ON 2-NORMED SETS

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Abstract. In this paper we will consider properties of the 2-normed sets and we will construct two 2-normed sets of linear operators.

1. Introduction

In [2] S. Gähler introduced the following definition of a 2-normed space:

Definition 1.1 ([2]). Let $X$ be a real linear space of dimension greater than 1 and let $\| \cdot, \cdot \|$ be a real valued function on $X \times X$ satisfying the following four properties:

(G1) $\|x, y\| = 0$ if and only if the vectors $x$ and $y$ are linearly dependent;
(G2) $\|x, y\| = \|y, x\|$;
(G3) $\|x, \alpha y\| = |\alpha| \cdot \|x, y\|$ for every real number $\alpha$;
(G4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

The function $\| \cdot, \cdot \|$ will be called a 2-norm on $X$ and the pair $(X, \| \cdot, \cdot \|)$ - a linear 2-normed space.

In [4] and [5] we gave a generalization of the Gähler’s 2-normed space. Namely a generalized 2-norm need not be symmetric and satisfy the first condition of the above definition.

Definition 1.2 ([4]). Let $X$ and $Y$ be real linear spaces. Denote by $\mathcal{D}$ a non-empty subset $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$\mathcal{D}_x = \{y \in Y; (x, y) \in \mathcal{D}\}$ and $\mathcal{D}^x = \{x \in X; (x, y) \in \mathcal{D}\}$

are linear subspaces of the space $Y$ and $X$, respectively.
A function \( \| \cdot, \cdot \| : \mathcal{D} \to [0, \infty) \) will be called a generalized 2-norm on \( \mathcal{D} \) if it satisfies the following conditions:

\begin{enumerate}[(N1)]
    \item \( \| x, \alpha y \| = |\alpha| \cdot \| x, y \| = \| \alpha x, y \| \) for any real number \( \alpha \) and all \( (x, y) \in \mathcal{D} \);
    \item \( \| x, y + z \| \leq \| x, y \| + \| x, z \| \) for \( x \in X \), \( y, z \in Y \) such that \( (x, y), (x, z) \in \mathcal{D} \);
    \item \( \| x + y, z \| \leq \| x, z \| + \| y, z \| \) for \( x, y \in X \), \( z \in Y \) such that \( (x, z), (y, z) \in \mathcal{D} \).
\end{enumerate}

The set \( \mathcal{D} \) is called a 2-normed set.

In particular, if \( \mathcal{D} = X \times Y \), the function \( \| \cdot, \cdot \| \) will be called a generalized 2-norm on \( X \times Y \) and the pair \( (X \times Y, \| \cdot, \cdot \|) \) a generalized 2-normed space.

Moreover, if \( X = Y \), then the generalized 2-normed space will be denoted by \( (X, \| \cdot, \cdot \|) \).

Assume that the generalized 2-norm satisfies, in addition, the symmetry condition. Then we will define the 2-norm as follows:

**Definition 1.3** ([4]). Let \( X \) be a real linear space. Denote by \( \mathcal{X} \) a non-empty subset \( X \times X \) with the property \( \mathcal{X} = \mathcal{X}^{-1} \) and such that the set \( \mathcal{X}^y = \{ x \in X; (x, y) \in \mathcal{X} \} \) is a linear subspace of \( X \), for all \( y \in X \).

A function \( \| \cdot, \cdot \| : \mathcal{X} \to [0, \infty) \) satisfying the following conditions:

\begin{enumerate}[(S1)]
    \item \( \| x, y \| = \| y, x \| \) for all \( (x, y) \in \mathcal{X} \),
    \item \( \| x, \alpha y \| = |\alpha| \cdot \| x, y \| \) for any real number \( \alpha \) and all \( (x, y) \in \mathcal{X} \),
    \item \( \| x, y + z \| \leq \| x, y \| + \| x, z \| \) for \( x, y, z \in X \) such that \( (x, y), (x, z) \in \mathcal{X} \),
\end{enumerate}

will be called a symmetric 2-norm on \( X \). The set \( \mathcal{X} \) is called a symmetric 2-normed set. In particular, if \( \mathcal{X} = X \times X \), the function \( \| \cdot, \cdot \| \) will be called a generalized symmetric 2-norm on \( X \) and the pair \( (X, \| \cdot, \cdot \|) \) a generalized symmetric 2-normed space.

