2–ISOMETRIC OPERATORS

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Abstract. An operator $T$ on a complex Hilbert space is called a 2–isometry if $T^*T^2 - 2T^*T + I = 0$. Our underlying purpose in this article is to investigate some algebraic and spectral properties of 2–isometries.

1. Introduction

Let $H$ be a complex Hilbert space. By an operator on $H$, we shall mean a bounded linear transformation from $H$ to $H$. Let $\sigma(T)$, $\pi(T)$, $\pi_0(T)$, $\pi_{00}(T)$ and $w(T)$, respectively denote the spectrum, the approximate point spectrum, the point spectrum, the set of eigenvalues with finite multiplicity and the Weyl spectrum of an operator $T$. We use the symbol $\partial \sigma(T)$ for the boundary of $\sigma(T)$. If for an operator $T$, $w(T) = \sigma(T) \sim \pi_{00}(T)$, then we say that the Weyl's theorem holds for $T$. The spectral radius and the numerical radius of $T$ will be denoted by $r(T)$ and $|W(T)|$ respectively. If $r(T) = |W(T)|$, then $T$ is called a spectraloid operator. By saying that an operator $T$ is power bounded, we mean that there exists some $M > 0$ such that $\|T^n\| \leq M$ for each positive integer $n$. According to [1], an operator $T$ is defined to be a 2–isometry if $T^*T^2 - 2T^*T + I = 0$. In the present note, we explore some properties of 2–isometries.

Clearly every isometry is a 2–isometry. According to [1, Proposition 1.23], an invertible 2–isometry turns out to be a unitary operator. It is obvious from the definition that every 2–isometry is left invertible. In particular if both $T$ and $T^*$ are 2–isometries then $T$ is invertible and so must be unitary.

2. Results

Theorem 2.1. A power of a 2–isometry is again a 2–isometry.
Proof. Let $T$ be a 2–isometry. We prove the assertion by using the mathematical induction. Since $T$ is a 2–isometry, the result is true for $n = 1$. Now assume that the result is true for $n = k$, i.e.,

\[(2.1)\]

\[T^{*2k}T^{2k} - 2T^{*k}T^k + I = 0.\]

Then

\[
T^{*2(k+1)}T^{2(k+1)} - 2T^{*k+1}T^{k+1} + I = T^{*2}(T^{*2k}T^{2k}) - 2T^{*k+1}T^{k+1} + I \quad \text{(by (2.1))}
\]

\[
= 2T^{*k+2}T^{k+2} - T^{*2}T^2 - 2T^{*k+1}T^{k+1} + I \quad \text{(by (2.1))}
\]

\[
= 2T^{*k}(T^{*2}T^2 - T^{*}T)T^k - T^{*2}T^2 + I = 2T^{*k}(T^{*}T - I)T^k - T^{*2}T^2 + I \quad \text{(by (2.1))}
\]

\[
= T^{*2}T^2 - 2T^{*}T + I = 0.
\]

This shows that the result is true for $n = k + 1$: thus $T^n$ is a 2–isometry for each $n$.

It is well known and obvious that a unilateral weighted shift is an isometry iff all its weights lie on the unit circle. In the next result, we obtain a necessary and sufficient condition under which a non–isometric unilateral weighted shift is a 2–isometry.

Theorem 2.2. A non–isometric unilateral weighted shift $T$ with weights $\{\alpha_n\}$ is a 2–isometry if and only if

(i) $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ for each $n$;

(ii) $|\alpha_n| \neq 1$ for each $n$.

Proof. Suppose $T$ is a 2–isometry. If $\{e_n\}$ is an orthonormal base for $H$, then $T e_n = \alpha_n e_{n+1}$ and hence (i) follows. Suppose (ii) is false. Select the least positive integer $k$ such that $|\alpha_k| = 1$. If $k > 1$, then (i) gives $|\alpha_{k-1}| = 1$ which is contrary to the selection of $k$. Therefore $|\alpha_1| = 1$. Using the induction argument and (i), one can show that $|\alpha_n| = 1$ for each positive integer $n$. But this will contradict our assumption that $T$ is non–isometric. Hence we conclude that (ii) is true. The converse assertion is obvious.

