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ON 2-NORMED SETS

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Abstract. In this paper we will consider properties of the 2-normed sets and we will construct two 2-normed sets of linear operators.

1. Introduction

In [2] S. Gähler introduced the following definition of a 2-normed space:

Definition 1.1 ([2]). *Let X be a real linear space of dimension greater than 1 and let $\| \cdot , \cdot \|$ be a real valued function on $X \times X$ satisfying the following four properties:*

(G1) $\|x, y\| = 0$ *if and only if the vectors x and y are linearly dependent;*
(G2) $\|x, y\| = \|y, x\|$*;*
(G3) $\|x, \alpha y\| = |\alpha| \cdot \|x, y\|$ *for every real number α;*
(G4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for every $x, y, z \in X$.*

The function $\| \cdot , \cdot \|$ will be called a 2-norm on X and the pair $(X, \| \cdot , \cdot \|)$ - a linear 2-normed space.

In [4] and [5] we gave a generalization of the Gähler's 2-normed space. Namely a generalized 2-norm need not be symmetric and satisfy the first condition of the above definition.

Definition 1.2 ([4]). *Let X and Y be real linear spaces. Denote by $\mathcal{D}$ a non-empty subset $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets*

$$\mathcal{D}_x = \{y \in Y; (x, y) \in \mathcal{D}\} \text{ and } \mathcal{D}^{\,y} = \{x \in X; (x, y) \in \mathcal{D}\}$$

are linear subspaces of the space Y and X, respectively.

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A function $\|\cdot,\cdot\|\colon \mathcal{D} \to [0,\infty)$ *will be called a generalized 2-norm on* $\mathcal{D}$ *if it satisfies the following conditions:*

(N1) $\|x,\alpha y\| = |\alpha|\cdot\|x,y\| = \|\alpha x,y\|$ *for any real number* α *and all* $(x,y) \in \mathcal{D}$*;*

(N2) $\|x,y+z\| \le \|x,y\|+\|x,z\|$ *for* $x \in X,\ y,z \in Y$ *such that* $(x,y),\ (x,z) \in \mathcal{D}$*;*

(N3) $\|x+y,z\| \le \|x,z\|+\|y,z\|$ *for* $x,y \in X,\ z \in Y$ *such that* $(x,z),\ (y,z) \in \mathcal{D}$.

The set $\mathcal{D}$ *is called a 2-normed set.*

In particular, if $\mathcal{D} = X \times Y$*, the function* $\|\cdot,\cdot\|$ *will be called a generalized 2-norm on* $X \times Y$ *and the pair* $(X \times Y, \|\cdot,\cdot\|)$ *a generalized 2-normed space.*

Moreover, if $X = Y$*, then the generalized 2-normed space will be denoted by* $(X, \|\cdot,\cdot\|)$.

Assume that the generalized 2-norm satisfies, in addition, the symmetry condition. Then we will define the 2-norm as follows:

Definition 1.3 ([4]). *Let* X *be a real linear space. Denote by* $\mathcal{X}$ *a non-empty subset* $X \times X$ *with the property* $\mathcal{X} = \mathcal{X}^{-1}$ *and such that the set* $\mathcal{X}^y = \{x \in X; (x,y) \in \mathcal{X}\}$ *is a linear subspace of* X*, for all* $y \in X$.

A function $\|\cdot,\cdot\|\colon \mathcal{X} \to [0,\infty)$ *satisfying the following conditions:*

(S1) $\|x,y\| = \|y,x\|$ *for all* $(x,y) \in \mathcal{X}$*,*

(S2) $\|x,\alpha y\| = |\alpha|\cdot\|x,y\|$ *for any real number* α *and all* $(x,y) \in \mathcal{X}$*,*

(S3) $\|x,y+z\| \le \|x,y\|+\|x,z\|$ *for* $x,y,z \in X$ *such that* $(x,y),\ (x,z) \in \mathcal{X}$*,*

will be called a generalized symmetric 2-norm on $\mathcal{X}$*. The set* $\mathcal{X}$ *is called a symmetric 2-normed set. In particular, if* $\mathcal{X} = X \times X$*, the function* $\|\cdot,\cdot\|$ *will be called a generalized symmetric 2-norm on* X *and the pair* $(X, \|\cdot,\cdot\|)$ *- a generalized symmetric 2-normed space.*

In this paper we give some properties of certain 2-normed sets.