In this paper we give some properties of certain 2-normed sets.

2. A 2-normed set

A 2-normed set has interesting properties, for example it can be a small in some sense and a big in other one. We will show them in this section.

**Example 2.1.** ([4]) Let \( s \) be a linear space of all sequences of real numbers. Let

\[
\| x, y \| = \sum_{n=1}^{\infty} |\xi_n| \cdot |\eta_n|
\]

for all \( x, y \in s \) such that \( x = \{ \xi_n; n \in N \}, y = \{ \eta_n; n \in N \} \). Then \( \| \cdot, \cdot \| : s \times s \to [0, \infty] \). By \( \mathcal{X} \) we will denote the set \( \{(x, y) \in s \times s; \|x, y\| < \infty\} \). Thus we have the properties:
(1) \( \mathcal{X} = \mathcal{X}^{-1} \);
(2) \( \mathcal{X}^y \) is a linear subspace of \( s \) for every \( y \in s \).

Then the set \( \mathcal{X} \) is a symmetric 2-normed set and the function \( \| \cdot, \cdot \| : \mathcal{X} \to [0, \infty) \) is a generalized symmetric 2-norm on \( \mathcal{X} \).

In the sequel \( s \) denotes the linear space of all real sequences with the usual metric \( \rho \) given by

\[
\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}
\]

for \( x = \{\xi_n; n \in N\} \in s, y = \{\eta_n; n \in N\} \in s \). Then the function \( d \) by the formula \( d((x, y), (x', y')) = \rho(x, x') + \rho(y, y') \) for \( x, x', y, y' \in s \), is a metric in \( s \times s \).

**Theorem 2.2.** The 2-normed set \( \mathcal{X} \) is a dense \( F_\sigma \)-set of the first Baire category in the space \( (s \times s, d) \).

**Proof.** At first we will show that \( \mathcal{X} \) is dense in \( (s \times s, d) \). Consider the set

\[
l^2 = \left\{ x = \{\xi_j; j \in N\} \in s; \sum_{j=1}^{\infty} |\xi_j|^2 < \infty \right\}.
\]

Let \( x = \{\xi_j; j \in N\} \in l^2 \) and \( y = \{\eta_j; j \in N\} \in l^2 \). Obviously \( (x, y) \in l^2 \times l^2 \). By the well-known Hölder’s inequality we have

\[
\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left( \sum_{j=1}^{\infty} |\xi_j|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{j=1}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}} < \infty,
\]

and in the consequence

\[
(2.1) \quad l^2 \times l^2 \subset \mathcal{X}.
\]

Because \( l^2 \) is dense in \( s \), (it follows from Theorem 2.1 in [1]), therefore \( l^2 \times l^2 \) is dense in \( s \times s \). Using the inclusion \( (2.1) \) we see that \( \mathcal{X} \) is dense in \( s \times s \), too.

We shall prove that \( \mathcal{X} \) is an \( F_\sigma \)-set in \( s \times s \). For \( k, n \in N \) we put

\[
A_{kn} = \left\{ (x, y) \in s; \sum_{j=1}^{n} |\xi_j \eta_j| > k \right\}.
\]

Then

\[
(2.2) \quad s \times s \setminus \mathcal{X} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_{kn}.
\]
Now, we shall show that $A_{kn}, (k, n \in N),$ is open in $s \times s$. Let $(x_0, y_0) \in A_{kn}$, where $x_0 = \{\xi_j^o; j \in N\}$ and $y_0 = \{\eta_j^o; j \in N\}$. Choose $\varepsilon > 0$ in such a way that
\[
\sum_{j=1}^{n} |\xi_j^o - \xi_j^o| - \varepsilon > k.
\]

The function $\varphi: \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined as follows
\[
\varphi(a, b) = \sum_{j=1}^{n} |\alpha_j^o - \beta_j^o|, \quad \text{for } a = (\alpha_1^o, \ldots, \alpha_n^o), b = (\beta_1^o, \ldots, \beta_n^o),
\]
is continuous at the point $(a_o, b_o)$ such that $a_o = (\xi_1^o, \ldots, \xi_n^o), b_o = (\eta_1^o, \ldots, \eta_n^o)$. Then there exists $\delta > 0$ such that $|\varphi(a, b) - \varphi(a_o, b_o)| < \varepsilon$ when $\|a - a_o\| < \delta$ and $\|b - b_o\| < \delta$.

