

ON REDUNDANCE OF ONE OF THE AXIOMS OF A GENERALIZED NORMED SPACE

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ABSTRACT. It is shown that one of the Zalar's axioms of a generalized normed space is redundant. More precisely, Jordan-von Neumann type theorem for a pre-Hilbert module is also valid if that axiom is omitted.

The concept of an H^* -algebra was introduced by W. Ambrose in [1] in order to obtain an abstract characterization of Hilbert-Schmidt operators.

DEFINITION 1. Let $(A, |\cdot|)$ be a complex Banach algebra with an inner product $\langle \cdot, \cdot \rangle$ such that $\langle a, a \rangle = |a|^2$ for all $a \in A$. If for each $a \in A$ there exists some $a^* \in A$ such that

$$\langle ab, c \rangle = \langle b, a^*c \rangle \text{ and } \langle ba, c \rangle = \langle b, ca^* \rangle \text{ for all } b, c \in A,$$

then A is called an H^* -algebra.

Proper H^* -algebra is an H^* -algebra which satisfies the following equivalent conditions:

- (i) $aA = 0 \Rightarrow a = 0$;
- (ii) $Aa = 0 \Rightarrow a = 0$.

An H^* -algebra A is proper if and only if each $a \in A$ has a unique adjoint $a^* \in A$. From now on, A will denote a proper H^* -algebra.

An element $a \in A$ is called

- (i) self-adjoint if $a = a^*$;
- (ii) a projection if $a = a^* = a^2 \neq 0$;
- (iii) normal if $a^*a = aa^*$;
- (iv) positive ($a \geq 0$) if $\langle ax, x \rangle \geq 0$ for every $x \in A$.

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It is not difficult to establish that every $a \geq 0$ is self-adjoint. Projections $e, f \in A$ are called mutually orthogonal if $\langle e, f \rangle = \langle f, e \rangle = 0$ which is equivalent to $ef = fe = 0$. A primitive projection is a projection which can not be expressed as the sum of two mutually orthogonal projections. If $e \in A$ is a primitive projection, then $eAe = \mathbb{C}e$ holds. In this article, by a projection base in A we mean a maximal family of mutually orthogonal primitive projections. If $\{e_i\}$ is a projection base in A , then $|a|^2 = \sum_i |ae_i|^2$ holds for every $a \in A$. The existence of a projection base in A , as well as other recalled facts, was proved in the pioneering paper [1].

According to [3], every normal element $a \in A$ can be expressed as $\sum_i \mu_i e_i$, where $\{\mu_i\}$ is a family of complex numbers and $\{e_i\}$ is a projection base which will be called a projection base associated with a .

The trace-class for A was defined to be the set $\tau(A) = \{ab : a, b \in A\}$. It is a self-adjoint ideal of A which is dense in A . There is a norm $\tau(\cdot)$ on the trace-class such that $(\tau(A), \tau(\cdot))$ is a Banach algebra. The inequality $\tau(ab) \leq |a||b|$ holds for all $a, b \in A$.

LEMMA 2. For all $a, b \in A$, $\tau(a^2 - b^2) \leq |a - b||a + b|$.

PROOF. The expression (used in [4])

$$a^2 - b^2 = \frac{1}{2}(a - b)(a + b) + \frac{1}{2}(a + b)(a - b)$$

implies

$$\begin{aligned} \tau(a^2 - b^2) &\leq \frac{1}{2}\tau((a - b)(a + b)) + \frac{1}{2}\tau((a + b)(a - b)) \leq \\ &\leq \frac{1}{2}|a - b||a + b| + \frac{1}{2}|a + b||a - b| = |a - b||a + b|. \end{aligned}$$

□

If $a \in \tau(A)$ is positive, then there is a unique positive element $b \in A$ such that $b^2 = a$; b is called a square root of a . For every $a \in A$, a square root of a^*a is denoted by $[a]$.

LEMMA 3. For every $a \in A$, $|[a]| = |a|$.

PROOF.

$$|[a]|^2 = \sum_i |[a]e_i|^2 = \sum_i \langle [a]^2 e_i, e_i \rangle = \sum_i \langle a^* a e_i, e_i \rangle = \sum_i |a e_i|^2 = |a|^2$$

□

Hilbert A -modules were first defined by P. P. Saworotnow in [2]. A Hilbert A -module arises as a generalization of a complex Hilbert space when the complex field is replaced by a proper H^* -algebra. We shall omit the axiom of completeness and consider pre-Hilbert A -modules.