2. A 2-normed set

A 2-normed set has interesting properties, for example it can be a small in some sense and a big in other one. We will show them in this section.

Example 2.1. ([4]) Let s be a linear space of all sequences of real numbers. Let

$$\|x,y\| = \sum_{n=1}^{\infty} |\xi_n|\cdot|\eta_n|$$

for all $x,y \in s$ such that $x = \{\xi_n; n \in N\}, y = \{\eta_n; n \in N\}$. Then $\|\cdot,\cdot\|\colon s\times s \to [0,\infty]$. By $\mathcal{X}$ we will denote the set $\{(x,y) \in s \times s;\ \|x,y\| < \infty\}$. Thus we have the properties:

(1) $\mathcal{X} = \mathcal{X}^{-1}$;
(2) $\mathcal{X}^y$ is a linear subspace of s for every $y \in s$.

Then the set $\mathcal{X}$ is a symmetric 2-normed set and the function $\| \cdot , \cdot \| \colon \mathcal{X} \to [0, \infty)$ is a generalized symmetric 2-norm on $\mathcal{X}$.

In the sequel s denotes the linear space of all real sequences with the usual metric ϱ given by

$$\varrho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}$$

for $x = \{\xi_n; n \in N\} \in s, y = \{\eta_n; n \in N\} \in s$. Then the function d by the formula $d((x, y), (x^{'}, y^{'})) = \varrho(x, x^{'}) + \varrho(y, y^{'})$ for $x, x^{'}, y, y^{'} \in s$, is a metric in $s \times s$.

Theorem 2.2. *The 2-normed set $\mathcal{X}$ is a dense F_σ- set of the first Baire category in the space $(s \times s, d)$.*

Proof. At first we will show that $\mathcal{X}$ is dense in $(s \times s, d)$. Consider the set

$$l^2 = \left\{ x = \{\xi_j; j \in N\} \in s;\ \sum_{j=1}^{\infty} |\xi_j|^2 < \infty \right\}.$$

Let $x = \{\xi_j; j \in N\} \in l^2$ and $y = \{\eta_j; j \in N\} \in l^2$. Obviously $(x, y) \in l^2 \times l^2$. By the well-known Hölder's inequality we have

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{j=1}^{\infty} |\xi_j|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}} < \infty,$$

and in the consequence

$$l^2 \times l^2 \subset \mathcal{X}. \tag{2.1}$$

Because l^2 is dense in s, (it follows from Theorem 2.1 in [1]), therefore $l^2 \times l^2$ is dense in $s \times s$. Using the inclusion (2.1) we see that $\mathcal{X}$ is dense in $s \times s$, too.

We shall prove that $\mathcal{X}$ is an F_σ - set in $s \times s$. For $k, n \in N$ we put

$$A_{kn} = \left\{ (x, y) \in s;\ \sum_{j=1}^{n} |\xi_j \eta_j| > k \right\}.$$

Then

$$s \times s \setminus \mathcal{X} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_{kn}. \tag{2.2}$$

Now, we shall show that A_{kn}, $(k,n\in N)$, is open in $s\times s$. Let $(x_o,y_o)\in A_{kn}$, where $x_o=\{\xi_j^o; j\in N\}$ and $y_o=\{\eta_j^o; j\in N\}$. Choose $\varepsilon>0$ in such a way that

$$\sum_{j=1}^{n}|\xi_j^o\eta_j^o|-\varepsilon>k.$$

The function $\varphi\colon \mathcal{R}^n\times\mathcal{R}^n\to[0,\infty)$ defined as follows

$$\varphi(a,b)=\sum_{j=1}^{n}|\alpha_j\beta_j|, \text{ for } a=(\alpha_1,\ldots,\alpha_n), b=(\beta_1,\ldots,\beta_n),$$

is continuous at the point (a_o,b_o) such that $a_o=(\xi_1^o,\ldots,\xi_n^o)$, $b_o=(\eta_1^o,\ldots,\eta_n^o)$. Then there exists $\delta>0$ such that $|\varphi(a,b)-\varphi(a_o,b_o)|<\varepsilon$ when $\|a-a_o\|<\delta$ and $\|b-b_o\|<\delta$.