Let us take
\[
r = \frac{1}{2^n} \cdot \frac{\delta}{\sqrt{n}} + \frac{\delta}{\sqrt{n}} \quad \text{and} \quad K^o (x_0, y_0) \subset s \times s.
\]

For $(x, y) \in K^o (x_0, y_0, r)$, where $x = \{\xi_j; j \in N\}, y = \{\eta_j; j \in N\}$, the following inequality is true: $d((x, y), (x_o, y_o)) < r$. Thus $\varrho(x, x_o) < r$ and $\varrho(y, y_o) < r$. By the first inequality we have
\[
\sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_j^o - \xi_j^o|}{1 + |\xi_j^o - \xi_j^o|} < \frac{1}{2^n} \cdot \frac{\delta}{\sqrt{n}} + \frac{\delta}{\sqrt{n}}.
\]

Since
\[
\frac{1}{2^j} \cdot \frac{|\xi_j - \xi_j^o|}{1 + |\xi_j - \xi_j^o|} < \frac{1}{2^n} \cdot \frac{\delta}{\sqrt{n}} + \frac{\delta}{\sqrt{n}} \quad \text{for each } j = 1, 2, \ldots, n.
\]

From this we get
\[
\frac{|\xi_j - \xi_j^o|}{1 + |\xi_j - \xi_j^o|} < \frac{2^j}{2^n} \cdot \frac{\delta}{\sqrt{n}} + \frac{\delta}{\sqrt{n}} < \frac{\delta}{\sqrt{n}} \quad \text{for } j = 1, 2, \ldots, n.
\]

Because the function $f(t) = \frac{t}{1+t}$ for $t \geq 0$, is an increasing function, then
\[
|\xi_j - \xi_j^o| < \frac{\delta}{\sqrt{n}} \quad \text{for } j = 1, 2, \ldots, n.
\]

By analogy we obtain the inequality
\[
|\eta_j - \eta_j^o| < \frac{\delta}{\sqrt{n}} \quad \text{for } j = 1, 2, \ldots, n.
\]
So for \( a = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, b = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n \) we have

\[
\|a - a_o\| = \sqrt{\sum_{j=1}^n |\xi_j - \xi_o^j|^2} < \delta \quad \text{and} \quad \|b - b_o\| < \delta.
\]

As a consequence \( |\varphi(a, b) - \varphi(a_o, b_o)| < \varepsilon \), i.e.

\[
\left| \sum_{j=1}^n |\xi_j \eta_j| - \sum_{j=1}^n |\xi_o^j \eta_o^j| \right| < \varepsilon.
\]

Thus

\[
\sum_{j=1}^n |\xi_j \eta_j| > \sum_{j=1}^n |\xi_o^j \eta_o^j| - \varepsilon > k.
\]

Hence \((x, y) \in A_{kn} \). And we have proved that \( \mathcal{K}^0((x_o, y_o), r) \subset A_{kn} \), this means that \( A_{kn} \) is open. Using the equality (2.2) we see that \( s \times s \setminus \mathcal{X} \) is an \( G_\delta \)-set. Therefore \( \mathcal{X} \) is an \( F_\sigma \)-set.

Finally, we shall show that \( \mathcal{X} \) is a set of the first Baire category in \((s \times s, d)\). Let \( r > 0 \) and \((x_o, y_o) \in s \times s\), where \( x_o = \{\xi_o^j; j \in N\} \) and \( y_o = \{\eta_o^j; j \in N\} \). Then there exists an \( n_o \in N \) such that

\[
\sum_{j=n_o+1}^{\infty} \frac{1}{2^j} < \frac{r}{2}.
\]

Choose \( x_1 = \{\xi_1^j; j \in N\} \) and \( y_1 = \{\eta_1^j; j \in N\} \) in the following way:

\[
\xi_1^j = \xi_o^j \quad \text{for } j = 1, 2, \ldots, n_o \quad \text{and} \quad \xi_1^j = 1 \quad \text{for } j \geq n_o + 1,
\]

\[
\eta_1^j = \eta_o^j \quad \text{for } j = 1, 2, \ldots, n_o \quad \text{and} \quad \eta_1^j = 1 \quad \text{for } j \geq n_o + 1.
\]