DEFINITION 4. Let H be a right module over A . If a $\tau(A)$ -valued mapping $(\cdot, \cdot)$ on $H \times H$ has the following properties:

- (i) $(f, g + h) = (f, g) + (f, h)$ for all $f, g, h \in H$;
- (ii) $(f, ga) = (f, g)a$ for all $f, g \in H$ and $a \in A$;
- (iii) $(f, g)^* = (g, f)$ for all $f, g \in H$;
- (iv) $(f, f) \geq 0$ for every $f \in H$ and $(f, f) = 0 \Rightarrow f = 0$;

then it is called a generalized inner product and $(H, (\cdot, \cdot))$ is called a pre-Hilbert A -module.

Naturally, the question of analogous generalization of a complex normed space appeared. The notion of a generalized normed space was introduced by B. Zalar in [4].

DEFINITION 5. Let H be a right module over A . Let $N : H \rightarrow A$ be a mapping with the following properties:

- (i) $N(f) \geq 0$ for every $f \in H$;
- (ii) $N(f) = 0 \Rightarrow f = 0$;
- (iii) $N(fa) = [N(f)a]$ for all $f \in H$ and $a \in A$;
- (iv) $|N(f + g)| \leq |N(f)| + |N(g)|$ for all $f, g \in H$;
- (v) If $\{f_\alpha\}$ is a generalized sequence in H such that for all $\varepsilon > 0$ there exists α_0 such that for all $\alpha, \beta \geq \alpha_0$ we have $|N(f_\alpha - f_\beta)| < \varepsilon$, then $\{N(f_\alpha)\}$ is a generalized Cauchy sequence in A .

Then N is called an A -valued generalized norm (A -norm) and (H, N) is called a generalized normed space or a normed A -module.

If A is the H^* -algebra of complex numbers, then H is a usual complex normed space.

The main result in [4] is a generalization of the well-known Jordan-von Neumann theorem which characterizes pre-Hilbert spaces among normed spaces:

THEOREM 6. A normed A -module (H, N) satisfies the parallelogram law

$$N(f + g)^2 + N(f - g)^2 = 2N(f)^2 + 2N(g)^2 \text{ for all } f, g \in H$$

if and only if H is a pre-Hilbert A -module with respect to the generalized inner product $(\cdot, \cdot)$ such that $N(f)^2 = (f, f)$ holds for all $f \in H$.

Let us note that axiom (v) represents some sort of A -norm continuity. Zalar questioned if that axiom was redundant. The aim of this article is to show that axiom (v) can be omitted, by proving

THEOREM 7. Let H be a right module over A . If $N : H \rightarrow A$ is a mapping that satisfies the axioms (i)-(iv) in Definition 5 and the parallelogram law, then it also satisfies (v).

The proof of this theorem consists of several steps.

STEP 1. If $f \in H$ and $e \in A$ is a primitive projection, then there exists a nonnegative number λ such that $N(fe) = \lambda e$.

PROOF. Axiom (iii) implies

$$N(fe)^2 = [N(f)e]^2 = eN(f)^2e \in eAe = \mathbf{C}e.$$

Hence there exists $\mu \in \mathbf{C}$ such that $N(fe)^2 = \mu e$. But

$$\mu|e|^2 = \langle \mu e, e \rangle = \langle N(fe)^2 e, e \rangle = |N(fe)e|^2$$

implies $\mu \geq 0$. If we put $\lambda = \sqrt{\mu}$, then we have $N(fe)^2 = \lambda^2 e = (\lambda e)^2$. Since $N(fe) \geq 0$ and $\lambda e \geq 0$, it follows that $N(fe) = \lambda e$. $\square$

STEP 2. For all $f \in H$ and $a \in A$, $|N(fa)| = |N(f)a|$.

PROOF. Directly from axiom (iii) and Lemma 3. $\square$

STEP 3. If $\{e_i\}$ is a projection base in A , then $\sum_i N(fe_i) \in A$ for every $f \in H$ and $|\sum_i N(fe_i)| = |N(f)|$ holds.