Let us take

$$r=\frac{1}{2^n}\cdot\frac{\frac{\delta}{\sqrt{n}}}{1+\frac{\delta}{\sqrt{n}}} \text{ and } \mathcal{K}^o\Big((x_o,y_o),r\Big)\subset s\times s.$$

For $(x,y)\in\mathcal{K}^o\Big((x_o,y_o),r\Big)$, where $x=\{\xi_j; j\in N\}, y=\{\eta_j; j\in N\}$, the following inequality is true: $d((x,y),(x_o,y_o))<r$. Thus $\varrho(x,x_o)<r$ and $\varrho(y,y_o)<r$. By the first inequality we have

$$\sum_{j=1}^{\infty}\frac{1}{2^j}\cdot\frac{|\xi_j-\xi_j^o|}{1+|\xi_j-\xi_j^o|}<\frac{1}{2^n}\cdot\frac{\frac{\delta}{\sqrt{n}}}{1+\frac{\delta}{\sqrt{n}}}.$$

Since

$$\frac{1}{2^j}\cdot\frac{|\xi_j-\xi_j^o|}{1+|\xi_j-\xi_j^o|}<\frac{1}{2^n}\cdot\frac{\frac{\delta}{\sqrt{n}}}{1+\frac{\delta}{\sqrt{n}}} \quad \text{for each } j=1,2,\ldots,n.$$

From this we get

$$\frac{|\xi_j-\xi_j^o|}{1+|\xi_j-\xi_j^o|}<\frac{2^j}{2^n}\cdot\frac{\frac{\delta}{\sqrt{n}}}{1+\frac{\delta}{\sqrt{n}}}<\frac{\frac{\delta}{\sqrt{n}}}{1+\frac{\delta}{\sqrt{n}}} \quad \text{for } j=1,2,\ldots,n.$$

Because the function $f(t)=\frac{t}{1+t}$ for $t\geq 0$, is an increasing function, then

$$|\xi_j-\xi_j^o|<\frac{\delta}{\sqrt{n}} \text{ for } j=1,2,\ldots,n.$$

By analogy we obtain the inequality

$$|\eta_j-\eta_j^o|<\frac{\delta}{\sqrt{n}} \text{ for } j=1,2,\ldots,n.$$

So for $a = (\xi_1, \ldots, \xi_n) \in \mathcal{R}^n, b = (\eta_1, \ldots, \eta_n) \in \mathcal{R}^n$ we have

$$\|a - a_o\| = \sqrt{\sum_{j=1}^{n} |\xi_j - \xi_j^o|^2} < \delta \text{ and } \|b - b_o\| < \delta.$$

As a consequence $|\varphi(a,b) - \varphi(a_o, b_o)| < \varepsilon$, i.e.

$$\left| \sum_{j=1}^{n} |\xi_j \eta_j| - \sum_{j=1}^{n} |\xi_j^o \eta_j^o| \right| < \varepsilon.$$

Thus

$$\sum_{j=1}^{n} |\xi_j \eta_j| > \sum_{j=1}^{n} |\xi_j^o \eta_j^o| - \varepsilon > k.$$

Hence $(x,y) \in A_{kn}$. And we have proved that $\mathcal{K}^o\Big((x_o, y_o), r\Big) \subset A_{kn}$, this means that A_{kn} is open. Using the equality (2.2) we see that $s \times s \setminus \mathcal{X}$ is an G_δ - set. Therefore $\mathcal{X}$ is an F_σ - set.

Finally, we shall show that $\mathcal{X}$ is a set of the first Baire category in $(s \times s, d)$. Let $r > 0$ and $(x_o, y_o) \in s \times s$, where $x_o = \{\xi_j^o; j \in N\}$ and $y_o = \{\eta_j^o; j \in N\}$. Then there exists an $n_o \in N$ such that

$$\sum_{j=n_o+1}^{\infty} \frac{1}{2^j} < \frac{r}{2}.$$

Choose $x_1 = \{\xi_j^1; j \in N\}$ and $y_1 = \{\eta_j^1; j \in N\}$ in the following way:

$$\xi_j^1 = \xi_j^o \text{ for } j = 1, 2, \ldots, n_o \text{ and } \xi_j^1 = 1 \text{ for } j \geq n_o + 1,$$

$$\eta_j^1 = \eta_j^o \text{ for } j = 1, 2, \ldots, n_o \text{ and } \eta_j^1 = 1 \text{ for } j \geq n_o + 1.$$