Then for each \( k \in N \) there exists \( n \in N \) such that

\[
\sum_{j=1}^n |\xi_1^j \eta_1^j| > k.
\]
From this we get \((x_1, y_1) \in s \times s \setminus \mathcal{X}\). Further
\[
d((x_1, y_1), (x_o, y_o)) = d(x_1, x_o) + d(y_1, y_o)
\]
\[
= \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_j^1 - \xi_j^o|}{1 + |\xi_j^1 - \xi_j^o|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\eta_j^1 - \eta_j^o|}{1 + |\eta_j^1 - \eta_j^o|}
\]
\[
= \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_j^o - 1|}{1 + |\xi_j^o - 1|} + \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} \cdot \frac{|\eta_j^o - 1|}{1 + |\eta_j^o - 1|}
\]
\[
< \sum_{j=n_o+1}^{\infty} \frac{1}{2^j} + \sum_{j=n_o+1}^{\infty} \frac{1}{2^j}
\]
\[
< \frac{r}{2} + \frac{r}{2} = r.
\]
Hence \((x_1, y_1) \in \mathcal{K}^o((x_o, y_o), r)\). In the consequence \(\mathcal{K}^o((x_o, y_o), r) \cap (s \times s \setminus \mathcal{X}) \neq \emptyset\). This means that \(s \times s \setminus \mathcal{X}\) is dense in \(s \times s\). Using the foregoing results we see that \(\mathcal{X}\) is a set of the first Baire category in \((s \times s, d)\). This ends the proof. \(\square\)

### 3. 2-NORMED SETS OF LINEAR OPERATORS

Let \(X, Y\) be real linear spaces. The set \(X \times Y\) is the linear space with respect to the operations:
\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \text{ for all } x_1, x_2, y_1, y_2 \in Y
\]
and
\[
\alpha \cdot (x, y) = (\alpha x, \alpha y) \text{ for all } x, y \in Y, \alpha \in \mathcal{R}.
\]

By \(L(X, Y)\) denote the linear space of all linear operators from \(X\) with values in \(Y\). It is easy to see that for each linear operator \(F: X \to Y \times Y\) there exists a pair of operators \(f, g \in L(X, Y)\) such that \(F(x) = (f(x), g(x))\) for all \(x \in X\). And conversely, the operator \(F: X \to Y \times Y\) defined by the formula \(F(x) = (f(x), g(x))\) for all \(x \in X\), where \(f, g \in L(X, Y)\), is linear.

Further we will consider properties of the set of these pairs of linear operators satisfying certain additional conditions.

**Definition 3.1.** Let \(X\) be a real normed space and \(\mathcal{Y} \subset Y \times Y\) be a 2-normed set, where \(Y\) denotes a real linear space. A set \(\mathcal{M}\) is defined as follows:
\[
\mathcal{M} = \{(f, g) \in L(X, Y)^2; \forall x \in X \ (f(x), g(x)) \in \mathcal{Y} \land \exists M > 0 \forall x \in X \ |f(x), g(x)| \leq M \cdot \|x\|^2\}.
\]

**Lemma 3.2.** The set \(\mathcal{M}\) defined in Definition 3.1 has the following property:
(a) If \(\mathcal{Y}\) is a symmetric 2-normed set, then \(\mathcal{M} = \mathcal{M}^{-1}\).
(b) For every \( f, g \in L(X,Y) \) sets \( \mathcal{M}^g = \{ f' \in L(X,Y); (f', g) \in \mathcal{M} \} \) and 
\[ \mathcal{M}_f = \{ g' \in L(X,Y); (f, g') \in \mathcal{M} \} \] 
are linear subspaces of the space \( L(X,Y) \). In the case when \( \mathcal{Y} \) is a symmetric 2-normed set we have the 
equality \( \mathcal{M}^f = \mathcal{M}_f \).

**Proof.** The condition (a) follows from the definition of the set \( \mathcal{M} \).

(b) Let \( g \) be a linear operator from \( X \) to \( Y \). First, we shall show that \( \mathcal{M}^g \) is non-empty. Consider the linear operator \( f_o: X \to Y \) defined by the formula \( f_o(x) = 0 \) for all \( x \in X \). Because for each \( x \in X \) the set \( \mathcal{Y}^{g(x)} \) is linear subspace of \( Y \), then \( f_o(x) = 0 \in \mathcal{Y}^{g(x)} \). Thus \( (f_o(x), g(x)) \in \mathcal{Y}^{g(x)} \) for all \( x \in X \). Moreover for each positive number \( M \) and for all \( x \in X \) the inequality \( \|f_o(x), g(x)\| = \|0, g(x)\| = 0 \leq M \cdot \|x\|^2 \) is satisfied. In the consequence \((f_o, g) \in \mathcal{M}, \) i.e. \( f_o \in \mathcal{M}^g \). Thus \( \mathcal{M}^g \neq \emptyset \).

