Galois algebra over \( C^G \) with Galois group \( G \) (see [1, Theorem 2]). Therefore both \( B \) and \( B^G \cdot V_B(B^G) \) are Galois extensions of \( B^G \) with Galois group \( G \) such that \( B^G \cdot V_B(B^G) \subset B \). This implies that \( B = B^G \cdot V_B(B^G) \) such that \( V_B(B^G) \) is a Galois algebra over \( C^G \) with Galois group \( G \); and so \( V_B(B^G) \) an Azumaya \( C \)-algebra and both \( V_B(B^G) \) and \( C \) are finitely generated projective modules over \( C^G \).

Next we claim that \( V_B(B^G) = C \). In fact, since \( C \otimes_{C^G} (B \ast G) \cong M_n(B) \), rank_{C^G}(B \ast G) = rank_C(M_n(B)). This implies that rank_{C^G}(C) \cdot rank_C(B) \cdot n = rank_C(M_n(B)). But V_B(B^G) is a Galois algebra over C^G with Galois group G, so rank_{C^G}(V_B(B^G)) = n. Therefore rank_{C^G}(V_B(B^G)) = n = rank_C(M_n(B)). Noting that V_B(B^G) is an Azumaya C-algebra and a finitely generated projective C^G-module, we conclude that V_B(B^G) = C; and so C is a Galois algebra with Galois group induced by and isomorphic with G. Consequently, \( B \) is a DeMeyer-Kanzaki Galois extension with Galois group \( G \).

(1) \( \iff \) (3) The proof is similar to (1) \( \iff \) (2). \( \square \)
REFERENCES


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Received: 06.09.2002.