Then for each $k \in N$ there exists $n \in N$ such that

$$\sum_{j=1}^{n} |\xi_j^1 \eta_j^1| > k.$$

From this we get $(x_1, y_1) \in s \times s \setminus \mathcal{X}$. Further

$$\begin{aligned} d((x_1,y_1),(x_o,y_o)) &= \varrho(x_1,x_o) + \varrho(y_1,y_o) \\ &= \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_j^1 - \xi_j^o|}{1+|\xi_j^1 - \xi_j^o|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\eta_j^1 - \eta_j^o|}{1+|\eta_j^1 - \eta_j^o|} \\ &= \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_j^o - 1|}{1+|\xi_j^o - 1|} + \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} \cdot \frac{|\eta_j^o - 1|}{1+|\eta_j^o - 1|} \\ &< \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} + \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Hence $(x_1, y_1) \in \mathcal{K}^o\Big((x_o, y_o), r\Big)$. In the consequence $\mathcal{K}^o\Big((x_o, y_o), r\Big) \cap (s \times s \setminus \mathcal{X}) \neq \emptyset$. This means that $s \times s \setminus \mathcal{X}$ is dense in $s \times s$. Using the foregoing results we see that $\mathcal{X}$ is a set of the first Baire category in $(s \times s, d)$. This ends the proof. ∎

3. 2-normed sets of linear operators

Let X, Y be real linear spaces. The set $Y \times Y$ is the linear space with respect to the operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \text{ for all } x_1, x_2, y_1, y_2 \in Y$$

and

$$\alpha \cdot (x, y) = (\alpha x, \alpha y) \text{ for all } x, y \in Y, \alpha \in \mathcal{R}.$$

By $L(X, Y)$ denote the linear space of all linear operators from X with values in Y. It is easy to see that for each linear operator $F\colon X \to Y \times Y$ there exists a pair of operators $f, g \in L(X, Y)$ such that $F(x) = (f(x), g(x))$ for all $x \in X$. And conversely, the operator $F\colon X \to Y \times Y$ defined by the formula $F(x) = (f(x), g(x))$ for all $x \in X$, where $f, g \in L(X, Y)$, is linear.

Further we will consider properties of the set of these pairs of linear operators satisfying certain additional conditions.

Definition 3.1. *Let X be a real normed space and $\mathcal{Y} \subset Y \times Y$ be a 2-normed set, where Y denotes a real linear space. A set $\mathcal{M}$ is defined as follows:*

$$\begin{aligned} \mathcal{M} = \{&(f,g) \in L(X,Y)^2; \\ &\forall x \in X \ (f(x), g(x)) \in \mathcal{Y} \wedge \exists M > 0 \ \forall x \in X \ \|f(x), g(x)\| \leq M \cdot \|x\|^2\}. \end{aligned}$$

Lemma 3.2. *The set $\mathcal{M}$ defined in Definition 3.1 has the following property:*

(a) *If $\mathcal{Y}$ is a symmetric 2-normed set, then $\mathcal{M} = \mathcal{M}^{-1}$.*

(b) *For every* $f, g \in L(X,Y)$ *sets* $\mathcal{M}^g = \{f^{'} \in L(X,Y); (f^{'}, g) \in \mathcal{M}\}$ *and* $\mathcal{M}_f = \{g^{'} \in L(X,Y); (f, g^{'}) \in \mathcal{M}\}$ *are linear subspaces of the space* $L(X,Y)$. *In the case when* $\mathcal{Y}$ *is a symmetric 2-normed set we have the equality* $\mathcal{M}^f = \mathcal{M}_f$.

Proof. The condition (a) follows from the definition of the set $\mathcal{M}$.

(b) Let g be a linear operator from X in Y. First, we shall show that $\mathcal{M}^g$ is non- empty. Consider the linear operator $f_o \colon X \to Y$ defined by the formula $f_o(x) = 0$ for all $x \in X$. Because for each $x \in X$ the set $\mathcal{Y}^{\,g(x)}$ is linear subspace of Y, then $f_o(x) = 0 \in \mathcal{Y}^{\,g(x)}$. Thus $(f_o(x), g(x)) \in \mathcal{Y}$ for all $x \in X$. Moreover for each positive number M and for all $x \in X$ the inequality $\|f_o(x), g(x)\| = \|0, g(x)\| = 0 \le M \cdot \|x\|^2$ is satisfied. In the consequence $(f_o, g) \in \mathcal{M}$, i.e. $f_o \in \mathcal{M}^g$. Thus $\mathcal{M}^g \neq \emptyset$.