Let \( f_1, f_2 \in \mathcal{M}^g \). It follows that \((f_1, g), (f_2, g) \in \mathcal{M} \). Thus for all \( x \in X \) we have \((f_1(x), g(x)), (f_2(x), g(x)) \in \mathcal{Y}^{g(x)} \). It means that \( f_1(x), f_2(x) \in \mathcal{Y}^{g(x)} \). Hence for all \( x \in X \) \( \|f_1(x) + f_2(x), g(x)\| \leq \|f_1(x), g(x)\| + \|f_2(x), g(x)\| \leq M_1 \cdot \|x\|^2 + M_2 \cdot \|x\|^2 = (M_1 + M_2) \cdot \|x\|^2 \) for all \( x \in X \).

Finally, we showed that there exists the positive number \( M = M_1 + M_2 \) such that for all \( x \in X \) the inequality \( \|(f_1 + f_2)(x), g(x)\| \leq M \cdot \|x\|^2 \) is true. Thus \((f_1 + f_2, g) \in \mathcal{M}, \) i.e. \( f_1 + f_2 \in \mathcal{M}^g \).

Let now \( \alpha \in \mathbb{R}, \) \( f \in \mathcal{M}^g \). It follows \((f, g) \in \mathcal{M}, \) i.e. for all \( x \in X \) we have \((f(x), g(x)) \in \mathcal{Y} \). Thus \( f(x) \in \mathcal{Y}^{g(x)} \) and because \( \mathcal{Y}^{g(x)} \) is a linear subspace of \( Y \), then \( \alpha \cdot f(x) \in \mathcal{Y}^{g(x)} \). We obtain that \((\alpha f(x), g(x)) \in \mathcal{Y} \) for all \( x \in X \). Moreover there exists \( M > 0 \) such that \( \|f(x), g(x)\| \leq M \cdot \|x\|^2 \) for all \( x \in X \). Hence

\[
\|(\alpha f)(x), g(x)\| = \|\alpha f(x), g(x)\| = |\alpha| \cdot \|f(x), g(x)\| \\
\leq |\alpha| \cdot M \cdot \|x\|^2 \text{ for all } x \in X.
\]

As a consequence there exists a positive number \( M' = |\alpha| \cdot M \) such that

\[
\|(\alpha f)(x), g(x)\| \leq M' \cdot \|x\|^2 \text{ for all } x \in X.
\]

It implies that \( \alpha f \in \mathcal{M}^g \).

We proved that \( \mathcal{M}^g \) is a linear subspace of \( L(X,Y) \) for all \( g \in L(X,Y) \). Analogously we show that \( \mathcal{M}^f \) is a linear subspace of \( L(X,Y) \) for all \( f \in L(X,Y) \). The condition (a) implies simply the equality \( \mathcal{M}^f = \mathcal{M}_f \).
**Definition 3.3.** For \((f, g) \in \mathcal{M}\) we introduce a number
\[
\|f, g\| = \inf \{ M > 0; \forall x \in X \| f(x), g(x) \| \leq M \cdot \|x\|^2 \}.
\]

**Theorem 3.4.** If \((f, g) \in \mathcal{M}\), then
(a) \(\|f, g\| \leq M\) for all \(M \in \mathcal{P}(f,g)\), where
\[
\mathcal{P}(f,g) = \{ M' > 0; \forall x \in X \| f(x), g(x) \| \leq M' \cdot \|x\|^2 \};
\]
(b) \(\|f(x), g(x)\| \leq \|f, g\| \cdot \|x\|^2\) for all \(x \in X\);
(c) \[
\|f, g\| = \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| = 1 \}
= \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| \leq 1 \}
= \sup \left\{ \frac{\|f(x), g(x)\|}{\|x\|^2}; x \in X \wedge \|x\| \neq 0 \right\}.
\]
(d) If \(Y\) is a symmetric 2-normed set, then \(\|f, g\| = \|g, f\|\) for \((f, g) \in \mathcal{M}\).