Corollary 2.3. Let $T$ be a non–isometric unilateral weighted shift with weights $\{\alpha_n\}$. If $T$ is a 2–isometry, then the following assertions hold.

(i) $\{|\alpha_n|\}$ is a strictly decreasing sequence of real numbers converging to $1$.

(ii) $\sqrt{2} > |\alpha_n| > 1$ for each $n > 1$. 

Proof. (i) Suppose $|\alpha_{n+1}| \geq |\alpha_n|$ for some $n$. Then by Theorem 2.2 (i), we find $0 \geq (1 - |\alpha_n|^2)^2$ or $|\alpha_n| = 1$. But this contradicts Theorem 2.2 (ii). Thus $\{\alpha_n\}$ is a strictly decreasing sequence of real numbers and so must be convergent. By Theorem 2.2 (i), we infer that $|\alpha_n| \to 1$.

(ii) Rewriting equality (i) of Theorem 2.2 as

$$ (2.2) \quad |\alpha_{n+1}|^2 - 2 + 1/|\alpha_n|^2 = 0 $$

we get $\sqrt{2} > |\alpha_n|$ for each $n > 1$. By (i) and Theorem 2.2 (ii), $|\alpha_n| > 1$. This finishes the proof of (ii).

**Theorem 2.4.** A power bounded $2$–isometry is an isometry.

**Proof.** Let $T$ be a power bounded $2$–isometry. Then there exists a positive real number $M$ such that

$$ (2.3) \quad \|T^n\| \leq M $$

for $n = 1, 2, 3, \ldots$. The definition of a $2$–isometry yields

$$ (2.4) \quad \|T^2\|^2 + 1 = 2\|T\|^2. $$

Since $T^n$ is also a $2$–isometry by Theorem 2.1, an induction argument shows that

$$ (2.5) \quad \|T^{2^n}\|^2 = 2^n\|T\|^2 - (2^n - 1) $$

for every positive integer $n$. Now (2.3) and (2.5) will give

$$ M^2/2^n \geq \|T\|^2 - 1 + 1/2^n \geq 0. $$

Letting $n \to \infty$, we find $\|T\| = 1$. In particular, $I \geq T^*T$. Since $T^*T \geq I$ [1, Proposition 1.5], we conclude $T^*T = I$.

**Remark 2.5.** Above theorem can be used to show that unlike isometries, the class of $2$–isometries is not bounded. To see this, use Theorem 2.2 to construct a $2$–isometry $T$, which is not an isometry. Then by Theorem 2.4, we see that for each $M > 0$, there corresponds a positive integer $n$ such that $\|T^n\| > M$. Since Theorem 2.1 says that $T^n$ is also a $2$–isometry, we conclude that the class of $2$–isometries contains operators with arbitrarily large norm.

**Corollary 2.6.** A $2$–isometry similar to a spectraloid operator is an isometry.

**Proof.** Let $T$ be a $2$–isometry. Suppose it is similar to a spectraloid operator $A$. Then $r(T^n) = r(A^n) = |W(A^n)|$ for $n = 1, 2, 3, \ldots$. Since $r(T) = 1$, [1], we find $1 = |W(A^n)|$ and hence $\|A^n\| \leq 2$ for each $n$. Now the similarity of $T$ and $A$ shows that $T$ is power bounded; thus the result follows from the preceding theorem.
Remark 2.7. Above corollary shows that unlike the class of isometries, the class of 2–isometries fails to be a subclass of spectraloid operators.

Corollary 2.8. If $T$ is a 2–isometry, then $1 \in \sigma(T^*T)$.

