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THE DIOPHANTINE EQUATION $P(x) = n!$ AND A RESULT OF M. OVERHOLT

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Abstract. In this note, we show that the ABC-conjecture implies that a diophantine equation of the form $P(x) = n!$ with P a polynomial with integer coefficients and degree $d \geq 2$ has only finitely many integer solutions $(x,\ n)$ with $n > 0$.

Let $P \in \mathbf{Z}[X]$ be any polynomial with integer coefficients of degree $d \geq 2$. In this note, we look at the diophantine equation

$$P(x) = n!, \tag{1}$$

where x is an integer. Berend and Osgood (see [1]) showed that the density of the set of positive integers n for which there exists an integer x such that equation (1) is satisfied is zero. Of course, there are several polynomials for which equation (1) is known to have either very few solutions or none. For example, if $P(X) := X^d$, then equation (1) has no solutions with $|x| > 1$. Erdős and Obláth (see [5]) showed that equation (1) has no solutions with $|x| > 1$ if $P(X) := X^d \pm 1$ and $d \geq 3$ is prime, but finding all the solutions of the equation

$$x^2 - 1 = n! \tag{2}$$

is a famously unsolved problem (see **D25** in [6]) which was first posed by Brocard in 1876 (see [3]) and also later by Ramanujan in 1913. Recent computations by Berndt and Galway (see [2]) showed that the largest value of n in the range $n < 10^9$ for which equation (2) has a positive integer solution x is $n = 7$. Thus, while it is very likely that equation (2) has only finitely many positive integer solutions $(x,\ n)$ (and maybe none with $n > 7$), the fact

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that this is so has been proven only conditionally by Overholt (see [7]), who showed that a *weak form of the ABC-conjecture* implies that equation (2) has only finitely many solutions. The weak form of the ABC-conjecture employed by Overholt is the statement that there exists a constant $e > 0$ so that for all integers $x,\ y$ with $x^3 \neq y^2$ the inequality

$$|x^3 - y^2| < N(x^3 - y^2)^e \tag{3}$$

holds, where in the above inequality for a non-zero integer k we use $N(k)$ for the *algebraic radical* of k, namely $N(k) := \prod_{p|k} p$. In a similar vein, Dabrowski (see [4]) showed that if A is any fixed non-zero integer, then with $P(X) := X^2 + A$ equation (1) has (unconditionally) only finitely many integer solutions $(x,\ n)$ when A is not a perfect square, and used Overholt's method to show that the weak ABC-conjecture implies that equation (1) has only finitely many positive integer solutions $(x,\ n)$ as well when A is a perfect square.

In this note, we generalize Overholt's result by pointing out that the full ABC-conjecture implies that equation (1) has only finitely many integer solutions $(x,\ n)$ with $n > 0$, where $P(X)$ is an arbitrary polynomial of degree $d \geq 2$.

Proposition 1. *The ABC-conjecture implies that equation* (1) *has only finitely many solutions* $(x,\ n)$.

We begin by recalling that the ABC-conjecture is the following statement.

The ABC-Conjecture. *For any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ depending only on ε such that whenever A, B and C are three coprime and non-zero integers with $A + B = C$, then*

$$\max(|A|,\ |B|,\ |C|) < C(\varepsilon)N(ABC)^{1+\epsilon}. \tag{4}$$

Proof of the Proposition 1. Let

$$P(X) := a_0X^d + a_1X^{d-1} + \cdots + a_d. \tag{5}$$

We multiply both sides of equation (1) by $d^d a_0^{d-1}$ and rewrite it as

$$y^d + b_1y^{d-1} + \cdots + b_d = cn!, \tag{6}$$

where $c := d^d a_0^{d-1}$, $y := a_0dx$ and $b_i := d^i a_0^{i-1}$ for $i = 1,\ 2,\ \ldots,\ d$. Notice that b_1 is a multiple of d. Thus, we can make the change of variable $z := y + b_1/d$ and rewrite equation (6) as

$$z^d + c_2z^{d-2} + \cdots + c_d = cn! \tag{7}$$

where c_i are some integer coefficients which can be easily computed in terms of a_i and d for $i = 2,\ \ldots,\ n$. Let

$$Q(X) := X^d + c_2X^{d-2} + \cdots + c_d. \tag{8}$$

From here on, we denote by $C_1,\ C_2,\ \ldots$ computable positive constants depending only on the coefficients a_i of the polynomial $P(X)$ for $i = 0,\ 1,\ \ldots,\ d$ and, eventually, a small $\varepsilon > 0$ to be fixed later.

Notice that, when $|z|$ is large, one has

$$\frac{|z|^d}{2} < |Q(z)| < 2|z|^d. \tag{9}$$

Using equation (7) and inequalities (9), we get that there exist two constants C_1 and C_2 such that

$$|d \log |z| - \log(n!)| < C_1, \qquad \text{for } |z| > C_2, \tag{10}$$

whenever $(z,\ n)$ is a solution of (7). From now on, we assume that $|z| > C_2$. For technical reasons which will become more transparent later, we also assume that C_2 is large enough with respect to C_1 such that whenever z and n are integers with $|z| > C_2$ satisfying inequality (10), then $n > c$.