Let $f_1, f_2 \in \mathcal{M}^g$. It follows that $(f_1, g), (f_2, g) \in \mathcal{M}$. Thus for all $x \in X$ we have $(f_1(x), g(x)), (f_2(x), g(x)) \in \mathcal{Y}$. It means that $f_1(x), f_2(x) \in \mathcal{Y}^{\,g(x)}$. Because the set $\mathcal{Y}^{\,g(x)}$ is a linear subspace of the space Y, then $f_1(x) + f_2(x) \in \mathcal{Y}^{\,g(x)}$, and in the consequence $((f_1 + f_2)(x), g(x)) \in \mathcal{Y}$ for all $x \in X$.

Moreover there exists $M_1 > 0$ such that $\|f_1(x), g(x)\| \le M_1 \cdot \|x\|^2$ for all $x \in X$. And there exists also $M_2 > 0$ satisfying the inequality $\|f_2(x), g(x)\| \le M_2 \cdot \|x\|^2$ for all $x \in X$. Hence

$$\begin{aligned}\|f_1(x) + f_2(x), g(x)\| &\le \|f_1(x), g(x)\| + \|f_2(x), g(x)\| \\ &\le M_1 \cdot \|x\|^2 + M_2 \cdot \|x\|^2 = (M_1 + M_2) \cdot \|x\|^2\end{aligned}$$

for all $x \in X$.

Finally, we showed that there exists the positive number $M = M_1 + M_2$ such that for all $x \in X$ the inequality $\|(f_1 + f_2)(x), g(x)\| \le M \cdot \|x\|^2$ is true. Thus $(f_1 + f_2, g) \in \mathcal{M}$, i.e. $f_1 + f_2 \in \mathcal{M}^g$.

Let now $\alpha \in \mathcal{R}$, $f \in \mathcal{M}^g$. It follows $(f, g) \in \mathcal{M}$, i.e. for all $x \in X$ we have $(f(x), g(x)) \in \mathcal{Y}$. Thus $f(x) \in \mathcal{Y}^{\,g(x)}$ and because $\mathcal{Y}^{\,g(x)}$ is a linear subspace of Y, then $\alpha \cdot f(x) \in \mathcal{Y}^{\,g(x)}$. We obtain that $(\alpha f(x), g(x)) \in \mathcal{Y}$ for all $x \in X$. Moreover there exists $M > 0$ such that $\|f(x), g(x)\| \le M \cdot \|x\|^2$ for all $x \in X$. Hence

$$\begin{aligned}\|(\alpha f)(x), g(x)\| &= \|\alpha f(x), g(x)\| = |\alpha| \cdot \ \|f(x), g(x)\| \\ &\le |\alpha| \cdot \ M \cdot \|x\|^2 \text{ for all } x \in X.\end{aligned}$$

As a consequence there exists a positive number $M' = |\alpha| \cdot M$ such that

$$\|(\alpha f)(x), g(x)\| \le M' \cdot \|x\|^2 \text{ for all } x \in X.$$

It implies that $\alpha f \in \mathcal{M}^g$.

We proved that $\mathcal{M}^g$ is a linear subspace of $L(X,Y)$ for all $g \in L(X,Y)$. Analogously we show that $\mathcal{M}^f$ is a linear subspace of $L(X,Y)$ for all $f \in L(X,Y)$. The condition (a) implies simply the equality $\mathcal{M}^f = \mathcal{M}_f$. $\square$

Definition 3.3. *For* $(f,g) \in \mathcal{M}$ *we introduce a number*

$$\|f,g\| = \inf\{M > 0; \forall x \in X \ \|f(x),g(x)\| \le M \cdot \|x\|^2\}.$$

Theorem 3.4. *If* $(f,g) \in \mathcal{M}$, *then*

(a) $\|f,g\| \le M$ *for all* $M \in \mathcal{P}^{(f,g)}$, *where*

$$\mathcal{P}^{(f,g)} = \{M' > 0; \forall x \in X \ \|f(x),g(x)\| \le M' \cdot \|x\|^2\};$$

(b) $\|f(x),g(x)\| \le \|f,g\| \cdot \|x\|^2$ *for all* $x \in X$;

(c)

$$\begin{aligned}\|f,g\| &= \sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| = 1\}\\ &= \sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| \le 1\}\\ &= \sup\left\{\frac{\|f(x),g(x)\|}{\|x\|^2};\ x \in X \wedge \|x\| \ne 0\right\}.\end{aligned}$$

(d) *If* $\mathcal{Y}$ *is a symmetric 2-normed set, then* $\|f,g\| = \|g,f\|$ *for* $(f,g) \in \mathcal{M}$.