**Proof.** Conditions (a) and (d) are evident.

(b) Because \((f, g) \in \mathcal{M}\), there exists a positive number \(M\) such that
\[
\|f(x), g(x)\| \leq M \cdot \|x\|^2\text{ for each } x \in X.
\]
Thus \(\|f(x), g(x)\| \leq \inf \{ M \cdot \|x\|^2; M \in \mathcal{P}(f,g) \}\), and \(\|f(x), g(x)\| \leq \|f, g\| \cdot \|x\|^2\).

(c) By virtue of the condition (b) we have
\[
(3.1) \quad \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| = 1 \} \leq \|f, g\|.
\]
Let \(A = \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| = 1 \}\). Consider a point \(x \in X, x \neq 0\).
Then
\[
\|f(x), g(x)\| = \left\| f \left( \frac{1}{\|x\|} \cdot x \cdot \|x\| \right), g \left( \frac{1}{\|x\|} \cdot x \cdot \|x\| \right) \right\|
= \|x\|^2 \cdot \left\| f \left( \frac{x}{\|x\|} \right), g \left( \frac{x}{\|x\|} \right) \right\|.
\]
For \(y = \frac{x}{\|x\|}\) we obtain \(\|y\| = 1\) and \(\|f(y), g(y)\| \leq A\). Thus \(\|f(x), g(x)\| \leq \|x\|^2 \cdot A\) for \(x \neq 0\). If \(x = 0\), then \(\|f(x), g(x)\| = 0 = \|x\|^2 \cdot A\). Consequently \(\|f(x), g(x)\| \leq \|x\|^2 \cdot A\) for all \(x \in X\), i.e. \(A \in \mathcal{P}(f,g)\).

From (a) we have \(\|f, g\| \leq A\) which with (3.1) gives the equality \(\|f, g\| = A\). By the condition (b) we have also
\[
\sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| \leq 1 \} \leq \|f, g\|.
\]
Moreover the inequality
\[
\sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| = 1 \} \leq \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| \leq 1 \}
\]
is true. Thus we have the equality
\[
\|f, g\| = \sup \{ \|f(x), g(x)\|; x \in X \wedge \|x\| \leq 1 \}.
\]
Let us take $x \in X, x \neq 0$. By (b) we have
\[ \frac{\|f(x), g(x)\|}{\|x\|^2} \leq \|f, g\| \]
and further
\[ \sup \left\{ \frac{\|f(x), g(x)\|}{\|x\|^2}; \ x \in X \land \|x\| \neq 0 \right\} \leq \|f, g\|. \]

Let $B = \sup \left\{ \frac{\|f(x), g(x)\|}{\|x\|^2}; \ x \in X \land \|x\| \neq 0 \right\}$. If $\|x\| = 0$, then $\|f(x), g(x)\| = 0$. Thus $\|f(x), g(x)\| \leq B \cdot \|x\|^2$ for every $x \in X$, which means that $B \in P \ (f, g)$ and $\|f, g\| \leq B$. This ends the proof.

**Theorem 3.5.** The set $M$ is a 2-normed set with the 2-norm defined by the formula
\[ \|f, g\| = \sup \{\|f(x), g(x)\|; \ x \in X \land \|x\| = 1\} \text{ for } (f, g) \in M. \]

In the case that $Y$ is a symmetric 2-normed set, then the set $M$ is also symmetric.

**Proof.** By virtue of Lemma 3.2 the set $M$ satisfies conditions from Definition 1.2. Let $(f, g) \in M$. Then there exists $M > 0$ such that $\|f(x), g(x)\| \leq M \cdot \|x\|^2$ for $x \in X$. Thus $\sup\{\|f(x), g(x)\|; \ x \in X \land \|x\| = 1\} \leq M < \infty$ and so the function $\| \cdot , \cdot \|$ has finite non-negative values. Moreover the following conditions are true:

(N1) Let $x \in X, \|x\| = 1, \alpha \in \mathbb{R}$. Then
\[ \|f(x), (\alpha g)(x)\| = \|f(x), \alpha g(x)\| = |\alpha| \cdot \|f(x), g(x)\|
\leq |\alpha| \cdot \sup\{\|f(x), g(x)\|; \ x \in X \land \|x\| = 1\}
= |\alpha| \cdot \|f, g\|. \]