PROOF. Step 1 implies the existence of a family $\{\lambda_i\}$ of nonnegative numbers such that $N(fe_i) = \lambda_i e_i$ holds for every i . Consequently, the family $\{N(fe_i)\}$ is orthogonal. Moreover,

$$\sum_i \left| N(fe_i) \right|^2 \stackrel{\text{Step 2}}{=} \sum_i |N(f)e_i|^2 = |N(f)|^2.$$

Thus, $\sum_i N(fe_i) \in A$ and $|\sum_i N(fe_i)| = |N(f)|$. $\square$

STEP 4. If $f, g \in H$ and $e \in A$ is a primitive projection, then the inequality

$$|N(fe + ge)| \leq |N(fe) + N(ge)|$$

holds.

PROOF. According to axiom (iv),

$$|N(fe + ge)| \leq |N(fe)| + |N(ge)|.$$

Let $\alpha, \beta \geq 0$ be such that $N(fe) = \alpha e$ and $N(ge) = \beta e$ hold. We have

$$\begin{aligned} |N(fe)| + |N(ge)| &= |\alpha e| + |\beta e| = \alpha|e| + \beta|e| = \\ &= |(\alpha + \beta)e| = |\alpha e + \beta e| = |N(fe) + N(ge)|. \end{aligned}$$

$\square$

STEP 5. For all $f, g \in H$,

$$|N(f + g)|^2 + |N(f - g)|^2 = |N(f) + N(g)|^2 + |N(f) - N(g)|^2.$$

PROOF. Let $\{e_i\}$ be a projection base in A . For every i , we have

$$\begin{aligned} |N(f+g)e_i|^2 + |N(f-g)e_i|^2 &= \langle N(f+g)^2 e_i, e_i \rangle + \langle N(f-g)^2 e_i, e_i \rangle = \\ &= \langle (N(f+g)^2 + N(f-g)^2) e_i, e_i \rangle = \langle (2N(f)^2 + 2N(g)^2) e_i, e_i \rangle = \\ &= 2\langle N(f)^2 e_i, e_i \rangle + 2\langle N(g)^2 e_i, e_i \rangle = 2|N(f)e_i|^2 + 2|N(g)e_i|^2. \end{aligned}$$

Summing over the set of all i gives

$$|N(f+g)|^2 + |N(f-g)|^2 = 2|N(f)|^2 + 2|N(g)|^2.$$

Since the norm $|\cdot|$ in A satisfies the parallelogram law, the right side is equal to $|N(f) + N(g)|^2 + |N(f) - N(g)|^2$. $\square$

STEP 6. If $\{e_i\}$ is a projection base in A and $f, g \in H$, then

$$\left| \sum_i N(fe_i) - \sum_i N(ge_i) \right| \leq |N(f-g)|$$

holds.

PROOF. Since Step 5 implies

$$|N(fe_i + ge_i)|^2 + |N(fe_i - ge_i)|^2 = |N(fe_i) + N(ge_i)|^2 + |N(fe_i) - N(ge_i)|^2$$

and Step 4

$$|N(fe_i + ge_i)| \leq |N(fe_i) + N(ge_i)|,$$

therefore for every i we have

$$|N(fe_i) - N(ge_i)| \leq |N(fe_i - ge_i)| = |N((f-g)e_i)| \stackrel{\text{Step 2}}{=} |N(f-g)e_i|$$

and hence

$$\sum_i |N(fe_i) - N(ge_i)|^2 \leq \sum_i |N(f-g)e_i|^2 = |N(f-g)|^2.$$

According to Step 3, $\sum_i N(fe_i) \in A$ and $\sum_i N(ge_i) \in A$. Applying Step 1, we conclude that for every i there exist $\alpha_i \geq 0$ and $\beta_i \geq 0$ such that $N(fe_i) = \alpha_i e_i$ and $N(ge_i) = \beta_i e_i$. Now we have

$$\begin{aligned} \left| \sum_i N(fe_i) - \sum_i N(ge_i) \right|^2 &= \left| \sum_i \alpha_i e_i - \sum_i \beta_i e_i \right|^2 = \\ &= \left| \sum_i (\alpha_i - \beta_i) e_i \right|^2 = \sum_i |\alpha_i e_i - \beta_i e_i|^2 = \\ &= \sum_i |N(fe_i) - N(ge_i)|^2 \leq |N(f-g)|^2. \end{aligned}$$