Proof. Conditions (a) and (d) are evident.

(b) Because $(f,g) \in \mathcal{M}$, then there exists a positive number M such that

$$\|f(x),g(x)\| \le M \cdot \|x\|^2 \text{ for each } x \in X.$$

Thus $\|f(x),g(x)\| \le \inf\{M \cdot \|x\|^2; M \in \mathcal{P}^{(f,g)}\}$, and $\|f(x),g(x)\| \le \|f,g\| \cdot \|x\|^2$.

(c) By virtue of the condition (b) we have

$$\sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| = 1\} \le \|f,g\|. \tag{3.1}$$

Let $A = \sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| = 1\}$. Consider a point $x \in X, x \ne 0$. Then

$$\begin{aligned}\|f(x),g(x)\| &= \left\|f\left(\frac{1}{\|x\|} \cdot x \cdot \|x\|\right), g\left(\frac{1}{\|x\|} \cdot x \cdot \|x\|\right)\right\|\\ &= \|x\|^2 \cdot \left\|f\left(\frac{x}{\|x\|}\right), g\left(\frac{x}{\|x\|}\right)\right\|.\end{aligned}$$

For $y = \frac{x}{\|x\|}$ we obtain $\|y\| = 1$ and $\|f(y),g(y)\| \le A$. Thus $\|f(x),g(x)\| \le \|x\|^2 \cdot A$ for $x \ne 0$. If $x = 0$, then $\|f(x),g(x)\| = 0 = \|x\|^2 \cdot A$. Consequently $\|f(x),g(x)\| \le \|x\|^2 \cdot A$ for all $x \in X$, i.e. $A \in \mathcal{P}^{(f,g)}$.

From (a) we have $\|f,g\| \le A$ which with (3.1) gives the equality $\|f,g\| = A$. By the condition (b) we have also

$$\sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| \le 1\} \le \|f,g\|.$$

Moreover the inequality

$$\sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| = 1\} \le \sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| \le 1\}$$

is true. Thus we have the equality

$$\|f,g\| = \sup\{\|f(x),g(x)\|;\ x \in X \wedge \|x\| \le 1\}.$$

Let us take $x \in X, x \neq 0$. By (b) we have

$$\frac{\|f(x), g(x)\|}{\|x\|^2} \leq \|f, g\|$$

and further

$$\sup\left\{\frac{\|f(x), g(x)\|}{\|x\|^2};\ x \in X \wedge \|x\| \neq 0\right\} \leq \|f, g\|.$$

Let $B = \sup\left\{\frac{\|f(x),g(x)\|}{\|x\|^2};\ x \in X \wedge \|x\| \neq 0\right\}$. If $\|x\| = 0$, then $\|f(x), g(x)\| = 0$. Thus $\|f(x), g(x)\| \leq B \cdot \|x\|^2$ for every $x \in X$, which means that $B \in \mathcal{P}^{(f,g)}$ and $\|f, g\| \leq B$. This ends the proof. ◻

THEOREM 3.5. *The set $\mathcal{M}$ is a 2-normed set with the 2-norm defined by the formula*

$$\|f, g\| = \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \text{ for } (f, g) \in \mathcal{M}.$$

In the case that $\mathcal{Y}$ is a symmetric 2-normed set, then the set $\mathcal{M}$ is also symmetric.

PROOF. By virtue of Lemma 3.2 the set $\mathcal{M}$ satisfies conditions from Definition 1.2. Let $(f, g) \in \mathcal{M}$. Then there exists $M > 0$ such that $\|f(x), g(x)\| \leq M \cdot \|x\|^2$ for $x \in X$. Thus $\sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \leq M < \infty$ and so the function $\|\cdot, \cdot\|$ has finite non-negative values. Moreover the following conditions are true:

(N1) Let $x \in X, \|x\| = 1, \alpha \in \mathcal{R}$. Then

$$\begin{aligned}\|f(x), (\alpha g)(x)\| &= \|f(x), \alpha g(x)\| = |\alpha| \cdot \|f(x), g(x)\| \\ &\leq |\alpha| \cdot \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \\ &= |\alpha| \cdot \|f, g\|.\end{aligned}$$