Since $x$ is arbitrary, we obtain
\[ \sup\{\|f(x), (\alpha g)(x)\|; \ x \in X \land \|x\| = 1\} \leq |\alpha| \cdot \|f, g\| \]
and consequently the inequality
\[ (3.2) \quad \|f, \alpha g\| \leq |\alpha| \cdot \|f, g\|. \]

Let $\alpha \neq 0$. Using (3.2) we have
\[ \|f, g\| = \left\| f, \frac{1}{|\alpha|} \cdot \alpha g \right\| \leq \frac{1}{|\alpha|} \cdot \|f, \alpha g\| \text{ and } |\alpha| \cdot \|f, g\| \leq \|f, \alpha g\| \text{ for } \alpha \neq 0. \]

If however $\alpha = 0$, then $|\alpha| \cdot \|f, g\| = 0 = \|f, \alpha g\|$. And we showed that $|\alpha| \cdot \|f, g\| \leq \|f, \alpha g\|$ for all $\alpha \in \mathbb{R}$, which with (3.2) gives the equality
\[ \|f, \alpha g\| = |\alpha| \cdot \|f, g\|. \]

The proof of the equality $\|\alpha f, g\| = |\alpha| \cdot \|f, g\|$ is analogous, therefore it is omitted.
(N2) Let us take \( f, g, h \in L(X, Y) \) such that \( (f, g), (f, h) \in \mathcal{M} \). Consider \( x \in X, \|x\| = 1 \). Then the following inequalities are true:
\[
\|f(x), (g + h)(x)\| = \|f(x), g(x) + h(x)\| \leq \|f(x), g(x)\| + \|f(x), h(x)\|
\]
\[
\leq \sup\{\|f(x), g(x)\|; x \in X \land \|x\| = 1\}
\]
\[
+ \sup\{\|f(x), h(x)\|; x \in X \land \|x\| = 1\}
\]
\[
= \|f, g\| + \|f, h\|
\]
It implies the condition
\[
\sup\{\|f(x), (g + h)(x)\|; x \in X \land \|x\| = 1\} \leq \|f, g\| + \|f, h\|
\]
i.e. \( \|f, g + h\| \leq \|f, g\| + \|f, h\| \). Similarly we obtain:
(N3) \( \|f + g, h\| \leq \|f, h\| + \|g, h\| \).

Now assume that \( \mathcal{Y} \) is a symmetric 2-normed set. Then \( \mathcal{M} = \mathcal{M}^{-1} \) and the condition
\[
\|f, g\| = \sup\{\|f(x), g(x)\|; x \in X \land \|x\| = 1\}
\]
\[
= \sup\{\|g(x), f(x)\|; x \in X \land \|x\| = 1\} = \|g, f\|
\]
is satisfied. Thus by Definition 1.3 the set \( \mathcal{M} \) is a symmetric 2-normed set. This finishes the proof.

Taking linear spaces \( X \times X, Y \times Y \) we can consider linear operators \( (f, g) \) from \( X \times X \) into \( Y \times Y \), defined by the formula \( (f, g)(x, y) = (f(x), g(y)) \) for every \( x, y \in X \), where \( f, g \in L(X, Y) \). Further we will show properties of the set of these operators satisfying certain additional conditions.

**Definition 3.6.** Let \( X \) be a real normed space and \( \mathcal{Y} \subset Y \times Y \) be a 2-normed set, where \( Y \) denotes a real linear space. A set \( \mathcal{N} \) is defined as follows:
\[
\mathcal{N} = \left\{ (f, g) \in L(X, Y)^2; \forall x, y \in X \ (f(x), g(y)) \in \mathcal{Y} \land \exists M > 0 \forall x, y \in X \ |f(x), g(y)| \leq M \cdot \|x\| \cdot \|y\| \right\}.
\]

**Lemma 3.7.** The set \( \mathcal{N} \) defined in Definition 3.6 has the following property:
(a) If \( \mathcal{Y} \) is a symmetric 2-normed set, then \( \mathcal{N} = \mathcal{N}^{-1} \).
(b) For every \( f, g \in L(X, Y) \) sets \( \mathcal{N}^g = \{ f' \in L(X, Y); (f', g) \in \mathcal{N} \} \) and \( \mathcal{N}_f = \{ g' \in L(X, Y); (f, g') \in \mathcal{N} \} \) are linear subspaces of the space \( L(X, Y) \). If \( \mathcal{Y} \) is a symmetric 2-normed set, then \( \mathcal{N}^f = \mathcal{N}_f \).