$\square$

STEP 7. For all $f, g \in H$,

$$|N(f) - N(g)|^2 \leq |N(f-g)| \cdot (|N(f)| + |N(g)|).$$

PROOF. Let $\{e_i\}$ be a projection base in A associated with $N(f)^2 - N(g)^2$ and let $\{\mu_i\}$ be a family of complex numbers such that $N(f)^2 - N(g)^2 = \sum_i \mu_i e_i$. Step 1 implies the existence of families $\{\alpha_i\}$ and $\{\beta_i\}$ of nonnegative numbers such that $N(fe_i) = \alpha_i e_i$ and $N(ge_i) = \beta_i e_i$ hold for every i . Since

$$\begin{aligned} \left(\sum_i N(fe_i)\right)^2 - \left(\sum_i N(ge_i)\right)^2 &= \left(\sum_i \alpha_i e_i\right)^2 - \left(\sum_i \beta_i e_i\right)^2 = \\ &= \sum_i (\alpha_i e_i)^2 - \sum_i (\beta_i e_i)^2 = \sum_i N(fe_i)^2 - \sum_i N(ge_i)^2 = \\ &= \sum_i [N(f)e_i]^2 - \sum_i [N(g)e_i]^2 = \sum_i e_i N(f)^2 e_i - \sum_i e_i N(g)^2 e_i = \\ &= \sum_i e_i (N(f)^2 - N(g)^2) e_i = \sum_i \mu_i e_i = N(f)^2 - N(g)^2, \end{aligned}$$

we have

$$\begin{aligned} \tau(N(f)^2 - N(g)^2) &= \tau\left(\left(\sum_i N(fe_i)\right)^2 - \left(\sum_i N(ge_i)\right)^2\right) \stackrel{\text{Lemma 2}}{\leq} \\ &\leq \left|\sum_i N(fe_i) - \sum_i N(ge_i)\right| \cdot \left|\sum_i N(fe_i) + \sum_i N(ge_i)\right| \stackrel{\text{Step 6}}{\leq} \\ &\leq |N(f - g)| \cdot \left|\sum_i N(fe_i) + \sum_i N(ge_i)\right| \leq \\ &\leq |N(f - g)| \cdot \left(\left|\sum_i N(fe_i)\right| + \left|\sum_i N(ge_i)\right|\right) \stackrel{\text{Step 3}}{=} \\ &= |N(f - g)| \cdot (|N(f)| + |N(g)|). \end{aligned}$$

Lemma 10(3) from [4] completes the proof. $\square$

STEP 8. If $\{f_\alpha\}$ is a generalized sequence in H such that for all $\varepsilon > 0$ there exists α_0 such that for all $\alpha, \beta \geq \alpha_0$ we have $|N(f_\alpha - f_\beta)| < \varepsilon$, then $\{N(f_\alpha)\}$ is a generalized Cauchy sequence in A .

PROOF. There exists α' such that

$$\begin{aligned} \alpha \geq \alpha' &\Rightarrow |N(f_\alpha - f_{\alpha'})| < 1 \Rightarrow |N(f_\alpha)| = |N((f_\alpha - f_{\alpha'}) + f_{\alpha'})| \leq \\ &\leq |N(f_\alpha - f_{\alpha'})| + |N(f_{\alpha'})| < 1 + |N(f_{\alpha'})| \stackrel{\text{def}}{=} M. \end{aligned}$$

For a given $\varepsilon > 0$, there exists α'' with the property

$$\alpha, \beta \geq \alpha'' \Rightarrow |N(f_\alpha - f_\beta)| < \frac{\varepsilon^2}{2M}.$$

There is α_0 such that $\alpha' \leq \alpha_0$ and $\alpha'' \leq \alpha_0$ hold. Step 7 now gives

$$\alpha, \beta \geq \alpha_0 \Rightarrow |N(f_\alpha) - N(f_\beta)|^2 < \frac{\varepsilon^2}{2M} \cdot (M + M) = \varepsilon^2.$$

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