Let $R(X) \in \mathbf{Z}[X]$ be such that

$$Q(X) = X^d + R(X). \tag{11}$$

We may assume that $R(X)$ is non-zero. Indeed, if $R(X)$ is zero, then equation (7) reduces to

$$z^d = cn! \tag{12}$$

It is easy to see that equation (12) has no integer solutions $(z,\ n)$ with $n > 2c$. Indeed, when $n > 2c$, the interval $(n/2,\ n)$ contains a prime larger than c which will appear at the exponent 1 in the product $cn!$, therefore $cn!$ cannot be a perfect power. Thus, by the above argument and inequality (10), we conclude that equation (1) has only finitely many solutions when $R(X)$ is zero.

Assume now that $R(X)$ is non-zero and let $j \leq d$ be the largest integer with $c_j \neq 0$. Rewrite equation (7) as

$$z^j + c_2 z^{j-2} + \cdots + c_j = \frac{cn!}{z^{d-j}}. \tag{13}$$

Let $R_1(X) \in \mathbf{Z}[X]$ be the polynomial

$$R_1(X) := \frac{R(X)}{X^{d-j}} = c_2 X^{j-2} + \cdots + c_j. \tag{14}$$

Let C_3 and $C_4 \geq C_2$ be constants such that

$$0 < |R_1(z)| < C_3 |z|^{j-2}, \qquad \text{for } |z| > C_4. \tag{15}$$

Here, we can take $C_3 := |c_2| + 1$. From now on, we assume that $|z| > C_4$. Rewrite equation (13) as

$$z^j + R_1(z) = \frac{cn!}{z^{d-j}}. \tag{16}$$

Finally, let $D := \gcd(z^d,\ R_1(z))$. By formula (14), it is easy to see that all the prime divisors of D divide c_j. Simplifying by D in both sides of equation (16), we get

$$\frac{z^j}{D} + \frac{R_1(z)}{D} = \frac{cn!}{z^{d-j}D}. \tag{17}$$

We are now all set to apply the ABC-conjecture in inequality (17) with the obvious choices $A := \dfrac{z^j}{D}$, $B := \dfrac{R_1(z)}{D}$ and $C := \dfrac{cn!}{(z^{d-j}D)}$. We get

$$\frac{|z|^j}{D} < C_5 N\Big(\frac{z^j R_1(z) cn!}{D^3}\Big)^{1+\varepsilon}, \tag{18}$$

where C_5 depends only on ε. Let N be the algebraic radical appearing in the right hand side of inequality (18). In what follows, we bound N from above.

Notice that:

$$N\Big(\frac{z^j}{D}\Big) \leq N(z^j) \leq |z|; \tag{19}$$

$$N\Big(\frac{R_1(z)}{D}\Big) \leq \frac{|R_1(z)|}{D} < \frac{C_3|z|^{j-2}}{D}; \tag{20}$$

$$N\Big(\frac{cn!}{Dz^{d-j}}\Big) \leq N(cn!) = N(n!) = \prod_{p\leq n} p < 4^n. \tag{21}$$

In the above inequality (21) we used the fact that $n > c$ as well as the elementary estimate $\prod_{p\leq n} p < 4^n$. From (19)-(21), we get

$$N \leq N\Big(\frac{z^j}{D}\Big)N\Big(\frac{R_1(z)}{D}\Big)N\Big(\frac{cn!}{Dz^{d-j}}\Big) < \frac{C_3|z|^{j-1}4^n}{D}. \tag{22}$$

From inequalities (18) and (22), we get

$$\frac{|z|^j}{D} < C_6\Big(\frac{|z|^{j-1}4^n}{D}\Big)^{1+\varepsilon},$$

where $C_6 := C_5 C_3^{1+\varepsilon}$, or

$$|z|^j < C_6\frac{(|z|^{j-1}4^n)^{1+\varepsilon}}{D^\varepsilon} \leq C_6|z|^{(j-1)(1+\varepsilon)}4^{n(1+\varepsilon)},$$

or

$$|z|^{1+\varepsilon-\varepsilon j} < C_6 4^{n(1+\varepsilon)}. \tag{23}$$

We now choose $\varepsilon := \dfrac{1}{2d} \leq \dfrac{1}{2j}$, and notice that with this choice, inequality (23) implies that

$$|z|^{1/2} < |z|^{1+\varepsilon-\varepsilon j} < C_6 4^{n(1+\varepsilon)},$$

or

$$\log|z| < C_7 n + C_8, \tag{24}$$

where $C_7 := 2(1+\varepsilon)\log 4$ and $C_8 := 2\log C_6$. Hence,

$$d\log|z| < C_9 n + C_{10}, \tag{25}$$

where $C_9 := dC_7$ and $C_{10} := dC_8$. Combining inequalities (10) and (25), we get

$$\log(n!) < C_1 + d\log|z| < C_9 n + C_{11},$$

where $C_{11} := C_1 + C_{10}$, which, together with Stirling's formula for approximating $n!$ implies that $n < C_{12}$. Inequality (10) now tells us that $|z| < C_{13}$, therefore equation (1) has only finitely many integer solutions $(x,\ n)$. □

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