Since x is arbitrary, we obtain

$$\sup\{\|f(x), (\alpha g)(x)\|;\ x \in X \wedge \|x\| = 1\} \leq |\alpha| \cdot \|f, g\|$$

and consequently the inequality

$$\|f, \alpha g\| \leq |\alpha| \cdot \|f, g\|. \tag{3.2}$$

Let $\alpha \neq 0$. Using (3.2) we have

$$\|f, g\| = \left\|f, \frac{1}{\alpha} \cdot \alpha g\right\| \leq \frac{1}{|\alpha|} \cdot \|f, \alpha g\| \text{ and } |\alpha| \cdot \|f, g\| \leq \|f, \alpha g\| \text{ for } \alpha \neq 0.$$

If however $\alpha = 0$, then $|\alpha| \cdot \|f, g\| = 0 = \|f, \alpha g\|$. And we showed that $|\alpha| \cdot \|f, g\| \leq \|f, \alpha g\|$ for all $\alpha \in \mathcal{R}$, which with (3.2) gives the equality

$$\|f, \alpha g\| = |\alpha| \cdot \|f, g\|.$$

The proof of the equality $\|\alpha f, g\| = |\alpha| \cdot \|f, g\|$ is analogous, therefore it is omitted.

(N2) Let us take $f, g, h \in L(X, Y)$ such that $(f, g), (f, h) \in \mathcal{M}$. Consider $x \in X, \|x\| = 1$. Then the following inequalities are true:

$$\begin{aligned}\|f(x), (g+h)(x)\| &= \|f(x), g(x) + h(x)\| \le \|f(x), g(x)\| + \|f(x), h(x)\| \\ &\le \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \\ &\quad + \sup\{\|f(x), h(x)\|;\ x \in X \wedge \|x\| = 1\} \\ &= \|f, g\| + \|f, h\|.\end{aligned}$$

It implies the condition

$$\sup\{\|f(x), (g+h)(x)\|;\ x \in X \wedge \|x\| = 1\} \le \|f, g\| + \|f, h\|$$

i.e. $\|f, g+h\| \le \|f, g\| + \|f, h\|$. Similarly we obtain:

(N3) $\|f + g, h\| \le \|f, h\| + \|g, h\|$.

Now assume that $\mathcal{Y}$ is a symmetric 2-normed set. Then $\mathcal{M} = \mathcal{M}^{-1}$ and the condition

$$\begin{aligned}\|f, g\| &= \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \\ &= \sup\{\|g(x), f(x)\|;\ x \in X \wedge \|x\| = 1\} = \|g, f\|\end{aligned}$$

is satisfied. Thus by Definition 1.3 the set $\mathcal{M}$ is a symmetric 2-normed set. This finishes the proof. □

Taking linear spaces $X \times X$, $Y \times Y$ we can consider linear operators (f, g) from $X \times X$ into $Y \times Y$, defined by the formula $(f, g)(x, y) = (f(x), g(y))$ for every $x, y \in X$, where $f, g \in L(X, Y)$. Further we will show properties of the set of these operators satisfying certain additional conditions.

Definition 3.6. *Let X be a real normed space and $\mathcal{Y} \subset Y \times Y$ be a 2-normed set, where Y denotes a real linear space. A set $\mathcal{N}$ is defined as follows:*

$$\mathcal{N} = \Big\{(f, g) \in L(X, Y)^2; \forall x, y \in X\ (f(x), g(y)) \in \mathcal{Y} \\ \wedge \exists_{M>0} \forall x, y \in X\ \|f(x), g(y)\| \le M \cdot \|x\| \cdot \|y\|\Big\}.$$

Lemma 3.7. *The set $\mathcal{N}$ defined in Definition 3.6 has the following property:*

(a) *If $\mathcal{Y}$ is a symmetric 2-normed set, then $\mathcal{N} = \mathcal{N}^{-1}$.*
(b) *For every $f, g \in L(X, Y)$ sets $\mathcal{N}^g = \{f' \in L(X, Y); (f', g) \in \mathcal{N}\}$ and $\mathcal{N}_f = \{g' \in L(X, Y); (f, g') \in \mathcal{N}\}$ are linear subspaces of the space $L(X, Y)$. If $\mathcal{Y}$ is a symmetric 2-normed set, then $\mathcal{N}^f = \mathcal{N}_f$.*

The proof is similar to the proof of Lemma 3.2 so it is omitted.