Proof. Suppose to the contrary that $1 \notin (T^*T)$. Then the operator $A = T^*T - I$ is invertible. Moreover $A \geq 0$ [1, Proposition 1.5]. From the definition of a 2–isometry it follows that $\sigma^*A^*T = A$ or $(A^{1/2}T^2A^{-1/2})^* = A^{1/2}T^2A^{-1/2} = I$ where $A^{1/2}$ denotes the positive square root of $A$. Thus $T$ is similar to an isometry and so must be an isometry by virtue of Corollary 2.6. This contradicts our supposition that $1 \notin \sigma(T^*T)$.

In the rest of the article, we shall obtain some spectral properties of 2–isometries.

Theorem 2.9. Let $T$ be a 2–isometry. Then

(i) $z \in \pi(T)$ implies $z^* \in \pi(T^*)$.

(ii) $z \in \pi_0(T)$ implies $z^* \in \pi_0(T^*)$.

(iii) Eigenvectors of $T$ corresponding to distinct eigen–values are orthogonal.

Proof. (i) Let $z \in \pi(T)$. Choose a sequence $\{x_n\}$ of unit vectors such that $(T - zI)x_n \to 0$. Then $(T^*T - z^2I)x_n \to 0$ and $T^*Tx_n - z^2T^*x_n \to 0$. The hypothesis that $T$ is a 2–isometry yields $0 = T^*T - 2T^*T + I = T^*T^2 - z^2T + 2T^*T - 2z^2T + z^2T + I$. This will imply $z^2T^2x_n - 2z^2T^*x_n + x_n \to 0$. Since $\pi(T)$ is a subset of the unit circle [1], we find $(T^* - z^*I)^2x_n \to 0$. From this it follows that $z^* \in \pi(T^*)$.

(ii) The argument is similar to one given in (i).

(iii) Let $\lambda$ and $\mu$ be distinct eigen–values of $T$. Suppose $Tx = \lambda x$ and $Ty = \mu y$. Then $0 = ((T^*T - 2T^*T + I)x, y) = (T^2x, T^2y) - 2(Tx, Ty) + (x, y) = (\lambda^2 - 2\lambda\mu + 1)(x, y)$. Since $\lambda \neq \mu$ with $|\lambda| = 1 = |\mu|$, $\lambda^2 - 2\lambda\mu + 1 = (\lambda - \mu)^2 \neq 0$. This leads to $(x, y) = 0$ which proves the assertion.

Theorem 2.10. The spectrum of a 2–isometry is the closed unit disc provided it is non–unitary.

Proof. Let $T$ be a non–unitary 2–isometry. Then $0 \in \sigma(T) \sim \pi(T)$. Since $\partial\pi(T) \supseteq \pi(T)$, $0$ turns out to be an interior point of $\sigma(T)$. Therefore we can find the largest positive number $r$ such that $\{z : |z| < r\}$ is contained in $\sigma(T)$. It is possible to select a complex number $z$ in $\partial\sigma(T)$ such that $r = |z|$. Since $\partial\sigma(T) \subseteq \pi(T) \subseteq \{z : |z| = 1\}$ [1], $r = 1$. Consequently we find $\sigma(T) = \{z : |z| \leq 1\}$.

Corollary 2.11. If $T$ is a 2–isometry, then each isolated point in its spectrum is an eigen–value.

Proof. If $\sigma(T)$ has an isolated point, then it is clear from the above theorem that $T$ is unitary and hence the result follows.
Corollary 2.12. Let $T$ be a 2–isometry. If the Lebesgue planar measure of $\sigma(T)$ is zero, then $T$ is unitary.


Proof. The result holds if $T$ is unitary. Assume that $T$ is non-unitary. Then Theorem 2.10 shows that $\pi_{00}(T) = \emptyset$. Also by Theorem 2.9 (ii) and Lemma 3 of [2], $\sigma(T) \sim \pi_{00}(T) \subseteq w(T)$ and hence $\sigma(T) \subseteq w(T)$. This completes the argument.

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