The proof is similar to the proof of Lemma 3.2 so it is omitted.

**Definition 3.8.** For \( (f, g) \in \mathcal{N} \) we introduce a number
\[
\|f, g\| = \inf\{ M > 0; \forall x, y \in X \ |f(x), g(y)| \leq M \cdot \|x\| \cdot \|y\| \}.
\]
The following theorem gives properties of the number \( k_{f,g} \) for \((f,g) \in \mathcal{N}\), which are similar to the properties from Theorem 3.4.

**Theorem 3.9.** If \((f,g) \in \mathcal{N}\), then

(a) \( k_{f,g} \leq M \) for all \( M \in \mathcal{R}^2(f,g) \), where
\[
\mathcal{R}^2(f,g) = \{ M' > 0; \forall x, y \in X \| f(x), g(y) \| \leq M' \cdot \| x \| \cdot \| y \| \};
\]
(b) \( \| f(x), g(y) \| \leq \| f, g \| \cdot \| x \| \cdot \| y \| \) for all \( x, y \in X \);
(c) \[
\| f, g \| = \sup \{ \| f(x), g(y) \|; x, y \in X \land \| x \| = 1 \} = \sup \{ \| f(x), g(y) \|; x, y \in X \land \| x \| \leq 1, \| y \| \leq 1 \} = \sup \left\{ \frac{\| f(x), g(y) \|}{\| x \| \cdot \| y \|}; x, y \in X \lor \| x \| \neq 0, \| y \| \neq 0 \right\}.
\]
(d) \( \| f, g \| = \| g, f \| \), if \( Y \) is a symmetric 2-normed set.

**Theorem 3.10.** The set \( \mathcal{N} \) is a 2-normed set with the 2-norm defined by the formula
\[
\| f, g \| = \sup \{ \| f(x), g(y) \|; x, y \in X \land \| x \| = 1 \} \text{ for } (f, g) \in \mathcal{N}.
\]
If \( Y \) is a symmetric 2-normed set, then the set \( \mathcal{N} \) is also symmetric.

Proofs of Theorem 3.9 and Theorem 3.10 are analogous to proofs of Theorem 3.4 and Theorem 3.5, respectively, therefore they are omitted.

In this section we introduced two 2-normed sets \((\mathcal{M}, \| \cdot, \cdot \|_\mathcal{M})\) and \((\mathcal{N}, \| \cdot, \cdot \|_\mathcal{N})\), where \( \mathcal{N} \subset \mathcal{M} \). Let us remark that for every \((f,g) \in \mathcal{N}\) the inequality
\[
\| f, g \|_\mathcal{M} \leq \| f, g \|_\mathcal{N}
\]
is true.

Finally consider a normed space \((X, \| \cdot \|)\), in which is given a 2-norm in the Gähler’s sense independent of the norm. In [3] S. S. Kim, Y. J. Cho and A. White introduced the following definition of a 2-bounded operator.

**Definition 3.11 ([3]).** An operator \( T: (X, \| \cdot \|) \rightarrow (X, \| \cdot, \cdot \|) \) is said to be 2-bounded if there is a \( K \geq 0 \) such that
\[
\| T(x), y \| + \| x, T(y) \| \leq K \cdot \| x \| \cdot \| y \| \text{ for all } x, y \in X.
\]
Authors of [3] showed that the space \( BL(X,Y) \) of all 2-bounded linear operators from normed space \((X, \| \cdot \|)\) into a 2-normed space \((X, \| \cdot, \cdot \|)\) is a normed space with the norm \( \| \cdot \|_2 \) defined by the formula
\[
\| T \|_2 = \inf \{ K \geq 0; \| T(x), y \| + \| x, T(y) \| \leq K \cdot \| x \| \cdot \| y \| \text{ for all } x, y \in X \}.
\]
Considering a normed space \((X, \| \cdot \|)\), in which is defined also a 2-norm in the Gähler’s sense, we obtain that the set \( \mathcal{N}^{id} \) coincides with the space
where the operator \( id : X \to X \) is defined as follows: \( id(x) = x \) for all \( x \in X \). Thus results in [3] are the special case of the theory in the presented paper.

**References**


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