Definition 3.8. *For $(f, g) \in \mathcal{N}$ we introduce a number*

$$\|f, g\| = \inf\{M > 0; \forall x, y \in X\ \|f(x), g(y)\| \le M \cdot \|x\| \cdot \|y\|\}.$$

The following theorem gives properties of the number $\|f,g\|$ for $(f,g) \in \mathcal{N}$, which are similar to the properties from Theorem 3.4.

THEOREM 3.9. *If* $(f,g) \in \mathcal{N}$, *then*

(a) $\|f,g\| \leq M$ *for all* $M \in \mathcal{R}^{(f,g)}$, *where*

$$\mathcal{R}^{(f,g)} = \{M' > 0; \forall x,y \in X \|f(x),g(y)\| \leq M' \cdot \|x\| \cdot \|y\|\};$$

(b) $\|f(x),g(y)\| \leq \|f,g\| \cdot \|x\| \cdot \|y\|$ *for all* $x,y \in X$;

(c)

$$\begin{aligned}\|f,g\| &= \sup\{\|f(x),g(y)\|;\ x,y \in X \wedge \|x\| = \|y\| = 1\}\\ &= \sup\{\|f(x),g(y)\|;\ x,y \in X \wedge \|x\| \leq 1, \|y\| \leq 1\}\\ &= \sup\left\{\frac{\|f(x),g(y)\|}{\|x\| \cdot \|y\|};\ x,y \in X \wedge \|x\| \neq 0, \|y\| \neq 0\right\}.\end{aligned}$$

(d) $\|f,g\| = \|g,f\|$, *if* $\mathcal{Y}$ *is a symmetric 2-normed set.*

THEOREM 3.10. *The set* $\mathcal{N}$ *is a 2-normed set with the 2-norm defined by the formula*

$$\|f,g\| = \sup\{\|f(x),g(y)\|;\ x,y \in X \wedge \|x\| = \|y\| = 1\} \text{ for } (f,g) \in \mathcal{N}.$$

If $\mathcal{Y}$ *is a symmetric 2-normed set, then the set* $\mathcal{N}$ *is also symmetric.*

Proofs of Theorem 3.9 and Theorem 3.10 are analogous to proofs of Theorem 3.4 and Theorem 3.5, respectively, therefore they are omitted.

In this section we introduced two 2-normed sets $\left(\mathcal{M}, \|\cdot,\cdot\|_{\mathcal{M}}\right)$ and $\left(\mathcal{N}, \|\cdot,\cdot\|_{\mathcal{N}}\right)$, where $\mathcal{N} \subset \mathcal{M}$. Let us remark that for every $(f,g) \in \mathcal{N}$ the inequality

$$\|f,g\|_{\mathcal{M}} \leq \|f,g\|_{\mathcal{N}}$$

is true.

Finally consider a normed space $(X, \|\cdot\|)$, in which is given a 2-norm in the Gähler's sense independent of the norm. In [3] S. S. Kim, Y. J. Cho and A. White introduced the following definition of an 2-bounded operator.

DEFINITION 3.11 ([3]). *An operator* $T\colon (X, \|\cdot\|) \to (X, \|\cdot,\cdot\|)$ *is said to be 2-bounded if there is a* $K \geq 0$ *such that*

$$\|T(x),y\| + \|x,T(y)\| \leq K \cdot \|x\| \cdot \|y\| \text{ for all } x,y \in X.$$

Authors of [3] showed that the space $BL(X,Y)$ of all 2-bounded linear operators from normed space $(X, \|\cdot\|)$ into a 2-normed space $(X, \|\cdot,\cdot\|)$ is a normed space with the norm $\|\cdot\|_2$ defined by the formula

$$\|T\|_2 = \inf\{K \geq 0; \|T(x),y\| + \|x,T(y)\| \leq K \cdot \|x\| \cdot \|y\| \text{ for all } x,y \in X\}.$$

Considering a normed space $(X, \|\cdot\|)$, in which is defined also a 2-norm in the Gähler's sense, we obtain that the set $\mathcal{N}^{id}$ coincides with the space

$BL(X, Y)$, where the operator $id\colon X \to X$ is defined as follows: $id(x) = x$ for all $x \in X$. Thus results in [3] are the special case of the theory in the presented paper.

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